A Study of Riordan Arrays with Applications to Continued Fractions, Orthogonal Polynomials and Lattice Paths

by
Aoife Hennessy

A thesis submitted in partial fulfillment for the
degree of Doctor of Philosophy

in the
Department of Computing, Mathematics and Physics
Waterford Institute of Technology

Supervisor: Dr. Paul Barry
Head of the School of Science
Waterford Institute of Technology

October 2011

Declaration of Authorship

I, AOIFE HENNESSY, declare that this thesis titled, 'A Study of Riordan Arrays with Applications to Continued Fractions, Orthogonal Polynomials and Lattice Paths' and the work presented in it are my own. I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this Institute.
2. Where I have consulted the published work of others, this is always clearly attributed.
3. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
4. I have acknowledged all main sources of help.

Signed:
____

Date:
____

Abstract

We study integer sequences using methods from the theory of continued fractions, orthogonal polynomials and most importantly from the Riordan groups of matrices, the ordinary Riordan group and the exponential Riordan group. Firstly, we will introduce the Riordan group and their links through orthogonal polynomials to the Stieltjes matrix. Through the context of Riordan arrays we study the classical orthogonal polynomials, the Chebyshev polynomials. We use Riordan arrays to calculate determinants of Hankel and Toeplitz-plus-Hankel matrices, extending known results relating to the Chebyshev polynomials of the third kind to the other members of the family of Chebyshev polynomials. We then define the form of the Stieltjes matrices of important subgroups of the Riordan group. In the following few chapters, we develop the well established links between orthogonal polynomials, continued fractions and Motzkin paths through the medium of the Riordan group. Inspired by these links, we extend results to the Łukasiewicz paths, and establish relationships between Motzkin, Schröder and certain Łukasiewicz paths. We concern ourselves with the Binomial transform of integer sequences that arise from the study of Łukasiewicz and Motzkin paths and we also study the effects of this transform on lattice paths. In the latter chapters, we apply the Riordan array concept to the study of sequences related to MIMO communications through integer arrays relating to the Narayana numbers. In the final chapter, we use the exponential Riordan group to study the historical Euler-Seidel matrix. We calculate the Hankel transform of many families of sequences encountered throughout.

Acknowledgements

I wish to express my most sincere gratitude to Dr. Paul Barry, who has been extremely helpful and patient. It has been a privilege to work with Paul over the last few years. I could not have wished for a better supervisor.

To Peter, thank you for cooking for me, keeping me in wine, and putting up with me playing on my laptop for an unforgivable length of time!

Lastly, thanks to my family and friends for all their support, especially Mam, Keith, and all my mothering Aunties, whose kindness I can only hope to return.

Notation

- $\mathbb{R}$ The set of real numbers.
- $\mathbb{Z}$ The set of integers.
- $\mathbb{Z}^2$ The integer lattice.
- $\mathbb{Q}$ The set of rational numbers.
- $\mathbb{C}$ The set of complex numbers.
- **o.g.f.** Ordinary generating function.
- **e.g.f.** Exponential generating function.
- $\mathbf{c(x)}$ The generating function of the sequence of Catalan numbers.
- $\mathbf{c_n}$ The n^{th} Catalan number.
- $\mathbf{[x^n]f(x)}$ The coefficient of the x^n term of the power series $f(x)$.
- $\mathbf{0^n}$ The sequence $1, 0, 0, 0, \ldots,$ with o.g.f. 1.
- $\mathbf{\bar{f}(x)}$ or $\mathrm{Rev}(\mathbf{f(x)})$ The series reversion of the series $f(x)$, where $f(0) = 0$.
- $\mathbf{L}$ A Riordan array.
- $\overline{\mathbf{L}}$ The matrix with $\overline{\mathbf{L}}_{\mathbf{n,k}} = \mathbf{L_{n+1,k}}$.
- $(\mathbf{g}, \mathbf{f})$ An ordinary Riordan array.
- $[\mathbf{g}, \mathbf{f}]$ An exponential Riordan array.
- $\mathbf{S}$ The Stieltjes matrix.
- $\mathbf{H_f}$ The Hankel matrix of the coefficients of the power series $f(x)$ where the $(i, j)^{th}$ element of the power series $a_{i+j} = [x^{i+j}]f(x)$.
- $\mathbf{L} = \overline{\mathbf{L}}\mathbf{S}$ The Stieltjes equation.
- $\mathbf{B(n)}$ The Sequence of Bell numbers.
- $\mathbf{S(n, k)}$ The Stirling numbers of the second kind.

- $\mathbf{N_m(n,k)}$ The m^{th} Narayana triangle, $m = 0, 1, 2$.
- (Axxxxxx) A-number. The On-line Encyclopedia of Integer Sequences (OEIS [124]) reference for an integer sequence.
- δ The Kronecker delta, $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

Contents

Chapter 1

Introduction

This thesis is concerned with the connection between Riordan arrays, continued Fractions, orthogonal Polynomials and lattice paths. From the outset, the original questions proposed related to aspects of the algebraic structure of Hankel, Toeplitz and Toeplitz-plus-Hankel matrices which are associated with random matrices, and how such algebraic structure could be exploited to provide a more comprehensive analysis of their behaviour? Matrices which could be associated with certain families of orthogonal polynomials were of particular interest. We were concerned with how the presence of algebraic structure was reflected in the properties of these polynomials. The algebraic structure of interest was that of the Riordan group, named after the combinatorialist John Riordan. Riordan was an American mathematician who worked at Bell Labs for most of his working life. He had a strong influence on the development of combinatorics. In 1989, The Riordan group, named in his honour, was first introduced by Shapiro, Getu, Woan and Woodson in a seminal paper [119].

The Riordan group (exponential Riordan group) is a set of infinite lower triangular matrices, where each matrix is defined by a pair of generating functions

$$g(x) = g_0 + g_1x + g_2x^2 + \dots \quad (g(x) = g_0 + g_1\frac{x}{1!} + g_2\frac{x^2}{2!} + \dots), g_0 \neq 0$$

$$f(x) = f_1x + f_2x^2 + \dots \quad (f(x) = f_1\frac{x}{1!} + f_2\frac{x^2}{2!} + \dots)$$

The associated matrix is the matrix whose k^{th} column is generated by $g(x)f(x)^k$ $(g(x), \frac{f(x)^k}{k!})$. The matrix corresponding to the pair g, f is denoted $(g, f)([g, f])$ and is called a (exponential) Riordan array.

Shapiro and colleagues Paul Peart and Wen-Jin Woan at Howard University Washington, continue to carry out research into Riordan arrays and their applications. Riordan arrays are also an active area of research in the Universitá di Firenze in Italy, where Renzo Sprugnoli maintains a bibliography [117] of Riordan arrays research. We will introduce relevant results relating to the Riordan group in Chapter 2. In Chapter 3 we classify important subgroups of the Riordan group using the production matrices of the Riordan arrays. This preliminary classification of subgroups aids work in subsequent chapters of this thesis.

As previously stated, original questions proposed related to aspects of the algebraic structure of Hankel, Toeplitz and Toeplitz-plus-Hankel matrices which are associated with random matrices. This led us to study the work of Estelle Basor and Thorsten Ehrhardt [18]. Basor and Ehrhardt proved combinatorial identities relating to certain Hankel and associated Toeplitz-plus-Hankel matrices with a view to studying the asymptotics of those matrices. Through the algebraic structure of Riordan arrays we found a novel approach to developing these combinatorial identities. Using Riordan arrays we extended similar results to the family of Chebyshev polynomials. Part of this chapter has been submitted for publication [17]. A basis for this study is the Riordan matrix representation of Chebyshev polynomials. We note that Chebyshev polynomials recur in later chapters of this work, where again their links to Riordan arrays allow us to find new results. Further research on Riordan arrays and orthogonal polynomials resulted in the classification of Riordan arrays that determine classical orthogonal polynomials [14].

Further to this, another question originally proposed involved investigating aspects of random matrices with applications to the theory of communications. This was with a view to classifying systems that exhibit special algebraic structures and the investigation of combinatorial aspects related to these algebraic structures. This work is detailed in Chapter 8 and published in [13].

In the 1950's Eugene P. Wigner (1902 - 1995), a Hungarian born physicist who received the Nobel prize for physics in 1963, detailed the properties of an important set of random matrices. Wigner used random matrices in an attempt to model the energy levels of nuclear reactions in quantum physics. It was through the work of Wigner that combinatorial identities in random matrices first emerged. Wigner's work gave us one of the most important results in the field of random matrices, the Wigner semi-circle law:

Wigner's semi-circle law states that for an ensemble of $N \times N$ real symmetric matrices with entries chosen from a fixed probability density with mean 0 and variance 1, and finite higher moments. As $N \to \infty$, for all A(in the ensemble), $\mu_{A,N}(x)$, the eigenvalue probability distribution, converges to the semi-circle density

$$\frac{2}{\pi}\sqrt{1-x^2}.$$

We note that the density function $\sqrt{1-x^2}$ is the weight function of the Chebyshev polynomials of the second kind. We will see through the medium of Riordan arrays how these Chebyshev polynomials relate to the Catalan numbers.

The Catalan numbers is the sequence of numbers with first few elements $1, 1, 2, 5, 14 \ldots$, where the n^{th} element, c_n of the sequence is defined as

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \qquad n > 0.$$

which satisfy the recurrence formula

$$c_{n+1} = \sum_{i=1}^{n} c_i c_{n-i}$$

The generating function for the Catalan numbers, $C(x)$ is defined by

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

A summary of the properties of the Catalan numbers can be found at http://www-math.mit.edu/ rstan/ec/catadd.pdf, "The Catalan addendum", which is maintained by Richard Stanley [136].

Although Wigner does not explicitly name the Catalan numbers in his related paper [159], the Catalan numbers appear implicitly through his method of calculating the moments by the trace formula

$$m_k = \frac{1}{n} E[tr(A^k)].$$

In studying the trace of the matrix, Wigner eliminates non-relevant terms in the trace and concentrates on the relevant sequences, which he calls type sequences. It is through the type sequence that we see the appearance of the Catalan numbers. Wigner denotes the type sequence, t_v. He finds that

$$t_v = \sum_{k=1}^{v} t_{k-1} t_{v-k}$$

which is the recursive relationship for the Catalan numbers which we see written today as

$$c_{n+1} = \sum_{i=1}^{n} c_i c_{n-i}.$$

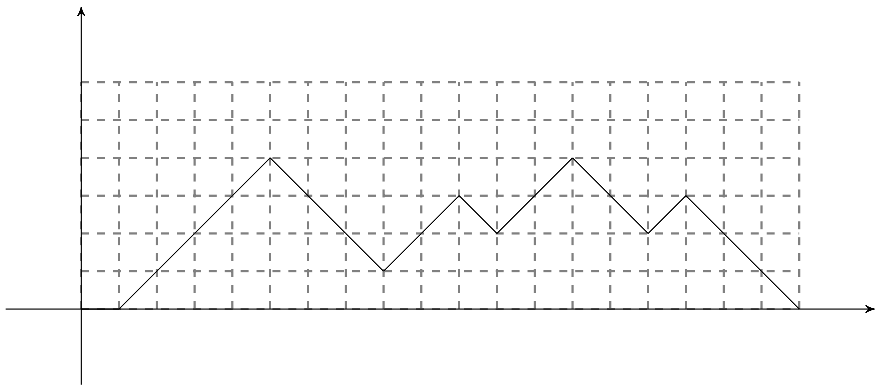


Figure 1.1: A Dyck path

In 1999, Emre Telatar [143], a researcher at Bell Labs, used the distribution associated to a particular random matrix family to calculate the capacity of multi-antenna channels. In [13] we calculate the channel capacity of a MIMO channel using Narayana polynomials which are formed from the Narayana triangle. Calculations in [13] draw on classical results from random matrix theory, in particular from the work of Vladimir Marchenko and Leonid Pastur [83].

The $(n,k)^{th}$ Narayana number, $N_{n,k}$ is defined as

$$N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}.$$

$$\sum_{k=0}^{n} N_{n,k} = c_{n+1}$$

where c_{n+1} is the $(n+1)^{th}$ Catalan number.

Further to this, Ioana Dimitriu [47] continued researching Wiger's eigenvalue distribution and used this to establish combinatorial links to random matrix theory. Dimitriu showed that the asymptotically relevant terms in the trace corresponded to Dyck paths. Inspired by these links we extended our research to lattice paths.

This work on lattice paths was influenced by the far-reaching research carried out by Xavier Viennot and Phillipe Flajolet. Both Viennot and Flajolet explored links between orthogonal polynomials, continued fractions and various combinatorial interpretations including lattice paths and integer partitions. Inspired by some of the work

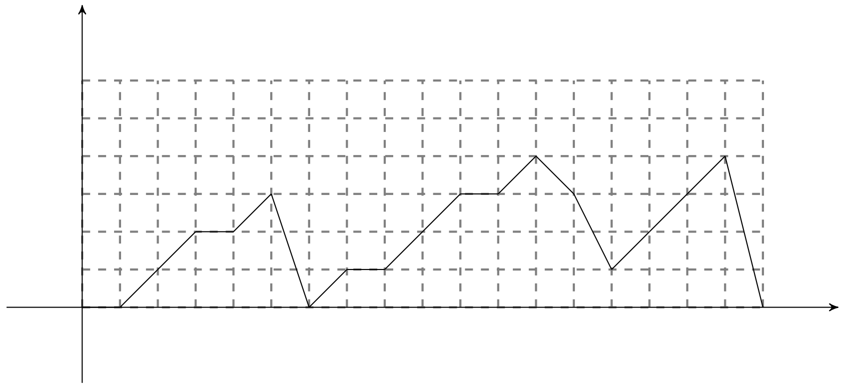


Figure 1.2: A Łukasiewicz path

carried out by Viennot [151] and Flajolet [56, 55] and through the medium of Riordan arrays we established links between orthogonal polynomials, continued fractions and certain lattice paths. Chapter 5 examines these links between Riordan arrays, their production matrices and associated lattice paths. A significant new result concerns Łukasiewicz paths and Riordan arrays possessing non-tridiagonal Stieltjes matrices. This allowed us to extend to general Riordan arrays results normally studied in the context of Riordan arrays with tridiagonal Stieltjes matrices. Riordan arrays having tridiagonal Stieltjes matrices correspond to Motzkin and Dyck paths. We generalized this fact to non-tridiagonal Stieltjes matrices in order to study the form of Łukasiewicz paths and to classify those paths that relate to general Riordan arrays. In studying paths relating to both tridiagonal and non-tridiagonal Stieltjes matrices we established relationships between various Motzkin and Łukasiewicz paths which resulted in the following bijections:

- The $(2,2)$-Łukasiewicz path and the Schröder paths.
- The $(1,1)$-Motzkin paths of length n and the $(1,0)$-Łukasiewicz paths of length $n+2$.
- The Motzkin paths of length n with no level step on the x axis and the Łukasiewicz paths with no level steps.

Extending on the research presented in Chapter 5, in Chapter 6 we studied a decomposition of Hankel matrices, using Riordan arrays which related to Łukasiewicz paths. Inspired by work carried out by Paul Peart and Wen-Jin Woan [103] we decomposed

Hankel matrices in terms of Riordan arrays relating to Łukasiewicz paths. Peart and Woan decomposed Hankel matrices into Riordan arrays with tridiagonal Stieltjes matrices. To begin, we related the Hankel decompositions from Peart and Woan to continued fraction expansions, orthogonal polynomials and Motzkin paths. We then studied the decomposition of Hankel matrices into Riordan arrays which related to non-tridiagonal Stieltjes matrices and consequently to Łukasiewicz paths. From this we established a Riordan array decomposition relating Łukasiewicz to Motkzin paths. Due to the invariance of the Hankel transform under the binomial transform, we studied the form of certain continued fraction expansions of generating functions arising after applying the binomial transform. We also explored the use of differential equations to study Łukasiewicz paths.

To conclude, and once again inspired by our interest in Hankel matrices, the final area of research is that of the classical Euler matrices. We detailed the link between these classical matrices and the Hankel matrices generated from the integer sequences that form the Euler-Seidel matrix. Chapter 9 is based on a published paper [15], and extends on these results.

Chapter 2

Preliminaries

In this chapter we review mainly known results related to integer sequences and Riordan arrays that will be referred to in the rest of the work. In the final section, we explore links between Motzkin and Łukasiewicz paths, Riordan arrays and orthogonal polynomials.

2.1 Integer sequences and generating functions

Formal power series [55] extend algebraic operations on polynomials to infinite series of the form

$$g = g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Let $\mathbb{K}(\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C})$ be a ring of coefficients. The ring of formal power series over $\mathbb{K}$ is denoted by $\mathbb{K}[[x]]$ and is the set $\mathbb{K}^{\mathbb{N}}$ of infinite sequences of elements of $\mathbb{K}$, with operations

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$

$$\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n = \sum_{n} \sum_{k=0}^{n} (a_k b_{n-k}) x^n.$$

Definition 2.1.1. *The* ordinary generating function (o.g.f.) *of a sequence a_n is the formal power series*

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Example. *The o.g.f. $c(x) = \sum_{n=0}^{\infty} C_n x^n$ of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ is given by*

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

Definition 2.1.2. *The* exponential generating function (e.g.f.) *of a sequence a_n is the formal power series*

$$g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Example. *The e.g.f. of the quadruple factorial numbers $\frac{(2n)!}{n!}$ is given by*

$$\frac{1}{\sqrt{1-4x}}.$$

Definition 2.1.3. *The* bivariate generating functions (b.g.f.'s)*, either ordinary or exponential of an array $a_{n,k}$ are the formal power series in two variables defined by*

$$a(x,y) = \sum_{n,k} a_{n,k} x^n y^k \quad (o.g.f.), \tag{2.1}$$

$$= \sum_{n,k} a_{n,k} \frac{x^n}{n!} y^k \quad (e.g.f.). \tag{2.2}$$

The Laplace transform allows us to relate an e.g.f. ϕ of a sequence to the corresponding o.g.f. $g(x)$. If we consider an e.g.f. $\phi(p) = \sum_{k=0}^{\infty} c_k \frac{p^k}{k!}$ then the Laplace transform of $\phi(p)$ allows us to find the o.g.f.:

$$F(x) = \frac{1}{x} g(\frac{1}{x}) = \sum_{k=0}^{\infty} c_k x^{-k-1} = \sum_{k=0}^{\infty} c_k \int_0^{\infty} \frac{p^k e^{-px}}{k!}\, dp = \int_0^{\infty} e^{-px} \phi(p)\, dp,$$

or

$$g(x) = \frac{1}{x} \int_0^{\infty} e^{-p/x} \phi(p)\, dp.$$

The coefficient of x^n is denoted by $[x^n]g(x)$, and from the definition of the e.g.f., we have $n![x^n]g(x) = [\frac{x^n}{n!}]g(x)$. For example $[x^n]\frac{1}{\sqrt{1-4x}} = \binom{2n}{n}$, the n^{th} central binomial coefficient. Here, we use the operator $[x^n]$ to extract the n^{th} coefficient of the power series $g(x)$ [88]. We adopt the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \ldots$ (A000007). The compositional inverse of a power series $g = \sum_{n=1} a_n x^n$ with $a_1 \neq 0$ is a series $f = \sum_{n=1} b_n x^n$ with $b_1 \neq 0$ such that $f \circ g(x) = f(g(x)) = \sum_{n \geq 1} b_n (g(x))^n = x$. We refer to the inverse of f as the series reversion $\bar{f}$. We note that in some texts the series reversion is referred to by the notation $f^{<-1>}$. Lagrange inversion [55] provides a simple method to calculate the coefficients of the series reversion.

Theorem 2.1.1. *Lagrange Inversion Theorem [55, Theorem A.2]*

Let $\phi(u) = \sum_{k=0}^{\infty} \phi_k u^k$ be a power series of $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then, the equation $y = z\phi(y)$ admits a unique solution in $\mathbb{C}[[u]]$ whose coefficients are given by

$$y(z) = \sum_{n=1}^{\infty} y_n z^n, \quad y_n = \frac{1}{n}[u^{n-1}]\phi(u)^n.$$

The Lagrange Inversion Theorem may be written as

$$[x^n]G(\bar{f}(x)) = \frac{1}{n}[x^{n-1}]G'(x)\left(\frac{x}{f(x)}\right)^n.$$

The simplest case is that of $G(x) = x$, in which we get

$$[x^n]\bar{f}(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{f(x)}\right)^n.$$

Example. *We have $xc(x) = \overline{x(1-x)}$ and so we have*

$$[x^n]xc(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{x(1-x)}\right)^n = \frac{1}{n}[x^{n-1}]\left(\frac{1}{1-x}\right)^n.$$

Thus,

$$[x^{n-1}]c(x) = \frac{1}{n}[x^{n-1}]\left(\frac{1}{1-x}\right)^n.$$

Changing $n-1$ to n gives us

$$[x^n]c(x) = \frac{1}{n+1}[x^n]\left(\frac{1}{1-x}\right)^{n+1}.$$

We thus have

$$\begin{aligned}
[x^n]c(x) &= \frac{1}{n+1}[x^n]\left(\frac{1}{1-x}\right)^{n+1} \\
&= \frac{1}{n+1}\sum_{j=0}^{\infty}\binom{-(n+j)}{j}(-x)^j \\
&= \frac{1}{n+1}\sum_{j=0}^{\infty}\binom{-(n+j)+j-1}{j}(-1)^j(-x)^j \\
&= \frac{1}{n+1}\sum_{j=0}^{\infty}\binom{n+j}{j}x^j \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

We now look at $[x^n]c(x)^k$. For this, we use $G(x) = x^k$ with $G'(x) = kx^{k-1}$ and apply the Lagrange Inversion Theorem to

$$xc(x) = \overline{x(1-x)}(x).$$

Thus we have

$$\begin{aligned}
[x^n](xc(x))^k &= [x^{n-k}]c(x)^k \\
&= \frac{1}{n}[x^{n-1}]kx^{k-1}\left(\frac{x}{x(1-x)}\right)^n \\
&= \frac{1}{n}[x^{n-1}]kx^{k-1}\left(\frac{1}{1-x}\right)^n.
\end{aligned}$$

Changing $n-k$ to n gives us

$$\begin{aligned}
[x^n]c(x)^k &= \frac{1}{n+k}[x^{n+k-1}]kx^{k-1}\left(\frac{1}{1-x}\right)^{n+k} \\
&= \frac{k}{n+k}\sum_{j=0}^{n_k-1}[x^j]x^{k-1}[x^{n+k-1-j}]\left(\frac{1}{1-x}\right)^{n+k} \\
&= \frac{k}{n+k}[x^{n+k-1-(k-1)}]\left(\frac{1}{1-x}\right)^{n+k} \\
&= \frac{k}{n+k}[x^n]\left(\frac{1}{1-x}\right)^{n+k}.
\end{aligned}$$

Thus

$$[x^n]c(x)^k = \frac{k}{n+k}[x^n]\left(\frac{1}{1-x}\right)^{n+k}.$$

We can simplify this using the Binomial Theorem. We get

$$\begin{aligned}
[x^n]c(x)^k &= \frac{k}{n+k}[x^n]\left(\frac{1}{1-x}\right)^{n+k} \\
&= \frac{k}{n+k}[x^n](1-x)^{-(n+k)} \\
&= \frac{k}{n+k}[x^n]\sum_{j=0}^{\infty}\binom{-(n+k)}{j}(-x)^j \\
&= \frac{k}{n+k}[x^n]\sum_{j=0}^{\infty}\binom{n+k+j-1}{j}(-1)^j(-x)^j \\
&= \frac{k}{n+k}[x^n]\sum_{j=0}^{\infty}\binom{n+k+j-1}{j}x^j \\
&= \frac{k}{n+k}\binom{n+k+n-1}{n} \\
&= \frac{k}{n+k}\binom{2n+k-1}{n}.
\end{aligned}$$

Thus we get

$$[x^n]c(x)^k = \frac{k}{n+k}[x^n]\left(\frac{1}{1-x}\right)^{n+k} = \frac{k}{n+k}\binom{2n+k-1}{n}.$$

Again, using Lagrange inversion, we have

$$[x^n](xc(x))^k = \frac{1}{n}k[x^{n-1}]x^{k-1}\left(\frac{1}{1-x}\right)^n.$$

Thus

$$\begin{aligned}
[x^n](xc(x))^k &= \frac{k}{n}[x^{n-1}]x^{k-1}\left(\frac{1}{1-x}\right)^n \\
&= \frac{k}{n}[x^{n-1-k+1}]\left(\frac{1}{1-x}\right)^n \\
&= \frac{k}{n}[x^{n-k}]\left(\frac{1}{1-x}\right)^n \\
&= \frac{k}{n}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-n}{j}(-x)^j \\
&= \frac{k}{n}\sum_{j=0}^{\infty}\binom{n+j-1}{j}x^j \\
&= \frac{k}{n}\binom{n+n-k-1}{n-k} \\
&= \frac{k}{n}\binom{2n-k-1}{n-k} \\
&= \frac{k}{n}\frac{n}{2n-k}\binom{2n-k}{n-k} \\
&= \frac{k}{2n-k}\binom{2n-k}{n-k}.
\end{aligned}$$

Adjusting this term for the case of $n = 0$, $k = 0$, we get [70]

$$[x^n](xc(x))^k = \frac{k+0^{n+k}}{2n-k+0^{2n-k}}\binom{2n-k}{n-k} = \frac{k+0^{n+k}}{2n-k+0^{2n-k}}\binom{2n-k}{n}.$$

By changing x to x^2 in the above, we can easily arrive at expressions for $[x^n]c(x^2)^k$ (this will give us the aerated versions of the sequences above). We prefer to use the Lagrange Inversion Theorem again.

Our starting point is the observation that

$$xc(x^2) = \overline{\frac{x}{1+x^2}}.$$

Thus we we have

$$[x^n](xc(x^2))^k = [x^{n-k}]c(x^2)^k = \frac{1}{n}[x^{n-1}]kx^{k-1}\left(x\frac{1+x^2}{x}\right)^n.$$

Thus we have (changing $n-k$ to n)

$$\begin{aligned}
[x^n]c(x^2)^k &= \frac{k}{n+k}[x^{n+k-1}]x^{k-1}(1+x^2)^{n+k} \\
&= \frac{k}{n+k}\sum_{j=0}^{n+k-1}[x^j]x^{k-1}[x^{n+k-1-j}](1+x^2)^{n+k} \\
&= \frac{k}{n+k}[x^n](1+x^2)^{n+k} \\
&= \frac{k}{n+k}[x^n]\sum_{j=0}^{n+k}\binom{n+k}{j}x^{2j} \\
&= \frac{k}{n+k}\binom{n+k}{\frac{n}{2}}\frac{1+(-1)^n}{2}.
\end{aligned}$$

Thus we have

$$[x^n]c(x^2)^k = \frac{k}{n+k}[x^n](1+x^2)^{n+k} = \frac{k}{n+k}\binom{n+k}{\frac{n}{2}}\frac{1+(-1)^n}{2}.$$

If the product of two power series f and g is 1 then f and g are termed *reciprocal sequences* and satisfy the following. For o.g.f.'s we have [161]

Definition 2.1.4. *A reciprocal series $g(x) = \sum_{n=0} a_n x^n$ with $a_0 = 1$, of a series $f(x) = \sum_{n=0} b_n x^n$ with $b_0 = 1$, is a power series where $g(x)f(x) = 1$, which can be calculated as follows*

$$\sum_{n=0}^{\infty} a_n x^n = -\sum_{n=0}^{\infty}\sum_{i=1}^{n} b_i a_{n-i} x^n, \qquad a_0 = 1 \tag{2.3}$$

and for e.g.f.'s we have

Definition 2.1.5. *A reciprocal series $g(x) = \sum_{n=0} a_n \frac{x^n}{n!}$ with $a_0 = 1$, of a power series $f(x) = \sum_{n=0} b_n \frac{x^n}{n!}$ with $b_0 = 1$, is a series where $g(x)f(x) = 1$, and can be calculated as follows*

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = -\sum_{n=0}^{\infty}\sum_{i=1}^{n}\binom{n}{k} b_i a_{n-i}\frac{x^n}{n!}, \qquad a_0 = 1. \tag{2.4}$$

2.2 The Riordan group

Riordan arrays give us an intuitive method of solving combinatorial problems, helping to build an understanding of many number patterns. They provide an effective method of proving combinatorial identities and solving numerical puzzles as in [86] rather than using computer based approaches [87, 141]. Riordan arrays, named after the combinatorist, John Riordan, [1] were first used in the 1990's by Shapiro et al [119] as a method of exploring combinatorial patterns in numbers of Pascal's triangle. Shapiro saw the natural extension of Pascal's triangle due to its shape, to a lower triangular matrix, making use of matrix representation of transformations on sequences, then using this to explore patterns in the numbers of Pascal's triangle. This has become a classical example of a Riordan array. It was while exploring these extensions of Pascal's triangle that it was realized that Riordan arrays have a group structure. Along with using Riordan arrays as a method of proving combinatorial identities [134] they have also been used in performing combinatorial sum inversions [133, 88]. In the past few years the idea of extending combinatorial theory to matrices as in Riordan arrays has been extended to represent succession rules and the ECO method [135] which have been translated into the notion of *Production matrices* [36]. Articles such as [37] have investigated the relationship between production matrices and Riordan arrays. Links between generating trees and Riordan matrices have also been explored [85].

The *Riordan group* [118, 131, 115, 134, 121, 35] is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = \sum_{n=0}^{\infty} g_n x^n$ with $g_0 = 1$ and $f(x) = \sum_{n=1}^{\infty} f_n x^n$ with $f_1 \neq 0$ [131]. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). This modifies to $g(x)\frac{f(x)^i}{i!}$ when we are concerned with exponential generating functions, leading to the exponential Riordan group. The matrix corresponding to the pair g, f is denoted by (g, f)(or $[g, f]$ in the exponential case). The group law is then given by

$$(g, f) \cdot (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of $f(\bar{f}(x)) = \bar{f}(f(x))$.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with o.g.f. $\mathcal{A}$

[1] John Riordan spent much of his life working at Bell Laboratories(Bell Labs). His published work includes "An Introduction to Combinatorics" published in 1978 and Combinatorial Identities, published in 1968.

(x), then the sequence $\mathbf{Ma}$ has o.g.f. $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

This result is called the fundamental theorem of Riordan arrays(FTRA).

Example. *For ordinary generating functions, the so-called binomial matrix* $\mathbf{B}$ *is the element* $(\frac{1}{1-x}, \frac{x}{1-x})$ *of the Riordan group. It has general element* $\binom{n}{k}$*, and hence as an array coincides with Pascal's triangle. More generally,* $\mathbf{B}^m$ *is the element* $(\frac{1}{1-mx}, \frac{x}{1-mx})$ *of the Riordan group, with general term* $\binom{n}{k}m^{n-k}$*. It is easy to show that the inverse* $\mathbf{B}^{-m}$ *of* $\mathbf{B}^m$ *is given by* $(\frac{1}{1+mx}, \frac{x}{1+mx})$*.*

Example. *For exponential generating functions, the binomial matrix* $\mathbf{B}$ *is the element* $[e^x, x]$ *of the Riordan group which as above, coincides with Pascal's triangle. More generally,* $\mathbf{B}^m$ *is the element* $[e^{mx}, x]$ *of the Riordan group. It is easy to show that the inverse* $\mathbf{B}^{-m}$ *of* $\mathbf{B}^m$ *is given by* $[e^{-mx}, x]$*.*

Multiplication of a matrix in the Riordan group by the binomial matrix (inverse Binomial matrix) is what we will refer to as the *Binomial transform (inverse Binomial tranform)*. In other words, $\mathbf{BA}$ will be called the binomial transform of A.

Example. *If* a_n *has g.f.* $g(x)$*, then the g.f. of the sequence*

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k}$$

is equal to

$$\frac{g(x)}{1-x^2} = \left(\frac{1}{1-x^2}, x \right) \cdot g(x).$$

The row sums of the matrix (g, f) have g.f.

$$(g, f) \cdot \frac{1}{1-x} = \frac{g(x)}{1-f(x)},$$

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \longrightarrow \begin{array}{ccccccc} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array}$$

Figure 2.1: Pascal's triangle as a element of the Riordan group

while the diagonal sums of (g, f) (sums of left-to-right diagonals in the north east direction) have g.f. $g(x)/(1-xf(x))$. These coincide with the row sums of the "generalized" Riordan array (g, xf). Thus the Fibonacci numbers F_{n+1} are the diagonal sums of the binomial matrix $\mathbf{B}$ given by $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & 0 & \dots \\ 1 & 5 & 10 & 10 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while they are the row sums of the "generalized" or "stretched" (using the nomenclature of [32]) Riordan array $\left(\frac{1}{1-x}, \frac{x^2}{1-x}\right)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 4 & 3 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We often work with "generalized" Riordan arrays, where we relax some of the conditions above. Thus for instance [32] discusses the notion of the "stretched" Riordan array. In this note, we shall encounter "vertically stretched" arrays of the form (g, h) where now $h_0 = h_1 = 0$ with $h_2 \neq 0$. Such arrays are not invertible, but we may explore their left inversion. In this context, standard Riordan arrays as described above are called "proper" Riordan arrays. We note for instance that for any proper Riordan array (g, f), its diagonal sums are just the row sums of the vertically stretched array (g, xf) and hence have g.f. $g/(1 - xf)$.

Each Riordan array $(g(x), f(x))$ has bivariate g.f. given by

$$\frac{g(x)}{1 - yf(x)}.$$

For instance, the binomial matrix $\mathbf{B}$ has g.f.

$$\frac{\frac{1}{1-x}}{1 - y\frac{x}{1-x}} = \frac{1}{1 - x(1+y)}.$$

Similarly, exponential Riordan arrays $[g(x), f(x)]$ have bivariate e.g.f. given by $g(x)e^{yf(x)}$.

For a sequence $a_0, a_1, a_2, \ldots$ with g.f. $g(x)$, the "aeration" of the sequence is the sequence $a_0, 0, a_1, 0, a_2, \ldots$ with interpolated zeros. Its g.f. is $g(x^2)$. The sequence $a_0, a_0, a_1, a_1, a_2, \ldots$ is called the "doubled" sequence. It has g.f. $(1 + x)g(x^2)$. The aeration of a matrix $\mathbf{M}$ with general term $m_{i,j}$ is the matrix whose general term is given by

$$m^r_{\frac{i+j}{2}, \frac{i-j}{2}} \frac{1 + (-1)^{i-j}}{2},$$

where $m^r_{i,j}$ is the i, j-th element of the reversal of $\mathbf{M}$:

$$m^r_{i,j} = m_{i,i-j}.$$

In the case of a Riordan array, the row sums of the aeration are equal to the diagonal sums of the reversal of the original matrix.

Example. *The Riordan array* $(c(x^2), xc(x^2))$ *is the aeration of* $(c(x), xc(x))$. *Here*

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the g.f. of the Catalan numbers. The reversal of $(c(x), xc(x))$ *is the matrix with general element*

$$[k \leq n+1]\binom{n+k}{k}\frac{n-k+1}{n+1},$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 0 & 0 & 0 & \dots \\ 1 & 3 & 5 & 5 & 0 & 0 & \dots \\ 1 & 4 & 9 & 14 & 14 & 0 & \dots \\ 1 & 5 & 14 & 28 & 42 & 42 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is the Catalan triangle, A009766. Then $(c(x^2), xc(x^2))$ *has general element*

$$\binom{n+1}{\frac{n-k}{2}}\frac{k+1}{n+1}\frac{(1+(-1)^{n-k}}{2},$$

and begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & 4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is the "aerated" Catalan triangle, A053121. Note that

$$(c(x^2), xc(x^2)) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}.$$

We note that the diagonal sums of the reverse of $(c(x), xc(x))$ *coincide with the row sums of* $(c(x^2), xc(x^2))$*, and are equal to the central binomial coefficients* $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ *A001405.*

2.3 Orthogonal polynomials

Orthogonal polynomials [27, 51, 63, 142, 144, 106] permeate many areas of mathematics which include algebra, combinatorics, numerical analysis, operator theory and random matrices. The study of classic orthogonal polynomials dates back to the 18^{th} century. By an *orthogonal polynomial sequence* $(p_n(x))_{n\geq 0}$ we shall understand an infinite sequence of polynomials $p_n(x)$ where $n \geq 0$, with real coefficients (often integer coefficients) that are mutually orthogonal on an interval $[x_0, x_1]$ (where $x_0 = -\infty$ is allowed, as well as $x_1 = \infty$), with respect to a weight function $w : [x_0, x_1] \to \mathbb{R}$:

$$\int_{x_0}^{x_1} p_n(x) p_m(x) w(x)\, dx = \delta_{nm} \sqrt{h_n h_m},$$

where

$$\int_{x_0}^{x_1} p_n^2(x) w(x)\, dx = h_n.$$

We assume that w is strictly positive on the interval (x_0, x_1). Referring to Favard's theorem [27], every such sequence obeys a so-called "three-term recurrence" :

$$p_{n+1}(x) = (a_n x + b_n) p_n(x) - c_n p_{n-1}(x)$$

for coefficients a_n, b_n and c_n that depend on n but not x. We note that if

$$p_j(x) = k_j x^j + k'_j x^{j-1} + \dots \qquad j = 0, 1, \dots$$

then

$$a_n = \frac{k_{n+1}}{k_n}, \qquad b_n = a_n \left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n} \right), \qquad c_n = a_n \left(\frac{k_{n-1} h_n}{k_n h_{n-1}} \right).$$

Since the degree of $p_n(x)$ is n, the coefficient array of the polynomials is a lower triangular (infinite) matrix. In the case of monic orthogonal polynomials the diagonal elements of this array will all be 1. In this case, we can write the three-term recurrence as

$$p_{n+1}(x) = (x - \beta_n) p_n(x) - \alpha_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \beta_0.$$

The *moments* associated to the orthogonal polynomial sequence are the numbers

$$\mu_n = \int_{x_0}^{x_1} x^n w(x)\, dx.$$

Theorem 2.3.1. *[27, Theorem 3.1] A necessary and sufficient condition for the existence of an orthogonal polynomial sequence is*

$$\Delta_n = det(\mu_{i+j})_{i,j\geq 0}^n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \dots & \vdots \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n} \end{vmatrix} \neq 0, n \geq 0.$$

The matrix of moments above is a Hankel matrix, a matrix where the entry $\mu_{n,k} = \mu_{n+k}$. We will refer to the Hankel transform of a matrix which is the integer sequence generated by the successive Hankel determinants of a Hankel matrix. We can find $p_n(x)$, α_n and β_n from a knowledge of these moments. To do this, let $\Delta_{n,x}$ be the same determinant as above, but with the last row replaced by $1, x, x^2, \dots$ thus

$$\Delta_{n,x} = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & x & \dots & x^n \end{vmatrix}.$$

Then

$$p_n(x) = \frac{\Delta_{n,x}}{\Delta_{n-1}}.$$

More generally, we let $H\begin{pmatrix} u_1 & \dots & u_k \\ v_1 & \dots & v_k \end{pmatrix}$ be the determinant of Hankel type with (i,j)-th term $\mu_{u_i+v_j}$. That is

$$H\begin{pmatrix} u_1 & \dots & u_k \\ v_1 & \dots & v_k \end{pmatrix} = \begin{vmatrix} \mu_{u_1+v_1} & \mu_{u_1+v_2} & \dots & \mu_{u_1+v_k} \\ \mu_{u_2+v_1} & \mu_{u_2+v_2} & \dots & \mu_{u_2+v_k} \\ \vdots & \vdots & \dots & \vdots \\ \mu_{u_k+v_1} & \mu_{u_k+v_2} & \dots & \mu_{u_k+v_k} \end{vmatrix}$$

Let

$$\Delta_n = H\begin{pmatrix} 0 & 1 & \dots & n \\ 0 & 1 & \dots & n \end{pmatrix}, \qquad \Delta' = H\begin{pmatrix} 0 & 1 & \dots & n-1 & n \\ 0 & 1 & \dots & n-1 & n+1 \end{pmatrix}.$$

Then we have

$$\beta_n = \frac{\Delta'_n}{\Delta_n} - \frac{\Delta'_{n-1}}{\Delta_{n-1}}, \qquad \alpha_n = \frac{\Delta_{n-2}\Delta_n}{\Delta^2_{n-1}}.$$

and the coefficient of x^{n-1} in $p_n(x)$ is $-(\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_n)$.

Consider the three-term recurrence equation associated to the family of orthogonal polynomials $\{p_n(x)\}_{n\geq 0}$:

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \alpha_n p_{n-1}(x).$$

Rearranging, this gives us

$$xp_n(x) = \alpha_n p_{n-1}(x) + \beta_n p_n(x) + p_{n+1}(x),$$

expanding for the first few n we have

$$\begin{aligned} xp_0(x) &= \alpha_0 p_{-1}(x) + \beta_0 p_0(x) + p_1(x), \\ xp_1(x) &= \alpha_1 p_0(x) + \beta_1 p_1(x) + p_2(x), \\ xp_2(x) &= \alpha_2 p_1(x) + \beta_2 p_2(x) + p_3(x), \\ & \cdots \end{aligned}$$

where $p_{-1}(x) = 0$. Hence we get the following matrix equation

$$x \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_0 & 1 & & & \\ \alpha_1 & \beta_1 & 1 & & \\ & \alpha_2 & \beta_2 & 1 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{pmatrix}.$$

Thus the matrix

$$\mathbf{J} = \begin{pmatrix} \beta_0 & 1 & & & \\ \alpha_1 & \beta_1 & 1 & & \\ & \alpha_2 & \beta_2 & 1 & \\ & & & & \ddots \end{pmatrix}$$

represents multiplication by x on the space of polynomials, when we use the family $\{p_n(x)\}_{n\geq 0}$ as a basis.

We have

$$p_n(x) = \begin{vmatrix} \beta_0 - x & 1 & & & & \\ \alpha_1 & \beta_1 - x & 1 & & & \\ & \alpha_2 & \beta_2 - x & 1 & & \\ & & & \ddots & & \\ & & & & \alpha_n & \beta_n - x \end{vmatrix},$$

that is, $p_n(x)$ is the characteristic polynomial of the n-th principal minor of J.

Example. *The Chebyshev polynomials of the second kind,* $p_n(x) = \frac{sin(n+1)\theta}{sin\,\theta}$, $x = cos\,\theta$, *are orthogonal polynomials with respect to the weight* $\sqrt{1-x^2}$ *on the interval (-1,1). They obey the three term recurrence*

$$p_{n+1}(x) = 2xp_n(x) - p_{n-1}(x)$$

and the associated monic polynomials have the associated infinite tridiagonal matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

2.4 Continued fractions and the Stieltjes matrix

Two types of continued fraction which can be used to define formal power series are the Jacobi *(J-fraction)* continued fraction and the Stieltjes *(S-fraction)* continued fraction. The *J-fraction* expansion for a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has the form

$$\cfrac{1}{1-\beta_0 x - \cfrac{\alpha_1 x^2}{1-\beta_1 x - \cfrac{\alpha_2 x^2}{1-\beta_2 x - \cfrac{\alpha_3 x^2}{\ddots}}}}, \tag{2.5}$$

and *S-fraction* expansion has the form

$$\sum_{n=0}^{\infty} a_n x^n = \cfrac{1}{1 - \cfrac{\alpha_1 x^2}{1 - \cfrac{\alpha_2 x^2}{1 - \cfrac{\alpha_3 x^2}{\ddots}}}}. \tag{2.6}$$

At this point we note an important result due to Heilermann [76] which relates continued fractions, as defined above, and orthogonal polynomials which we introduced in section 2.3.

Theorem 2.4.1. *[76, Theorem 11] Let $(a_n)_{n\geq 0}$ be a sequence of numbers with g.f. $\sum_{n=0}^{\infty} a_n x^n$ written in the form of*

$$\sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1 - \beta_0 x - \cfrac{\alpha_1 x^2}{1 - \beta_1 x - \cfrac{\alpha_2 x^2}{\ddots}}}.$$

Then the Hankel determinant h_n of order n of the sequence $(a_n)_{n\geq 0}$ is given by

$$h_n = a_0^n \alpha_1^{n-1} \alpha_2^{n-2} \dots \alpha_{n-1}^2 \alpha_n = a_0^n \prod_{k=1}^{n} \alpha_k^{n-k}$$

where the sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 0}$ are the coefficients in the recurrence relation

$$P_n(x) = (x - \beta_n) P_{n-1}(x) - \alpha_n P_{n-2}(x), \quad n = 1, 2, 3, 4, \dots$$

of the family of orthogonal polynomials P_n for which a_n forms the moment sequence.

The Hankel determinant [76] in the theorem above is a determinant of a matrix which has constant entries along antidiagonals. We previously encountered this matrix form in section 2.3, as the matrix of moments of orthogonal polynomials. The determinant has the form

$$det_{0\leq i,j\leq n}(a_{i+j}).$$

The sequence of these determinants is known as the *Hankel transform* of a_n and these determinants have been well studied due to the connection to both continued fractions and orthogonal polynomials [12, 33, 78, 109], both links arising from the above theorem.

Now, we introduce another theorem giving a matrix expansion relating to the coefficients of the *J-fraction* [156].

Theorem 2.4.2. *Stieltjes expansion theorem [156, Theorem 53.1]*

The coefficients in the J-fraction

$$\cfrac{1}{\beta_0 + x - \cfrac{\alpha_1}{\beta_1 + x - \cfrac{\alpha_2}{\beta_2 + x - \cfrac{\alpha_3}{\ddots}}}}$$

and its power series expansion

$$P\left(\frac{1}{x}\right) = \sum_{p=0}^{\infty} \frac{(-1)^p c_p}{x^{p+1}}$$

are connected by the relations

$$c_{p+q} = k_{o,p}k_{o,q} + a_1 k_{1,p}k_{1,q} + a_1 a_2 k_{2,p}k_{2,q} + \dots$$

where

$$k_{0,0} = 1, \quad k_{r,s} = 0 \quad if \quad r > s$$

and where the $k_{r,s}$ for $s \geq r$ are given recurrently by the matrix equation

$$\begin{pmatrix} k_{0,0} & 0 & 0 & 0 & \dots \\ k_{0,1} & k_{1,1} & 0 & 0 & \dots \\ k_{0,2} & k_{1,2} & k_{2,2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \beta_1 & 1 & 0 & 0 & \dots \\ \alpha_1 & \beta_2 & 1 & 0 & \dots \\ 0 & \alpha_2 & \beta_3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} k_{0,1} & k_{1,1} & 0 & 0 & \dots \\ k_{0,2} & k_{1,2} & k_{2,2} & 0 & \dots \\ k_{0,3} & k_{1,3} & k_{2,3} & k_{3,3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Relating this back to theorem **2.4.1**, the link between continued fractions and orthogonal polynomials can be seen once again, as we see the appearance of the tridiagonal matrix relating to orthogonal polynomials, which we introduced in section 2.3. Note in the theorem above, we obtain the form of the *J-fraction* in Theorem (2.4.1) if we replace the variable x by $\frac{1}{x}$ and divide by x.

In the context of Riordan arrays, we see the Stieltjes Expansion Theorem in [103], defined as follows

Definition 2.4.1. *Let* $\mathbf{L} = (l_{nk})_{n,k\geq 0}$ *be a lower triangular matrix with* $l_{i,i} = 1$ *for all* $i \geq 0$. *The Stieltjes matrix* $\mathbf{S_L}$ *associated with* $\mathbf{L}$ *is given by* $\mathbf{S_L} = \mathbf{L}^{-1}\overline{\mathbf{L}}$ *where* $\overline{\mathbf{L}}$ *is obtained from* $\mathbf{L}$ *by deleting the first row of* $\mathbf{L}$, *that is, the element in the* n^{th} *row and* k^{th} *column of* $\overline{\mathbf{L}}$ *is given by* $l_{n,k} = l_{n+1,k}$

Using the definition of the Stieltjes matrix above [103] leads to the following theorem relating the Riordan matrix to a Hankel matrix with a particular decomposition

Theorem 2.4.3. *[103, Theorem 1] Let* $\mathbf{H} = (h_{nk})_{n,k\geq 0}$ *be the Hankel matrix generated by the sequence* $1, a_1, a_2, a_3, \ldots$ *Assume that* $\mathbf{H} = \mathbf{LDU}$ *where*

$$\mathbf{L} = (l_{nk})_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ l_{1,0} & l_{1,0} & 0 & 0 & \ldots \\ l_{2,0} & l_{2,1} & 1 & 0 & \ldots \\ l_{3,0} & l_{3,1} & l_{3,2} & 1 & \ldots \\ l_{4,0} & l_{4,1} & l_{4,2} & l_{4,3} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} d_0 & 0 & 0 & 0 & \ldots \\ 0 & d_1 & 0 & 0 & \ldots \\ 0 & 0 & d_2 & 0 & \ldots \\ 0 & 0 & 0 & d_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad d_i \neq 0, \quad \mathbf{U} = \mathbf{L^T}$$

Then the Stieltjes matrix $\mathbf{S_L}$ *is tridiagonal, with the form*

$$\begin{pmatrix} \beta_0 & 1 & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & 1 & 0 & \dots \\ 0 & \alpha_2 & \beta_2 & 1 & \dots \\ 0 & 0 & \alpha_3 & \beta_3 & \dots \\ 0 & 0 & 0 & \alpha_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\beta_0 = a_1, \quad \alpha_1 = d_1, \quad \beta_k = l_{k+1,k} - l_{k,k+1}, \quad \alpha_{k+1} = \frac{d_{k+1}}{d_k}, \quad k \geq 0.$$

Now, we state two other relevant results from this paper, relating to generating functions which satisfy particular Stieltjes matrices. The first result relates to o.g.f.'s.

Theorem 2.4.4. *[103, Theorem 2] Let* $\mathbf{H}$ *be the Hankel matrix generating by the sequence* $1, a_1, a_2, \dots$ *and let* $\mathbf{H} = \mathbf{LDL^T}$*. Then* $\mathbf{S_L}$ *has the form*

$$\begin{pmatrix} a_1 & 1 & 0 & 0 & \dots \\ \alpha_1 & \beta & 1 & 0 & \dots \\ 0 & \alpha & \beta & 1 & \dots \\ 0 & 0 & \alpha & \beta & \dots \\ 0 & 0 & 0 & \alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

if and only if the o.g.f. $g(x)$ *of the sequence* $1, a_1, a_2, \dots$ *is given by*

$$g(x) = \frac{1}{1 - a_1 x - \alpha_1 x f}$$

where

$$f = x(1 + \beta f + \alpha f^2).$$

Peart and Woan [103] offer a proof of this in terms of the n^{th} row of the Riordan matrix. However the result can be deduced if we refer back to Theorem [76] relating to J - *fractions*. Referring to Theorem [76] the Stieltjes matrix above has the related *J-fraction*

$$g(x) = \cfrac{1}{1 - a_1 x - \cfrac{\alpha_1 x^2}{1 - \beta x - \cfrac{\alpha x^2}{\ddots}}}.$$

Now letting

$$f(x) = \cfrac{x}{1 - \beta x - \cfrac{\alpha x^2}{1 - \beta x - \cfrac{\alpha x^2}{\ddots}}},$$

we have

$$g(x) = \frac{1}{1 - a_1 x - \alpha_1 x f(x)}.$$

Solving both equations above give us the required result. Similarly for e.g.f.'s we have the following result

Theorem 2.4.5. *[103, Theorem 3] Let* $\mathbf{H}$ *be the Hankel matrix generated by the sequence* $1, a_1, a_2, \ldots$ *and let* $\mathbf{H} = \mathbf{L}\mathbf{D}\mathbf{L}^T$. *Then* $\mathbf{S_L}$ *has the form*

$$\begin{pmatrix} \beta_0 & 1 & 0 & 0 & \ldots \\ \alpha_1 & \beta_1 & 1 & 0 & \ldots \\ 0 & \alpha_2 & \beta_2 & 1 & \ldots \\ 0 & 0 & \alpha_3 & \beta_3 & \ldots \\ 0 & 0 & 0 & \alpha_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

if and only if the e.g.f. g(x) of the sequence $1, a_1, a_2, \ldots$ *is given by*

$$g(x) = \int (a_1 - \alpha_1 f) dx, \quad g(0) = 1$$

where

$$\frac{df}{dx} = 1 + \beta f + \alpha f^2, \quad f(0) = 0.$$

The proof again in [103] involves looking at the form of the n^{th} column of the Riordan array. However, intuitively this result can be seen from looking at the form of the matrix equation $\overline{\mathbf{L}} = \mathbf{LS}$. In the case that $L = [g(x), f(x)]$ is an exponential Riordan array, we have the following

Proposition 2.4.6. $\overline{\mathbf{L}} = \frac{d}{dx}(\mathbf{L})$.

Proof.

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} g_n(x)\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} g_n(x)\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} g_{n+1}(x)\frac{x^n}{(n)!}$$

Equating the first columns of matrices $\overline{\mathbf{L}}$ and $\mathbf{LS}$ we have

$$\frac{d}{dx}(g(x)) = \beta_0 g(x) + \alpha_1 g(x) f(x)$$

and second columns equate to

$$\frac{d}{dx}(f(x)) = \beta_1 f(x) + \alpha_2 f(x)^2.$$

which gives us the required result. □

The Stieltjes matrix as we have seen above is a tridiagonal infinite matrix which is associated with orthogonal polynomials. However in the context of the Riordan group, we are concerned with general polynomials, and therefore have a generalization of the Stieltjes matrix to the Riordan group. Referred to as a *production matrix* [36, 37], it is defined in the following terms.

Let $\mathbf{P}$ be an infinite matrix (most often it will have integer entries). Letting r_0 be the row vector

$$r_0 = (1, 0, 0, 0, \ldots),$$

we define $r_i = r_{i-1}P$ where $i \geq 1$. Stacking these rows leads to another infinite matrix which we denote by $\mathbf{A_P}$. Then $\mathbf{P}$ is said to be the *production matrix* for $\mathbf{A_P}$.

If we let

$$u^T = (1, 0, 0, 0, \ldots, 0, \ldots)$$

then we have

$$\mathbf{A_P} = \begin{pmatrix} u^T \\ u^T\mathbf{P} \\ u^T\mathbf{P}^2 \\ \vdots \end{pmatrix}$$

and

$$\mathbf{DA_P} = \mathbf{A_P}\mathbf{P}$$

where $\mathbf{D} = (\delta_{i,j+1})_{i,j\geq 0}$. In [103, 115] $\mathbf{P}$ is called the Stieltjes matrix associated to $\mathbf{A_P}$. The sequence formed by the row sums of $\mathbf{A_P}$ often has combinatorial significance and is called the *sequence associated to* $\mathbf{P}$. Its general term a_n is given by $a_n = u^T\mathbf{P}^n e$ where

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

In the context of ordinary Riordan arrays, the production matrix associated to a proper Riordan array takes on a special form:

Proposition 2.4.7. *[37] Let* $\mathbf{P}$ *be an infinite production matrix and let* $\mathbf{A_P}$ *be the matrix induced by* $\mathbf{P}$. *Then* $\mathbf{A_P}$ *is an (ordinary) Riordan matrix if and only if* $\mathbf{P}$ *is of the form*

$$\mathbf{P} = \begin{pmatrix} \xi_0 & \alpha_0 & 0 & 0 & 0 & 0 & \ldots \\ \xi_1 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \ldots \\ \xi_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \ldots \\ \xi_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ldots \\ \xi_4 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \ldots \\ \xi_5 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Moreover, columns 0 *and* 1 *of the matrix* $\mathbf{P}$ *are the* Z*- and* A*-sequences, respectively, of the Riordan array* $\mathbf{A_P}$.

We now introduce two results [36, 37, 35] concerning matrices that are production matrices for ordinary and exponential Riordan arrays which help us to recapture a knowledge of the Riordan array from the Stieltjes (production) matrices.

Proposition 2.4.8. *Let P be a Riordan production matrix and let $Z(x)$ and $A(x)$ be the generating functions of its first two columns, respectively. Then the bivariate g.f. $G(t,x)$ of the matrix A_P induced by P and the g.f. $f_P(x)$ of the sequence induced by P are given by*

$$G_P(t,x) = \frac{g(x)}{1-txf(x)}, \quad f_P(x) = \frac{g(x)}{1-xf(x)}, \tag{2.7}$$

where $h(x)$ is determined from the equation

$$f(x) = A(xf(x)) \tag{2.8}$$

and $g(x)$ is given by

$$g(x) = \frac{1}{1-xZ(xf(x))}. \tag{2.9}$$

As a consequence

$$A(x) = \frac{x}{\bar{f}(x)}$$

and

$$Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right)$$

Proposition 2.4.9. *[37, Proposition 4.1] [35] Let $L = (l_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \ldots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \ldots \tag{2.10}$$

be two formal power series that that

$$r(f(x)) = f'(x) \tag{2.11}$$

$$c(f(x)) = \frac{g'(x)}{g(x)}. \tag{2.12}$$

Then

$$(i) \qquad l_{n+1,0} = \sum_i i! c_i l_{n,i} \tag{2.13}$$

$$(ii) \qquad l_{n+1,k} = r_0 l_{n,k-1} + \frac{1}{k!} \sum_{i \geq k} i!(c_{i-k} + k r_{i-k+1}) l_{n,i} \tag{2.14}$$

or, assuming $c_k = 0$ *for* $k < 0$ *and* $r_k = 0$ *for* $k < 0$,

$$l_{n+1,k} = \frac{1}{k!} \sum_{i \geq k-1} i!(c_{i-k} + k r_{i-k+1}) l_{n,i}. \tag{2.15}$$

Conversely, starting from the sequences defined by (2.10), the infinite array $(l_{n,k})_{n,k \geq 0}$ *defined by (2.15) is an exponential Riordan array.*

A consequence of this proposition is that the production matrix $P = (p_{i,j})_{i,j \geq 0}$ for an exponential Riordan array obtained as in the proposition satisfies [37, 35]

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{i-j+1}) \qquad (c_{-1} = 0).$$

Furthermore, the bivariate e.g.f.

$$\phi_P(t,x) = \sum_{n,k} p_{n,k} t^k \frac{x^n}{n!}$$

of the matrix P is given by

$$\phi_P(t,x) = e^{tx}(c(x) + t r(x)),$$

where we have

$$r(x) = f'(\bar{f}(x)), \tag{2.16}$$

and

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}. \tag{2.17}$$

2.5 Lattice paths

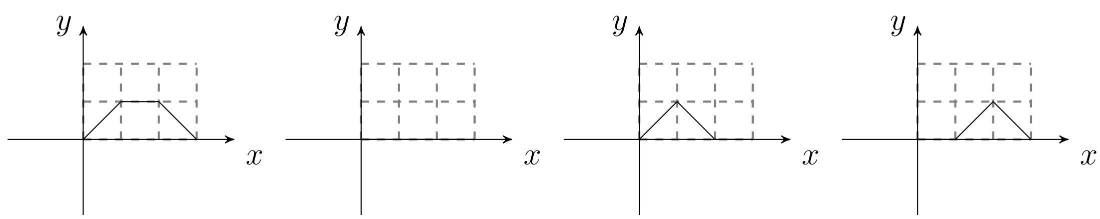
A lattice path [79] is a sequence of points in the integer lattice $\mathbb{Z}^2$. A pair of consecutive points is called a step of the path. A valuation is an integer function on the set of possible steps of $\mathbb{Z}^2 \times \mathbb{Z}^2$. A valuation of a path is the product of the valuations of its steps. We concern ourselves with two types of paths, Motzkin paths and Łukasiewicz paths [151], which are defined as follows:

Definition 2.5.1. *A Motzkin path [78] $\pi = (\pi(0), \pi(1), \dots, \pi(n))$, of length n, is a lattice path starting at $(0,0)$ and ending at $(n,0)$ that satisfies the following conditions*

1. *The elementary steps can be north-east(N-E), east(E) and south-east(S-E).*

2. *Steps never go below the x axis.*

Example. *The four Motzkin paths for $n = 3$ are*



Motzkin paths are counted by the Motzkin numbers, which have the g.f.

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

Dyck paths are Motzkin paths without the possibility of an East step.

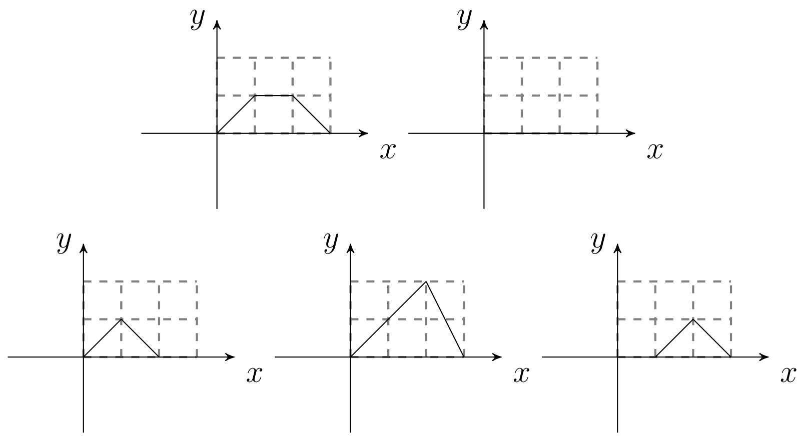
Definition 2.5.2. *A Łukasiewicz path [78] $\pi = (\pi(0), \pi(1), \dots, \pi(n))$, of length n, is a lattice path starting at $(0,0)$ and ending at $(n,0)$ that satisfies the following conditions*

1. *The elementary steps can be north-east(N-E) and east(E) as those in Motzkin paths.*

2. *South-east(S-E) steps from level k can fall to any level above or on the x axis, and are denoted as $\alpha_{n,k}$, where n is the length of the south-east step and k is the level where the step ends.*

3. *Steps never go below the x axis.*

Example. *The five Łukasiewicz paths for $n = 3$ are*

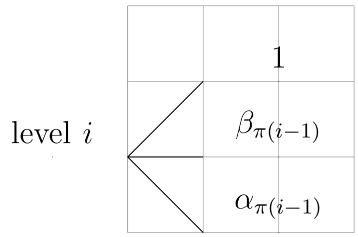


Theorem 2.5.1. *[79, Theorem 2.3] Let*

$$\mu_n = \sum_{\pi \in \mathbf{M}} v(\pi)$$

where the sum is over the set of Motzkin paths $\pi = (\pi(0)....\pi(n))$ of length n. Here $\pi(j)$ is the level after the j^{th} step, and the valuation of a path is the product of the valuations of its steps $v(\pi) = \prod_{i=1}^{n} v_i$ where

$$v_i = v(\pi(i-1), \pi(i)) = \begin{cases} 1 & \text{if the } i^{th} \text{ step rises} \\ \beta_{\pi(i-1)} & \text{if the } i^{th} \text{ step is horizontal} \\ \alpha_{\pi(i-1)} & \text{if the } i^{th} \text{ step falls} \end{cases}$$



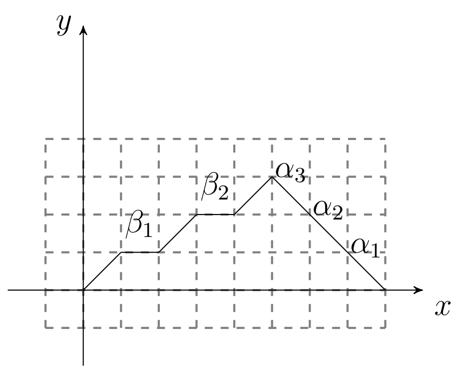
Then the g.f. of the sequence μ_n is given by

$$\mathbf{M}(x) = \sum_{n=0}^{\infty} \mu_n x^n.$$

A continued fraction expansion of the g.f. is then

$$M(x) = \cfrac{1}{1 - \beta_0 x - \cfrac{\alpha_1 x^2}{1 - \beta_1 x - \cfrac{\alpha_2 x^2}{1 - \beta_2 x - \cfrac{\alpha_3 x^2}{\ddots}}}}.$$

Example. *A counting of a Motzkin path*



$$v(\pi) = \beta_1 \beta_2 \alpha_1 \alpha_2 \alpha_3$$

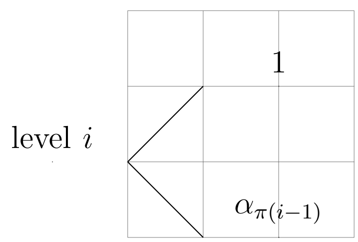
Similarly for Dyck paths we have

Theorem 2.5.2. *Let*

$$\mu_n = \sum_{\pi \in \mathbf{D}} v(\pi)$$

where the sum is over the set of Dyck paths $\pi = (\pi(0)....\pi(n))$ *of length* n. *Here* $\pi(j)$ *is the level after the* j^{th} *step, and the valuation of a path is the product of the valuations of its steps* $v(\pi) = \prod_{i=1}^{n} v_i$ *where*

$$v_i = v(\pi(i-1), \pi(i)) = \begin{cases} 1 & \text{if the } i^{th} \text{ step rises} \\ \alpha_{\pi(i-1)} & \text{if the } i^{th} \text{ step falls} \end{cases}$$



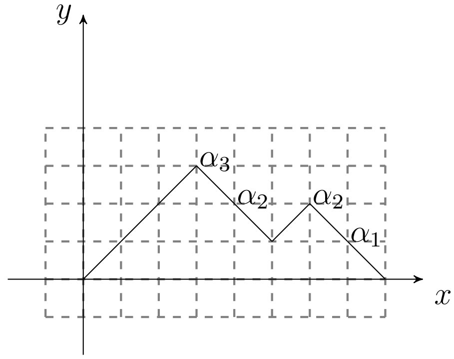
Then the g.f. of the sequence μ_n *is given by*

$$\mathbf{D}(x) = \sum_{n=0}^{\infty} \mu_n x^n.$$

A continued fraction expansion of the g.f. is then

$$D(x) = \cfrac{1}{1 - \cfrac{\alpha_1 x^2}{1 - \cfrac{\alpha_2 x^2}{1 - \cfrac{\alpha_3 x^2}{\ddots}}}}.$$

Example. *A counting of a Dyck path*



$$v(\pi) = \alpha_1\alpha_2^2\alpha_3$$

$M(x)$ corresponds to the *J-fraction* and $D(x)$ corresponds to the *S-fraction* as in eq. (2.5) and eq. (2.6) respectively.

Let

$$\mu_{n,k} = \sum_{\pi\in\mathbf{M}_{n,k}} v(\pi)$$

where $\mathbf{M}_{n,k}$ is the set of Motzkin paths of length n from level 0 to level k, and $v(\pi)$ is the valuation of the path as in Theorem 2.5.1. Now, [56] defines *vertical polynomials* $\mathbf{V}_n(x)$ by

$$\mathbf{V}_n(x) = \sum_{i=0}^{n} \mu_{n,i}x^i$$

so we now introduce the following theorem:

Theorem 2.5.3. *[151, Chapter 3, Proposition 7] Let $\{P_n(x)\}_{n\geq 0}$ be a set of polynomials satisfying the three term recurrence*

$$P_n(x) = (x - \beta_n)P_{n-1}(x) - \alpha_n P_{n-2}(x) \quad n = 1, 2, 3, 4, \dots$$

The vertical polynomials $\{V_n(x)\}_{n\geq 0}$ are the inverse of the orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$.

This means that the matrix $\mathbf{P} = (p_{n,k})_{0\leq k\leq n}$ is the inverse of the matrix $\mathbf{V} = (\mu_{n,k})_{0\leq k\leq n}$.

Example.

$$C(x) = \cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{\cdots}}}}.$$

The first few rows of $\mathbf{P}$ *are*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & \ldots \\ 1 & -3 & 1 & 0 & \ldots \\ -1 & 6 & -5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the Riordan array

$$\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right),$$

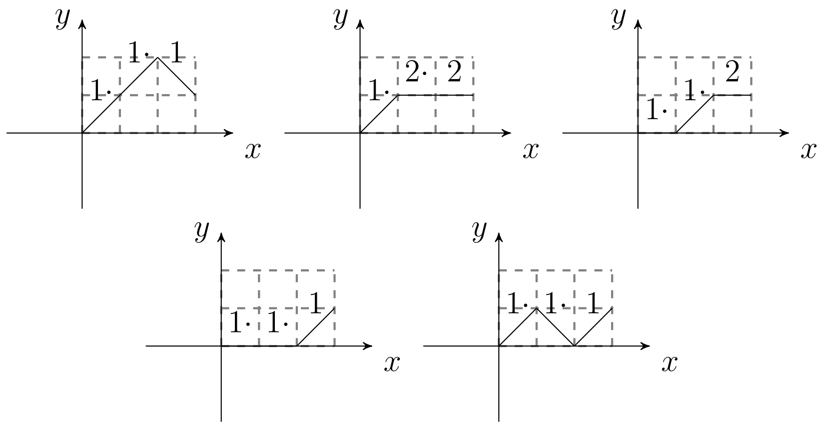
with inverse matrix $(\mu_{n,k})_{0\le k\le n}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 2 & 3 & 1 & 0 & \ldots \\ 5 & 9 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is the Riordan array

$$\left(c(x), c(x)-1\right).$$

To verify that $\mu_{3,1} = 9$*, we sum the weights of the following paths*



Thus we have $\mu_{3,1} = 1 + 4 + 2 + 1 + 1 = 9$.

We return to a theorem introduced previously [103], in which the Riordan array

$$\mathbf{L} = (l_{n,k})_{n,k\geq 0} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ l_{1,0} & 1 & 0 & 0 & 0 & \dots \\ l_{2,0} & l_{2,1} & 1 & 0 & 0 & \dots \\ l_{3,0} & l_{3,1} & l_{3,2} & 1 & 0 & \dots \\ l_{4,0} & l_{4,1} & l_{4,2} & l_{4,3} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

was shown to satisfy the equations

$$l_{n,0} = \beta_0 l_{n-1,0} + \alpha_1 l_{n-1,1},$$

and

$$l_{n,k} = l_{n-1,k-1} + \beta_k l_{n-1,k} + \alpha_k l_{n-1,k+1}. \tag{2.18}$$

We now understand this equation in terms of Motzkin paths.

1. $l_{n-1,k-1} \longrightarrow l_{n,k}$ requires an added north-east(N-E) step at the end of each path with the path value unchanged as the N-E step is 1.

2. $l_{n-1,k+1} \longrightarrow l_{n,k}$ requires an added south-east(S-E) step at the end of each path, changing the path value by α_k, the value defined in Theorem 2.5.1 for each S-E step.

3. $l_{n-1,k} \longrightarrow l_{n,k}$ requires an added east(E) step at the end of each path, changing the path value by β_k, the value defined in Theorem 2.5.1 for each E step.

Example. *Here we use the Riordan array,* $(\frac{c(x)-1}{x}, xc(x)-1)$ *to illustrate.*

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & \dots \\ 5 & 4 & 1 & 0 & 0 & \dots \\ 14 & 14 & 6 & 1 & 1 & \dots \\ 42 & 48 & 27 & 8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We calculate $l_{4,1}$ *using eq.* (2.18) *above, thus*

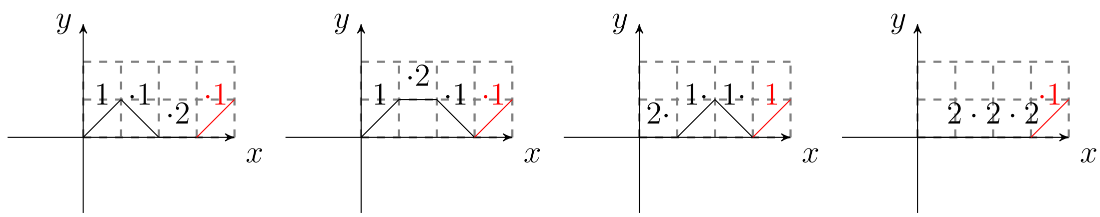
$$l_{4,1} = l_{3,0} + 2l_{3,1} + l_{3,2}.$$

We note that the level steps have weight two which can be seen from the continued fraction expansion of the g.f. of the Catalan numbers which has the form

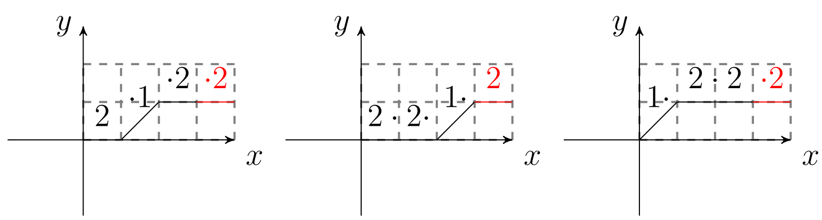
$$\frac{c(x)-1}{x} = \cfrac{1}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{\ddots}}}}.$$

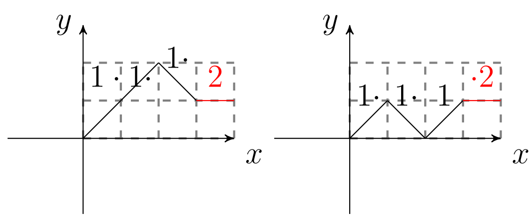
We now count each of the Motzkin paths $l_{3,0}, l_{3,1}, l_{3,2}$*, and adjust each path according to the steps laid out above. Each adjustment to the lattice path is highlighted in red.*

Firstly, the paths below are those of length three and final level zero, $l_{3,0}$*, and an added N-E step of weight one.*

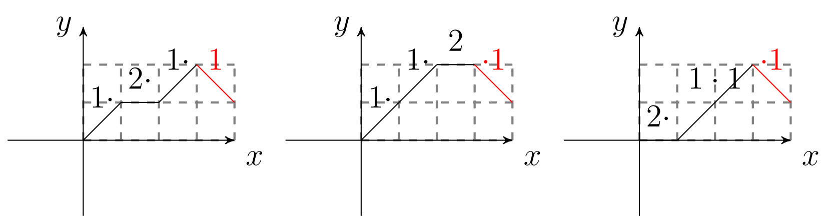


We now look at the paths of length three and final level one $l_{3,1}$*, and an added E step of weight two.*





Finally, the paths below are those of length three and final level two, $l_{3,2}$*, and an added S-E step of weight one.*



Summing over all paths above gives

$$l_{4,1} = l_{3,0} + 2l_{3,1} + l_{3,2} = 14 + 2(14) + 6 = 48.$$

Chapter 3

Chebyshev Polynomials

In this chapter we introduce the Chebyshev polynomials named after the 19^{th} century Russian mathematician Pafnuty Chebyshev, which have been studied in detail because of their relevance in many fields of mathematics. One use of Chebyshev polynomials is in the field of wireless communication where Chebyshev filters are based on the Chebyshev polynomials. We note that Chebyshev polynomials have also been used in the calculation of MIMO systems [71]. MIMO systems are of interest to us in Chapter 8. This chapter is broken down into two sections. In the first section we introduce the Chebyshev polynomials and the properties of interest to us and show the formation of the related Riordan arrays through their matrices of coefficients. We summarize these results in the table in Fig. 3.1. Inspired by Estelle Basor and Torsten Ehrhardt [18] we extend results relating determinants of Hankel plus Toeplitz matrices and Hankel matrices relating to the Chebyshev polynomials of the third kind, to the Chebyshev polynomials of the first and second kind using properties of the polynomials we have drawn to the readers attention in the first section. Note that we look at the polynomials in reverse order as the polynomials of the third kind are those from [18], so are a natural starting point for our study.

3.1 Introduction to Chebyshev polynomials

We begin this section by recalling some facts about the Chebyshev polynomials of the first, second and third kind [112].

The Chebyshev polynomials of the second kind, $U_n(x)$, can be defined by

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k}, \tag{3.1}$$

or alternatively as

$$U_n(x) = \sum_{k=0}^{n} \binom{\frac{n+k}{2}}{k} (-1)^{\frac{n-k}{2}} \frac{1+(-1)^{n-k}}{2} (2x)^k. \tag{3.2}$$

The g.f. is given by

$$\sum_{n=0}^{\infty} U_n(x) t^n = \frac{1}{1-2xt+t^2}.$$

The Chebyshev polynomials of the second kind, $U_n(x)$, which begin

$$1, 2x, 4x^2-1, 8x^3-4x, 16x^4-12x^2+1, 32x^5-32x^3+6x, \dots$$

have coefficient array

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 4 & 0 & 0 & 0 & \dots \\ 0 & -4 & 0 & 8 & 0 & 0 & \dots \\ 1 & 0 & -12 & 0 & 16 & 0 & \dots \\ 0 & 6 & 0 & -32 & 0 & 32 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \qquad \text{(A053117)}$$

This is the (generalized) Riordan array

$$\left(\frac{1}{1+x^2}, \frac{2x}{1+x^2}\right).$$

We note that the coefficient array of the modified Chebyshev polynomials $U_n(x/2)$ which begin

$$1, x, x^2-1, x^3-2x, x^4-3x^2+1, \dots,$$

is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & -2 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & -3 & 0 & 1 & 0 & \dots \\ 0 & 3 & 0 & -4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \qquad (\text{A049310})$$

This is the Riordan array

$$\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right),$$

with inverse

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & 4 & 0 & 1 & \dots \\ 5 & 0 & 9 & 0 & 5 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad (\text{A053121})$$

which is the Riordan array $\left(c(x^2), xc(x^2)\right)$.

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the g.f. of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Chebyshev polynomials of the second kind satisfy the recurrence relation,

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

and by the change of variable from x to $x/2$ we have

$$U_n(x/2) = xU_{n-1}(x/2) - U_{n-2}(x/2),$$

for the modified polynomials, with corresponding Stieltjes matrix

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The Chebyshev polynomials of the third kind can be defined as

$$V_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-1-k}{k} (2x)^{n-1-2k} \left((-1)^{\lfloor \frac{n}{2} \rfloor} - 1\right)$$

with g.f.

$$\sum_{k=0}^{\infty} V_n(x) t^n = \frac{1-t}{1-2xt+t^2}.$$

They relate to the Chebyshev polynomials of the second kind by the equation

$$V_n(x) = U_n(x) - U_{n-1}(x).$$

The Chebyshev polynomials of the third kind, $V_n(x)$ which begin

$$1, 2x-1, 4x^2-2x-1, 8x^3-4x^2-4x+1 \dots$$

have coefficient array

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 2 & 0 & 0 & 0 & 0 & \dots \\ -1 & -2 & 4 & 0 & 0 & 0 & \dots \\ 1 & -4 & -4 & 8 & 0 & 0 & \dots \\ 1 & -4 & -4 & -8 & 16 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This is the (generalized) Riordan array

$$\left(\frac{1-x}{1+x^2}, \frac{2x}{1+x^2}\right).$$

We note that the coefficient array of the modified Chebyshev polynomials $V_n(x/2)$ which begin

$$1, x-1, x^2-x-1, x^3-x^2-2x+1\ldots$$

is given by

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & -1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & -2 & -1 & 1 & 0 & 0 & \ldots \\ 1 & 2 & -3 & -1 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This is the Riordan array

$$\left(\frac{1-x}{1+x^2}, \frac{x}{1+x^2}\right),$$

with inverse,

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 3 & 3 & 1 & 1 & 0 & 0 & \ldots \\ 6 & 4 & 4 & 1 & 1 & 0 & \ldots \\ 10 & 10 & 5 & 5 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \qquad (\text{A061554})$$

which is the Riordan array

$$\left(\frac{1+xc(x^2)}{\sqrt{1-4x^2}}, xc(x^2)\right).$$

The Chebyshev polynomials of the third kind V_n satisfy the recurrence relation,

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x),$$

with corresponding Stieltjes matrix for $V_n(x/2)$, given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Again we note that the Chebyshev polynomials of the fourth kind, $W_n(x)$ are simply the third polynomials with a change of sign. Related Riordan arrays for these polynomials can be seen in the table below.

The Chebyshev polynomials of the first kind, $T_n(x)$, are defined by

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{(-1)^k}{n-k} (2x)^{n-2k} \tag{3.3}$$

for $n > 0$, and $T_0(x) = 1$. The first few Chebyshev polynomials of the first kind are

$$1, x, 2x^2 - 1, 4x^3 - 3x \ldots$$

and have g.f.

$$\sum_{n=0}^{\infty} T_n(x) t^n = \frac{1 - xt}{1 - 2xt + t^2},$$

They satisfy the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

The situation with the Chebyshev polynomials of the first kind differs slightly, since while the coefficient array of the polynomials $2T_n(x) - 0^n$, which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 0 & 0 & 0 & 0 & \ldots \\ -2 & 0 & 4 & 0 & 0 & 0 & \ldots \\ 0 & -6 & 0 & 8 & 0 & 0 & \ldots \\ 2 & 0 & -16 & 0 & 16 & 0 & \ldots \\ 0 & 10 & 0 & -40 & 0 & 32 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

is a (generalized) Riordan array, namely

$$\left(\frac{1 - x^2}{1 + x^2}, \frac{2x}{1 + x^2} \right),$$

that of $T_n(x)$, which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & -3 & 0 & 4 & 0 & 0 & \dots \\ 1 & 0 & -8 & 0 & 8 & 0 & \dots \\ 0 & 5 & 0 & -20 & 0 & 16 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (\text{A053120})$$

is not a generalized Riordan array. However the Riordan array

$$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & -3 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & -4 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & -5 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (\text{A108045})$$

is the coefficient array for the orthogonal polynomials given by $(2-0^n)T_n(x/2)$.

We see from the table in Fig. 3.1 that the inverse of the matrix of coefficients of the Chebyshev polynomials has Riordan array of the form $\Big(g(x), xc(x^2)\Big)$, with k^{th} column generated by $g(x)(xc(x^2))^k$. For this reason we introduce and prove the next identity before continuing to the next section.

Proposition 3.1.1.

$$(xc(x^2))^m = c(x^2)\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+2} - \sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+2} \tag{3.4}$$

Chebyshev polynomial	Stieltjes matrix	Coefficient array	Inverse coefficient array
$T_n(x) = 1, x, 2x^2-1, 4x^3-3x, \ldots$	$\begin{pmatrix} 0 & 2 & 0 & \ldots \\ 1 & 0 & 1 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$	$\left(\frac{1-x^2}{1+x^2}, \frac{2x}{1+x^2}\right)$	
$(2-0)T_n(x/2) = 1, x, x^2-2x^3 \ldots$	$\begin{pmatrix} 0 & 1 & 0 & \ldots \\ 2 & 0 & 1 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$	$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)$	$\left(\frac{1}{\sqrt{1-4x^2}}, xc(x^2)\right)$
$U_n(x) = 1, 2x, 4x^2-1, 8x^3-4x \ldots$	$\begin{pmatrix} 0 & 1 & 0 & \ldots \\ 1 & 0 & 1 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$	$\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)$	$\left(c(x^2), xc(x^2)\right)$
$V_n(x) = 1, 2x-1, 4x^2-2x-1 \ldots$	$\begin{pmatrix} 1 & 1 & 0 & \ldots \\ 1 & 0 & 1 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$	$\left(\frac{1-x}{1+x^2}, \frac{x}{1+x^2}\right)$	$\left(\frac{1+xc(x^2)}{\sqrt{1-4x^2}}, xc(x^2)\right)$
$W_n(x) = 1, 2x+1, 4x^2+2x-1 \ldots$	$\begin{pmatrix} -1 & 1 & 0 & \ldots \\ 1 & 0 & 1 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$	$\left(\frac{1+x}{1+x^2}, \frac{x}{1+x^2}\right)$	$\left(\frac{\sqrt{1-4x^2}-1}{\sqrt{1-4x^2}-2x-1}, xc(x^2)\right)$

Figure 3.1: Chebyshev polynomials and related Riordan arrays

Proof. Firstly, it is clear that the identity holds true for $m = 1$. We now assume true for m, and endeavor to prove by induction that

$$(xc(x^2))^{m+1} = c(x^2)\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor}(-1)^k\binom{m-k}{k}x^{2k-m+1} - \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1} \tag{3.5}$$

Expanding $(xc(x^2))(xc(x^2))^m$ we have

$$(xc(x^2))^2\sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+2} - xc(x^2)\sum_{k=0}^{\lfloor \frac{m-2}{2} \rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+2}$$

which expands further as

$$c(x^2)\left(\sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1} - \sum_{k=0}^{\lfloor \frac{m-2}{2} \rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+3}\right)$$
$$- \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1}.$$

Now, to sum over possible values of x, we change the summation of

$$\sum_{k=0}^{\lfloor \frac{m-2}{2} \rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+3},$$

to become

$$\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor-1}(-1)^{(k-1)}\binom{m-k-1}{k-1}x^{2k-m+1} - (-1)^{\lfloor \frac{m}{2} \rfloor}\binom{m-\lfloor \frac{m}{2} \rfloor-1}{\lfloor \frac{m}{2} \rfloor-1}x^{2\lfloor \frac{m}{2} \rfloor-m+1}.$$

Now with the above changed summation $(xc(x^2))^{m+1}$ becomes

$$c(x^2)\left(\sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1} + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor-1}(-1)^k\binom{m-k-1}{k-1}x^{2k-m+1} + \right.$$
$$\left.(-1)^{\lfloor \frac{m}{2} \rfloor}\binom{m-\lfloor \frac{m}{2} \rfloor-1}{\lfloor \frac{m}{2} \rfloor-1}x^{2\lfloor \frac{m}{2} \rfloor-m+1}\right) - \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1}.$$

Next, we will simplify further investigating separately the terms for m even and odd. For m odd, $(xc(x^2))^{m+1}$ becomes

$$c(x^2)\left(x^{-m+1}+\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor-1}(-1)^k\binom{m-k}{k}x^{2k-m+1}+(-1)^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-(\lfloor\frac{m-1}{2}\rfloor)-1}{\lfloor\frac{m-1}{2}\rfloor}x^{2(\lfloor\frac{m-1}{2}\rfloor)-m+1}\right.$$
$$\left.+(-1)^{\lfloor\frac{m}{2}\rfloor}\binom{m-\lfloor\frac{m}{2}\rfloor-1}{\lfloor\frac{m}{2}\rfloor-1}x^{2\lfloor\frac{m}{2}\rfloor-m+1}\right)-\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1}$$

which simplifies to

$$c(x^2)\left(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor-1}(-1)^k\binom{m-k}{k}x^{2k-m+1}+(-1)^{\frac{m-1}{2}}\left(\binom{\frac{m-1}{2}}{\frac{m-1}{2}}+\binom{\frac{m-1}{2}}{\frac{m-1}{2}-1}\right)x^{2\frac{m-1}{2}-m+1}\right)$$
$$-\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1}$$

thus

$$\begin{aligned}(xc(x^2))^{m+1} &= c(x^2)\left(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor-1}(-1)^k\binom{m-k}{k}x^{2k-m+1}+(-1)^{\frac{m-1}{2}}\left(\binom{\frac{m+1}{2}}{\frac{m-1}{2}}\right)x^{2\frac{m-1}{2}-m+1}\right)\\ &\qquad -\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1}\\ &= c(x^2)\left(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^k\binom{m-k}{k}x^{2k-m+1}\right)-\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1}.\end{aligned}$$

This gives eq. (**3.4**). Similarly, for m even, we have $(xc(x^2))^{m+1}$ as

$$c(x^2)\left(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor-1}(-1)^k\binom{m-k}{k}x^{2k-m+1}+(-1)^{\lfloor\frac{m}{2}\rfloor}x^{2\lfloor\frac{m}{2}\rfloor-m+1}\right)-\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1}$$

thus

$$(xc(x^2))^{m+1}=c(x^2)\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^k\binom{m-k}{k}x^{2k-m+1}-\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+1}$$

which also gives eq. (**3.4**), and completes the induction. □

Corollary 3.1.2. *For m even*

$$\sum_{r=1}^{\infty}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\frac{1}{2(2r-1+m-2k)}\binom{m-k-1}{k}\binom{2r+m-2k}{\frac{2r+m-2k}{2}}x^{2r} \tag{3.6}$$

and for m odd,

$$\sum_{r=1}^{\infty}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\frac{1}{2(-2k+m+2r)}\binom{m-k-1}{k}\binom{2r+m+1-2k}{\frac{2r+m+1-2k}{2}}x^{2r+1} \tag{3.7}$$

Now, we simplify eq. **3.4**,

$$\begin{aligned}
& c(x^2)\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m+2}-\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+2}\\
= & -\sum_{n=0}^{\infty}\frac{1}{1-2n}\binom{2n}{n}\frac{x^{2n}}{2}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m}+\frac{1}{2}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m}\\
& -\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+2}\\
= & -\left(\frac{1}{2}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m}+\sum_{n=1}^{\infty}\frac{1}{1-2n}\binom{2n}{n}\frac{x^{2n}}{2}\right)\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m}\\
& +\frac{1}{2}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m}-\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+2}\\
= & \sum_{n=0}^{\infty}\frac{1}{2(1+2n)}\binom{2n+2}{n+1}x^{2n+2}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m-k-1}{k}x^{2k-m}\\
& -\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+2}
\end{aligned}$$

now we have

$$\frac{1}{2x^m}\sum_{n=0}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{k=0}^{n}(-1)^k\frac{1}{1+2n-2k}\binom{m-k-1}{k}\binom{2n+2-2k}{n+1-k}x^{2n+2}$$

$$+\frac{1}{2x^m}\sum_{n=\lfloor\frac{m+1}{2}\rfloor}^{\infty}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\frac{1}{1+2n-2k}\binom{m-k-1}{k}\binom{2n+2-2k}{n+1-k}x^{2n+2}$$

$$-\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+2}$$

For $m > 1$

$$\frac{1}{2x^m}\sum_{n=0}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{k=0}^{n}(-1)^k\frac{1}{1+2n-2k}\binom{m-k-1}{k}\binom{2n+2-2k}{n+1-k}x^{2n+2}$$

$$=\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^k\binom{m-k-2}{k}x^{2k-m+2}$$

Now for $m > 1$, we have

$$(xc(x^2))^m=\frac{1}{2}\sum_{n=\lfloor\frac{m+1}{2}\rfloor}^{\infty}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\frac{1}{1+2n-2k}\binom{m-k-1}{k}\binom{2n+2-2k}{n+1-k}x^{2n+2-m} \tag{3.8}$$

Extracting odd and even terms gives the required result.

3.2 Toeplitz-plus-Hankel matrices and the family of Chebyshev polynomials

We extend results from [18], relating determinants of Hankel matrices to determinants of Toeplitz-plus-Hankel matrices, to the family of Chebyshev polynomials using Riordan arrays. Firstly, we take this opportunity to present the relevant result from [18]. We observe the relationship between the Toeplitz-plus-Hankel matrices, Hankel matrices and the matrix of the inverse of the coefficients of the Chebyshev polynomials of the third kind. We provide an alternative proof of this result through the medium of Riordan arrays. We then extend these results to Toeplitz-plus-Hankel matrices relating to the first and second Chebyshev polynomials.

3.2.1 Chebyshev polynomials of the third kind

Theorem 3.2.1. *[18, Proposition 2.1] Let* $\{a_n\}_{n=-\infty}^{\infty}$ *be a sequence of complex numbers such that* $a_n = a_{-n}$ *and let* $\{b_n\}_{n=1}^{\infty}$ *be a sequence defined by*

$$b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} (a_{1-n+2k} + a_{2-n+2k}). \tag{3.9}$$

Define the one-sided infinite matrices

$$\mathbf{A} = (a_{j-k} + a_{j+k+1})_{j,k=0}^{\infty} \quad \mathbf{B} = (b_{j+k+1})_{j,k=0}^{\infty}$$

and the upper triangular one-sided infinite matrix

$$\mathbf{D} = \begin{pmatrix} \xi(0,0) & \xi(1,1) & \xi(2,2) & \dots \\ & \xi(1,0) & \xi(2,1) & \dots \\ & & \xi(2,0) & \dots \\ & & & \ddots \end{pmatrix}$$

where $\xi(n,k) = \binom{n}{\lfloor \frac{n-k}{2} \rfloor}$*. Then* $\mathbf{B} = \mathbf{D}^T \mathbf{A} \mathbf{D}$*.*

We note that $\mathbf{D}^T = \mathbf{L}$ is the Riordan array

$$\mathbf{L} = \left(\frac{1 + xc(x^2)}{\sqrt{1-4x^2}}, xc(x^2) \right)$$

and

$$\left(\frac{1 + xc(x^2)}{\sqrt{1-4x^2}}, xc(x^2) \right)^{-1} = \left(\frac{1-x}{1+x^2}, \frac{x}{1+x^2} \right).$$

Referring back to the table in Fig. (3.1) we see that the Riordan array above is the Riordan array of the inverse of the coefficients of the Chebyshev polynomials of the third kind.

This result is the preliminary result in [18] showing the relationship between certain symmetric Toeplitz-plus-Hankel matrices and Hankel matrices. This preliminary result leads to the following result connecting determinants of these matrices.

Theorem 3.2.2. *[18, Theorem 2.2] Let $\{a_n\}_{n=-\infty}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ fulfill the assumptions of the previous proposition. For $N \geq 1$ define the matrices*

$$\mathbf{A}_N = (a_{j-k} + a_{j+k+1})_{j,k=0}^{N-1}, \qquad \mathbf{B}_N = (b_{j+k+1})_{j,k=0}^{N-1}$$

Then $det A_N = det B_N$.

With the aid of Riordan arrays, we define a transformation $\mathbb{B}$. We then show that this transformation is equivalent to the transformation $b_n = \sum_{k=0} \binom{n}{k}(a_{|n-2k|} + a_{|n-2k+1|})$.

The g.f. of the central binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is given by

$$\frac{1+(xc(x^2))^2}{1-xc(x^2)} = \frac{\sqrt{1-4x^2}+2x-1}{2x(1-2x)}.$$

We can show that the coefficient array of the orthogonal polynomials with weight $\frac{\sqrt{4-x^2}}{2-x}$ is the matrix

$$\left(\frac{1-x}{1+x^2}, \frac{x}{1+x^2}\right).$$

We have

$$\left(\frac{1-x}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} = \left(\frac{1+(xc(x^2))^2}{1-xc(x^2)}, xc(x^2)\right) := \mathbf{L},$$

with general term

$$\binom{n}{\frac{n-k}{2}}.$$

Then the **LDU** decomposition of the Hankel matrix $\mathbf{H} = \mathbf{H}_{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ for $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is given by

$$\mathbf{H} = \mathbf{L} \cdot \mathbf{I} \cdot \mathbf{L}^t = \mathbf{L}\mathbf{L}^t,$$

where the diagonal matrix is the identity since the Hankel transform of $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is all 1's. Now we have the following identity of Riordan arrays

$$\left(\frac{1}{1+x}, x\right) \cdot \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right) = \left(\frac{1-x}{1+x^2}, \frac{x}{1+x^2}\right).$$

Thus

$$\begin{aligned}\mathbf{L} &= \left(\frac{1-x}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \\ &= \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(\frac{1}{1+x}, x\right)^{-1} \\ &= \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot (1+x, x),\end{aligned}$$

where

$$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}$$

is the matrix with general term

$$\binom{n}{\frac{n-k}{2}} \frac{1+(-1)^{n-k}}{2}.$$

Thus we have

$$\mathbf{H} = \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot (1+x, x) \cdot (1+x, x)^t \left(\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}\right)^t,$$

where

$$(1+x, x) \cdot (1+x, x)^t$$

is the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 2 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now form the matrix

$$\mathbb{B} = \mathbf{L} \cdot (1+x, x)^t = \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot (1+x, x) \cdot (1+x, x)^t.$$

This matrix begins

$$\mathbb{B} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 3 & 2 & 1 & 0 & 0 & \ldots \\ 3 & 6 & 4 & 2 & 1 & 0 & \ldots \\ 6 & 10 & 8 & 5 & 2 & 1 & \ldots \\ 10 & 20 & 15 & 10 & 6 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that the matrix $\mathbb{B}$ is the matrix formed from expanding eq. (3.9), that is

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 3 & 2 & 1 & 0 & 0 & \ldots \\ 3 & 6 & 4 & 2 & 1 & 0 & \ldots \\ 6 & 10 & 8 & 5 & 2 & 1 & \ldots \\ 10 & 20 & 15 & 10 & 6 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix}.$$

We can regard this as the Riordan array

$$\left(\frac{1-x}{(1+x)(1+x^2)}, \frac{x}{1+x^2}\right)^{-1} = \left(\frac{(1+2x)c(x^2)}{\sqrt{1-4x^2}}, xc(x^2)\right),$$

with a column with elements $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ prepended. Alternatively, we can regard it as a "beheaded" version of the Riordan array

$$\left(\frac{1-x}{1+x}, \frac{x}{1+x^2}\right)^{-1} = \left(\frac{1+2x}{\sqrt{1-4x^2}}, xc(x^2)\right),$$

where the first column terms are divided by 2.

By definition, we have

$$\mathbf{H}_{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \mathbb{B} \cdot \left(\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}\right)^t = \mathbb{B} \cdot (c(x^2), xc(x^2))^t.$$

We can derive an expression for the general term of $\mathbb{B}$ in the following manner. Decompose $(1+x,x)\cdot(1+x,x)^t$ as the sum of two matrices:

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) + \left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

which is $(1+x,x)$ and a shifted version of $(1+x,x)$. To obtain $\mathbb{B}$ we multiply by $\left(\frac{1-x^2}{1+x^2},\frac{x}{1+x^2}\right)$. This gives us

$$\mathbb{B} = \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 3 & 3 & 1 & 1 & 0 & 0 & \dots \\ 6 & 4 & 4 & 1 & 1 & 0 & \dots \\ 10 & 10 & 5 & 5 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) + \left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 1 & 0 & 0 & \dots \\ 0 & 3 & 3 & 1 & 1 & 0 & \dots \\ 0 & 6 & 4 & 4 & 1 & 1 & \dots \\ 0 & 10 & 10 & 5 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

where the first member of the sum is the Riordan array

$$\left(\frac{1-x^2}{1+x^2},\frac{x}{1+x^2}\right)^{-1}\cdot(1+x,x) = \left(\frac{1-x^2}{(1+x)(1+x^2)},\frac{x}{1+x^2}\right)^{-1}.$$

This matrix has general term $\binom{n}{\lfloor\frac{n-k}{2}\rfloor}$, and hence $\mathbb{B}$ has general term

$$\binom{n}{\lfloor\frac{n-k}{2}\rfloor}+\binom{n}{\lfloor\frac{n-k+1}{2}\rfloor}-0^k\cdot\binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

Thus the $\mathbb{B}$ transform of a_n is given by

$$\sum_{k=0}^{n+1}\left(\binom{n}{\lfloor\frac{n-k}{2}\rfloor}+\binom{n}{\lfloor\frac{n-k+1}{2}\rfloor}-0^k\cdot\binom{n}{\lfloor\frac{n}{2}\rfloor}\right)a_k$$

which can also be written as

$$\sum_{k=0}^{n+1}\left(\binom{n}{\lfloor\frac{n-k}{2}\rfloor}(1-0^k)+\binom{n}{\lfloor\frac{n-k+1}{2}\rfloor}\right)a_k$$

since $0^k \cdot \binom{n}{\lfloor \frac{n}{2} \rfloor} = 0^k \cdot \binom{n}{\lfloor \frac{n-k}{2} \rfloor}$. We can also write this as

$$\sum_{k=0}^{n+1} \left(\binom{n}{\lfloor \frac{k-1}{2} \rfloor} (1 - 0^{n-k+1}) + \binom{n}{\lfloor \frac{k}{2} \rfloor} \right) a_{n-k+1}$$

If we now extend a_n to negative n by $a_{-n} = a_n$, we see that this last expression is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k} (a_{n-2k} + a_{n-2k+1}).$$

We now extend the results above to the first and second Chebyshev polynomials. We note that as seen in the table in Fig. (3.1) the third and fourth polynomials differ only in signs and as the relating Toeplitz-plus-Hankel and Hankel matrices also only differ in signs we will not extend results relating to the fourth Chebyshev polynomials.

3.2.2 Chebyshev polynomials of the second kind

Proposition 3.2.3. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers such that $a_n = a_{-n}$ and let $\{b_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} (a_{1-n+2k} - a_{3-n+2k}). \tag{3.10}$$

Define the one-sided infinite matrices

$$\mathbf{A} = (a_{j-k} - a_{j+k+2})_{j,k=0}^{\infty} \quad \mathbf{B} = (b_{j+k+1})_{j,k=0}^{\infty}$$

and the Riordan matrix

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 2 & 0 & 3 & 0 & 1 & 0 & 0 & \dots \\ 0 & 5 & 0 & 4 & 0 & 1 & 0 & \dots \\ 5 & 0 & 9 & 0 & 5 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(c(x^2), xc(x^2)\right).$$

Then $\mathbf{B} = \mathbf{LAL}^T$.

Proof. We have

$$\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} = \left(c(x^2), xc(x^2)\right) := \mathbf{L},$$

and we note from the table in Fig. (3.1) that $\mathbf{L}$ is the inverse of the matrix of coefficients of the Chebyshev polynomials of the second kind and has general term

$$\frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2}$$

$c(x^2)$ is the generating function of the sequence of aerated Catalan numbers

$$1, 0, 1, 0, 2, 0, 5, 0, 14, \dots$$

which can by represented by

$$C_{\frac{n}{2}}\frac{1+(-1)^n}{2} = \frac{1}{2\pi}\int_{-2}^{2} x^n\sqrt{4-x^2}\,dx.$$

The **LDU** decomposition of the Hankel matrix $\mathbf{H} = \mathbf{H}_{\left(\frac{1}{n+1}\binom{n+1}{\frac{n}{2}}\frac{(1+(-1)^n)}{2}\right)}$ is given by

$$\mathbf{H} = \mathbf{L}\cdot\mathbf{I}\cdot\mathbf{L}^t = \mathbf{LL}^t,$$

where the diagonal matrix is the identity since the Hankel transform of $\frac{1}{n+1}\binom{n+1}{\frac{n}{2}}\frac{(1+(-1)^n)}{2}$ is all 1's. Now we have the following identity of Riordan arrays

$$\left(\frac{1}{1-x^2}, x\right) \cdot \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right).$$

Thus

$$\begin{aligned} \mathbf{L} &= \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \\ &= \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(\frac{1}{1-x^2}, x\right)^{-1} \\ &= \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(1-x^2, x\right), \end{aligned}$$

where

$$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}$$

is the matrix with general term

$$\binom{n}{\frac{n+k}{2}}\frac{1+(-1)^{n-k}}{2}.$$

Thus we have

$$\mathbf{H} = \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(1-x^2, x\right) \cdot \left(1-x^2, x\right)^t \left(\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}\right)^t,$$

where

$$\left(1-x^2, x\right) \cdot \left(1-x^2, x\right)^t$$

is the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & -1 & 0 & 0 & \dots \\ -1 & 0 & 2 & 0 & -1 & 0 & \dots \\ 0 & -1 & 0 & 2 & 0 & -1 & \dots \\ 0 & 0 & -1 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now form the matrix

$$\mathbb{B} = \mathbf{L} \cdot \left(1 - x^2, x\right) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(1 - x^2, x\right) \cdot \left(1 - x^2, x\right)^t.$$

Let us look at the first few rows of the matrices forming the decomposition

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & 4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & -1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix $\mathbb{B}$ begins

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & -1 & 0 & \dots \\ 0 & 2 & 0 & -1 & 0 & -1 & \dots \\ 2 & 0 & 1 & 0 & -2 & 0 & \dots \\ 0 & 5 & 0 & -1 & 0 & -3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that the matrix $\mathbb{B}$ is the matrix formed from expanding eq. (3.10), that is

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & -1 & 0 & \dots \\ 0 & 2 & 0 & -1 & 0 & -1 & \dots \\ 2 & 0 & 1 & 0 & -2 & 0 & \dots \\ 0 & 5 & 0 & -1 & 0 & -3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix}.$$

We can derive an expression for the general term of $\mathbb{B}$ in the following manner. Decompose $(1-x^2, x) \cdot (1-x^2, x)^t$ as the sum of two matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is $(1-x^2, x)$ and a shifted version of $(1-x^2, x)$. To obtain $\mathbb{B}$ we multiply by

$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)$. This gives us

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & 4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 2 & 0 & 1 & \dots \\ 0 & 0 & 2 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 5 & 0 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the first member of the sum is the Riordan array

$$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot (1-x^2, x) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}.$$

This matrix has general term $\frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2}$, and hence $\mathbb{B}$ has general term

$$\frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2} - \frac{k-1}{n+1}\binom{n+1}{\frac{n-k+2}{2}}\frac{(1+(-1)^{n-k})}{2} + 0^k \cdot \frac{1}{n+1}\binom{n+1}{\frac{n}{2}}\frac{(1+(-1)^n)}{2}.$$

Thus the $\mathbb{B}$ transform of a_n is given by

$$\sum_{k=0}^{n+1}\left(\frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2} - \frac{k-1}{n+1}\binom{n+1}{\frac{n-k+2}{2}}\frac{(1+(-1)^{n-k})}{2}\right)a_k$$

$$+\sum_{k=0}^{n+1}\left(0^k \cdot \frac{1}{n+1}\binom{n+1}{\frac{n}{2}}\frac{(1+(-1)^n)}{2}\right)a_k,$$

which by a change of summation we rewrite as

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\left(\frac{n-2k+1}{n+1}\binom{n+1}{k}(a_{n-2k}-a_{n-2k+2}) + 0^{n-2k+2} \cdot \frac{1}{n+1}\binom{n+1}{\frac{n}{2}}\frac{(1+(-1)^n)}{2}a_{n-2k+2}\right).$$

Now, as $n-2k+2 \neq 0$ for any k we have

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{n-2k+1}{n+1}\binom{n+1}{k}(a_{n-2k}-a_{n-2k+2}).$$

If we now extend a_n to negative n by $a_{-n} = a_n$, we see that this last expression is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k} (a_{n-2k} - a_{n-2k+2}).$$

□

Corollary 3.2.4. *Let $\{a_n\}_{n=-\infty}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ fulfill the assumptions of the previous proposition. For $N \geq 1$ define the matrices*

$$\mathbf{A}_N = (a_{j-k} - a_{j+k+2})_{j,k=0}^{N-1}, \qquad \mathbf{B}_N = (b_{j+k+1})_{j,k=0}^{N-1}$$

Then $det A_N = det B_N$.

Proof. From proposition 3.2.3 we now have the decomposition for the $N \times N$ section of the infinite matrix decomposition satisfying $\mathbf{B_N} = \mathbf{L_N} \mathbf{A_N} \mathbf{L_N}^T$ and as $\mathbf{L} = \big(c(x^2), xc(x^2)\big)$, which is a matrix with all ones on the diagonal we have $\det \mathbf{B_N} = \det \mathbf{A_N}$. □

3.2.3 Chebyshev polynomials of the first kind

Proposition 3.2.5. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers such that $a_n = a_{-n}$ and let $\{b_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} a_{1-n+2k}. \tag{3.11}$$

Define the one-sided infinite matrices

$$\mathbf{A} = (a_{i-j} + a_{i+j})_{j,k=0}^{\infty} \quad \mathbf{B} = (b_{j+k+1})_{j,k=0}^{\infty}$$

and the Riordan matrix

$$\mathbf{L} = \left(\frac{1}{\sqrt{(1-4x^2)}}, xc(x^2)\right)$$

which begins

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 2 & 0 & 1 & 0 & 0 & \dots \\ 0 & 3 & 0 & 1 & 0 & \dots \\ 6 & 0 & 4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $\mathbf{B} = \mathbf{L}\mathbf{D}_1\mathbf{A}\mathbf{D}_1\mathbf{L}^T$, *where* $\mathbf{D}_1$ *is the diagonal matrix*

$$\mathbf{D}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \dots \\ 0 & 0 & 0 & \sqrt{2} & 0 & \dots \\ 0 & 0 & 0 & 0 & \sqrt{2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. We have

$$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} = \left(\frac{1}{\sqrt{(1-4x^2)}}, xc(x^2)\right) := \mathbf{L},$$

and we note from the table in Fig. (3.1) that $\mathbf{L}$ is the inverse of the matrix of coefficients of the Chebyshev polynomials of the first kind and has general term

$$\binom{n}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2}$$

The $\mathbf{LDU}$ decomposition of the Hankel matrix $\mathbf{H} = \mathbf{H}_{\left(\binom{n}{\frac{n}{2}}\frac{(1+(-1)^n)}{2}\right)}$ is given by

$$\mathbf{H} = \mathbf{L}\cdot\mathbf{D}\cdot\mathbf{L}^t = \mathbf{L}\mathbf{D}\mathbf{L}^t,$$

where $\mathbf{D}$ is the diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that $\mathbf{D}_1 \cdot \mathbf{D}_1 = \mathbf{D}$, except for the $(0,0)^{th}$ entry where an adjustment is made to allow for the entries in the first row and first column of the Hankel matrix $\mathbf{A}$. In the first row and column of $\mathbf{A}$ where $a_{i-j} = a_{i+j}$ the first row has entries $2a_i$ and the first column has entries $2a_j$. The $(0,0)^{th}$ entry of $\mathbf{D}_1 \cdot \mathbf{D}_1$ is $\frac{1}{2}$ to allow for this doubling factor. Now we have the following identity of Riordan arrays

$$(1-x^2, x) \cdot \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right) = \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right).$$

Thus

$$\begin{aligned} \mathbf{L} &= \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \\ &= \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot (1-x^2, x)^{-1} \\ &= \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(\frac{1}{1-x^2}, x\right), \end{aligned}$$

where

$$\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}$$

is the matrix with general term

$$\frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}.$$

Thus we have

$$\mathbf{H} = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(\frac{1}{1-x^2}, x\right) \cdot \mathbf{D} \cdot \left(\frac{1}{1-x^2}, x\right)^t \left(\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}\right)^t,$$

where

$$\left(\frac{1}{1-x^2}, x\right) \cdot \mathbf{D} \cdot \left(\frac{1}{1-x^2}, x\right)^t$$

is the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 2 & 0 & 2 & 0 & 2 & \dots \\ 1 & 0 & 3 & 0 & 3 & 0 & \dots \\ 0 & 2 & 0 & 4 & 0 & 4 & \dots \\ 1 & 0 & 3 & 0 & 5 & 0 & \dots \\ 0 & 2 & 0 & 4 & 0 & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now form the matrix

$$\mathbb{B} = \mathbf{L} \cdot \left(\frac{1}{1-x^2}, x\right) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(\frac{1}{1-x^2}, x\right) \cdot \mathbf{D}.$$

Let us look at the first few rows of the matrices forming the decomposition

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & 4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix $\mathbb{B}$ begins

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 6 & 0 & 2 & 0 & 0 & \dots \\ 6 & 0 & 8 & 0 & 2 & 0 & \dots \\ 0 & 20 & 0 & 10 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that the matrix $\mathbb{B}$ is the matrix formed from expanding eq. (3.11), that is

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 6 & 0 & 2 & 0 & 0 & \dots \\ 6 & 0 & 8 & 0 & 2 & 0 & \dots \\ 0 & 20 & 0 & 10 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \end{pmatrix}.$$

We can derive an expression for the general term of $\mathbb{B}$ in the following manner. Decompose $(\frac{1}{1-x^2}, x) \cdot \mathbf{D}$ as the sum of two matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is $(\frac{1}{1-x^2}, x)$ and a shifted version of $(\frac{1}{1-x^2}, x)$. To obtain $\mathbb{B}$ we multiply by $\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}$. This gives us

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 0 & 1 & 0 & 0 & \dots \\ 6 & 0 & 4 & 0 & 1 & 0 & \dots \\ 0 & 10 & 0 & 5 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 4 & 0 & 1 & 0 & \dots \\ 0 & 10 & 0 & 5 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the first member of the sum is the Riordan array

$$\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \cdot \left(\frac{1}{1-x^2}, x\right) = \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}.$$

This matrix has general term $\binom{n}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2}$, and hence $\mathbb{B}$ has general term

$$\binom{n}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2} + \binom{n}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2} - 0^k \cdot \binom{n}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2}.$$

Thus the $\mathbb{B}$ transform of a_n is given by

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2} + \binom{n}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2} - 0^k \cdot \binom{n}{\frac{n-k}{2}}\frac{(1+(-1)^{n-k})}{2}\right) a_k.$$

A change of summation gives

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k} + \binom{n}{k} - 0^{n-2k} \cdot \binom{n}{k}\right) a_{n-2k}.$$

If we now extend a_n to negative n by $a_{-n} = a_n$, we see that this last expression is equivalent to eq. (3.11)*, that is*

$$\sum_{k=0}^{n} \binom{n}{k}(a_{2k-n}).$$

□

Corollary 3.2.6. *Let* $\{a_n\}_{n=-\infty}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *fulfill the assumptions of the previous proposition. For* $N \geq 1$ *define the matrices*

$$\mathbf{A}_N = (a_{j-k} + a_{j+k})_{j,k=0}^{N-1}, \qquad \mathbf{B}_N = (b_{j+k+1})_{j,k=0}^{N-1}$$

Then $det\,\mathbf{A}_N = 2^{N-2} det\,\mathbf{B}_N$.

Proof. From proposition 3.2.5 we now have the decomposition for the $N \times N$ section of the infinite matrix decomposition satisfying $\mathbf{B}_N = \mathbf{L}_N (\mathbf{D}_1)_N \mathbf{A}_N (\mathbf{D}_1)_N \mathbf{L}_N^T$ and as

$$\mathbf{L} = \left(\frac{1}{\sqrt{(1-4x^2)}}, xc(x^2) \right),$$

which is a matrix with all ones on the diagonal, and

$$\det{(\mathbf{D}_1)_N} = \sqrt{2}^{N-2}$$

we have

$$\det \mathbf{B}_N = \sqrt{2}^{N-2} \det \mathbf{A}_N \sqrt{2}^{N-2} = 2^{N-2} \det \mathbf{A}_N.$$

□

Chapter 4

Properties of subgroups of the Riordan group

In this chapter we look at the form of Stieltjes matrices of certain subgroups of the Riordan group. The subgroups we concern ourselves with in this section [120] are

- The Appell subgroup,
  - Ordinary Appell subgroup: $\big(g(x), x\big)$
  - Exponential Appell subgroup: $\big[g(x), x\big]$
- The Asssociated subgroup,
  - Ordinary Asssociated subgroup: $\big(1, g(x)\big)$
  - Exponential Asssociated subgroup: $\big[1, g(x)\big]$
- The Bell subgroup
  - Ordinary Bell subgroup: $\left(\dfrac{g(x)}{x}, g(x)\right)$
  - Exponential Bell subgroup: $\left[\frac{d}{dx}\, g(x), g(x)\right] = \big[h(x), \int h(x)\big]$

- Hitting-time subgroup
  - Ordinary Hitting-time subgroup: $\left(\frac{x\frac{d}{dx}g(x)}{g(x)}, g(x)\right)$

4.1 The Appell subgroup

4.1.1 The ordinary Appell subgroup

Riordan arrays in the Appell subgroup have the form $(g(x), x)$ with inverse $\left(\frac{1}{g(x)}, x\right)$ and they satisfy the group law as

$$(g(x), x) \cdot (f(x), x) = (g(x)f(x), x).$$

We recall from [37] that given a Riordan array $A = (g(x), f(x))$, its Stieltjes (production) matrix P will be of the form

$$P = \begin{pmatrix} \xi_0 & \alpha_0 & 0 & 0 & 0 & 0 & \dots \\ \xi_1 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \dots \\ \xi_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \dots \\ \xi_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots \\ \xi_4 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \dots \\ \xi_5 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$A(x) = \frac{x}{\bar{f}(x)},$$

and

$$Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right),$$

where $A(x)$ is the g.f. of $\alpha_0, \alpha_1, \dots$ and $Z(x)$ is the g.f. of the first column of P, that is, of $\xi_0, \xi_1, \dots$.

Proposition 4.1.1. *Let $A = (g(x), x)$ be a member of the Appell subgroup of the Riordan group. Then its Stieltjes (production) matrix P is given by*

$$P = \begin{pmatrix} \xi_0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \xi_1 & 0 & 1 & 0 & 0 & 0 & \dots \\ \xi_2 & 0 & 0 & 1 & 0 & 0 & \dots \\ \xi_3 & 0 & 0 & 0 & 1 & 0 & \dots \\ \xi_4 & 0 & 0 & 0 & 0 & 1 & \dots \\ \xi_5 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$A(x) = x \quad \text{and} \quad Z(x) = \frac{1}{x}\left(1 - \frac{1}{g(x)}\right). \tag{4.1}$$

Proof. We have $f(x) = x$ and hence $\bar{f}(x) = \bar{x} = x$. Thus

$$A(x) = \frac{x}{\bar{f}(x)} = \frac{x}{x} = 1.$$

Also

$$\begin{aligned} Z(x) &= \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right) \\ &= \frac{1}{x}\left(1 - \frac{1}{g(x)}\right). \end{aligned}$$

□

Corollary 4.1.2. *Let* **P** *be the Stieltjes matrix of the Appell group element $(g(x), x)$. Then*

$$[x^n]Z(x) = \sum_{k=0}^{n} [x^k]\frac{1}{g(x)}[x^{n-k+1}]g(x),$$

where $Z(x)$ is the g.f. of the first column of P.

Proof. By the above, we have

$$Z(x) = \frac{1}{x}\left(1 - \frac{1}{g(x)}\right) = \frac{1}{g(x)}\frac{g(x)-1}{x}.$$

Hence

$$\begin{aligned}
[x^n]Z(x) &= [x^n]\frac{1}{g(x)}\frac{g(x)-1}{x} \\
&= \sum_{k=0}^{n}[x^k]\frac{1}{g(x)}[x^{n-k}]\frac{g(x)-1}{x} \\
&= \sum_{k=0}^{n}[x^k]\frac{1}{g(x)}[x^{n-k+1}](g(x)-1) \\
&= \sum_{k=0}^{n}[x^k]\frac{1}{g(x)}([x^{n-k+1}]g(x) - 0^{n-k+1}) \\
&= \sum_{k=0}^{n}[x^k]\frac{1}{g(x)}[x^{n-k+1}]g(x),
\end{aligned}$$

since there is a contribution from 0^{n-k+1} only when $k = n+1$. □

Corollary 4.1.3. *The tridiagonal Stieltjes matrices corresponding to Riordan arrays from the ordinary Appell subgroup have generating functions*

$$A(x) = 1, \quad Z(x) = \beta + \alpha x$$

Proof. Riordan arrays from the Appell subgroup with corresponding tridiagonal matrices have the form

$$\left(\frac{1}{1-\beta x - \alpha x^2}, x\right)$$

with inverse

$$(1 - \beta x - \alpha x^2, x).$$

Applying eq. (4.1) gives the result. □

Example. *The Riordan array* $\mathbf{L}$ *from the Appell subgroup with*

$$g(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - \cfrac{2x^2}{1 - x - \cfrac{3x^2}{1 - x - 4x^2}\cdots}}}$$

where $g(x)$ *is the g.f. for the Young tableaux numbers.* $\mathbf{L}$ *satisfies the matrix equation* $\mathbf{LS} = \overline{\mathbf{L}}$ *where the first few rows expand as*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 4 & 2 & 1 & 1 & 0 & 0 & \dots \\ 10 & 4 & 2 & 1 & 1 & 0 & \dots \\ 26 & 10 & 4 & 2 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ 3 & 0 & 0 & 0 & 1 & 0 & \dots \\ 7 & 0 & 0 & 0 & 0 & 1 & \dots \\ 23 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 4 & 2 & 1 & 1 & 0 & 0 & \dots \\ 10 & 4 & 2 & 1 & 1 & 0 & \dots \\ 26 & 10 & 4 & 2 & 1 & 1 & \dots \\ 76 & 26 & 10 & 4 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$Z(x)$ *generates the sequence which counts the number of indecomposable involutions of length* n *[124](A140456).*

Example. *The Riordan array* $\mathbf{L}$ *from the Appell subgroup with*

$$g(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - 2x - \cfrac{2x^2}{1 - 3x - \cfrac{3x^2}{1 - 4x - 4x^2}\cdots}}}$$

where $g(x)$ is the g.f. for the Bell numbers. $\mathbf{L}$ *satisfies the matrix equation* $\mathbf{LS} = \overline{\mathbf{L}}$ *where the first few rows expand as*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 5 & 2 & 1 & 1 & 0 & 0 & \dots \\ 15 & 5 & 2 & 1 & 1 & 0 & \dots \\ 52 & 15 & 5 & 2 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & 0 & 0 & 1 & 0 & \dots \\ 6 & 0 & 0 & 0 & 0 & 1 & \dots \\ 22 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 5 & 2 & 1 & 1 & 0 & 0 & \dots \\ 15 & 5 & 2 & 1 & 1 & 0 & \dots \\ 52 & 15 & 5 & 2 & 1 & 1 & \dots \\ 203 & 52 & 15 & 5 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Here $Z(x)$ generates the sequence which counts the number of set partitions of n which do not have a proper subset of parts with a union equal to a subset(number of irreducible set partitions of size n) [124] (A074664).

4.1.2 Exponential Appell subgroup

We now consider the exponential Appell subgroup, comprised of arrays of the form

$$[g(x), x].$$

We recall that for an exponential Riordan array $[g(x), f(x)]$, its Stieltjes (production) matrix will have bivariate g.f.

$$e^{xy}(c(x) + r(x)y), \tag{4.2}$$

where

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))},$$

and

$$r(x) = f'(\bar{f}(x)).$$

For the exponential Appell subgroup, we then have

Proposition 4.1.4. *Let $A = [g(x), x]$ be a member of the exponential Appell subgroup of the exponential Riordan group. Then its Stieltjes (production) matrix P will have bivariate g.f.*

$$e^{xy}\left(\frac{g'(x)}{g(x)} + y\right).$$

Proof. We have $f(x) = x$ and hence $\bar{f}(x) = x$ and $f'(x) = 1$. Thus $r(x) = 1$. Also, we have

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \frac{g'(x)}{g(x)}.$$

The result follows from eq. (4.2) . □

Corollary 4.1.5. *The e.g.f. of the first column of P is given by*

$$\frac{g'(x)}{g(x)} = \frac{d}{dx}\ln(g(x)). \tag{4.3}$$

We note that for $k \leq n$, the (n, k)-th element of P is given by

$$\binom{n}{k}\xi_{n-k},$$

where the elements of the first column are

$$\xi_0, \xi_1, \xi_2, \ldots .$$

Corollary 4.1.6. *The n-th element of the first column of P is given by*

$$n!\sum_{k=0}^{n}[x^k]\frac{1}{g(x)}[x^{n-k+1}]g(x).$$

Proof. The first column of P has e.g.f. given by $\frac{g'(x)}{g(x)}$, thus its n-th element is given by $n![x^n]\frac{g'(x)}{g(x)}$. Now

$$\begin{aligned}[x^n]\frac{g'(x)}{g(x)} &= \sum_{k=0}^{n}[x^k]\frac{1}{g(x)}[x^{n-k}]g'(x)\\ &= \sum_{k=0}^{n}[x^k]\frac{1}{g(x)}[x^{n-k+1}]g(x).\end{aligned}$$

□

Corollary 4.1.7. *Exponential Riordan arrays in the Appell subgroup with a corresponding tridiagonal Stieltes matrices have the form*

$$[e^{(\beta x+\alpha x^2/2)}, x] \tag{4.4}$$

with inverse $[e^{-(\beta x+\alpha x^2/2)}, x]$.

Proof. From eq. (4.3) we have

$$g(x)c(x) = \frac{d}{dx}\, g(x).$$

Expanding for Riordan arrays with corresponding tridiagonal Stieltjes matrices we then have

$$g(x)(\beta + \alpha x) = \frac{d}{dx}\, g(x).$$

Solving the differential equation gives

$$g(x) = e^{(\beta x+\alpha x^2/2)}.$$

The related Stieltjes matrix has first few entries

$$\begin{pmatrix} \beta & 1 & 0 & 0 & 0 & 0 & \dots \\ \alpha & \beta & 1 & 0 & 0 & 0 & \dots \\ 0 & 2\alpha & \beta & 1 & 0 & 0 & \dots \\ 0 & 0 & 3\alpha & \beta & 1 & 0 & \dots \\ 0 & 0 & 0 & 4\alpha & \beta & 1 & \dots \\ 0 & 0 & 0 & 0 & 5\alpha & \beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

□

We note that with $\alpha = 1, \beta = 0$ the Riordan array

$$[e^{x^2}, x] = \left[\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, x\right],$$

and $[e^{x^2}, x]^{-1} = [e^{-x^2}, x]$. The binomial transform of the Riordan array $[e^{\alpha x^2}, x]$ can be easily shown to be $[e^{\alpha x^2+x}, x]$ and the β^{th} binomial transform is the Riordan array $[e^{\alpha x^2+\beta x}, x]$ giving the Riordan array form as in eq. (4.4)

As we have seen in the previous section, the Stieltjes matrix of Riordan arrays of the form $[e^{\alpha x^2}, x]$ is tridiagonal. Let us look at the form of the n^{th} column through expanding the equation $\mathbf{S} = \mathbf{L}^{-1}\overline{\mathbf{L}}$.

Proof. The Stieltjes matrix $\mathbf{S} = \mathbf{L}^{-1}\overline{\mathbf{L}}$. Expanding for $L = [e^{\alpha x^2}, x]$ we have

$$\mathbf{L} = [e^{\alpha x^2}, x], \qquad \mathbf{L}^{-1} = [e^{-\alpha x^2}, x], \qquad \overline{\mathbf{L}} = \frac{d}{dx}[e^{\alpha x^2}, x].$$

thus we have

$$\begin{aligned}
\mathbf{L}^{-1}\overline{\mathbf{L}} &= e^{-\alpha x^2} \frac{d}{dx}\left(e^{\alpha x^2} \frac{x^n}{n!}\right) \\
&= e^{-\alpha x^2}\left(e^{\alpha x^2} \frac{x^{n-1}}{(n-1)!} + 2\alpha x e^{\alpha x^2} \frac{x^n}{n!}\right) \\
&= \frac{x^{n-1}}{n-1!} + 2\alpha \frac{x^{n+1}}{n!} \\
&= \frac{x^{n-1}}{(n-1)!} + 2\alpha(n+1)\frac{x^{n+1}}{(n+1)!}
\end{aligned}$$

□

Example. *Let us look at the Stieltjes matrix of the Riordan array $[e^{2x^2}, x]$. From the above, the n^{th} column has the form*

$$e^{-2x^2} \frac{d}{dx}\left(e^{2x^2} \frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!} + 4(n+1)\frac{x^{n+1}}{(n+1)!}$$

which in matrix notation is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ -4 & 0 & 1 & 0 & \dots \\ 0 & -12 & 0 & 1 & \dots \\ 48 & 0 & -24 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 4 & 0 & 1 & 0 & \dots \\ 0 & 12 & 0 & 1 & \dots \\ 48 & 0 & 24 & 0 & \dots \\ 0 & 240 & 0 & 40 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 4 & 0 & 1 & 0 & \dots \\ 0 & 8 & 0 & 1 & \dots \\ 0 & 0 & 12 & 0 & \dots \\ 0 & 0 & 0 & 16 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proposition 4.1.8. *The Stieltjes matrix of successive Binomial transforms of* $[e^{-\alpha x^2}, x]$ *has* n^{th} *column*

$$\frac{x^{n-1}}{(n-1)!} + \beta\frac{x^n}{n!} + 2\alpha(n+1)\frac{x^{n+1}}{n!}.$$

Proof.

$$\begin{aligned} e^{-\alpha x^2+\beta x}\frac{d}{dx}\left(e^{\alpha x^2+\beta x}\frac{x^n}{n!}\right) &= e^{-\alpha x^2+\beta x}\left(e^{\alpha x^2+\beta x}\frac{x^{n-1}}{(n-1)!} + 2\alpha x e^{\alpha x^2+\beta x}\frac{x^n}{n!} + \beta e^{\alpha x^2+\beta x}\frac{x^n}{n!}\right) \\ &= \frac{x^{n-1}}{(n-1)!} + \beta\frac{x^n}{n!} + 2\alpha(n+1)\frac{x^{n+1}}{n!}. \end{aligned}$$

□

Example. *Once again with* $\alpha = 1/2$, *the* n^{th} *column of the Stieltjes matrix corresponding to the* β^{th} *binomial transform of the Riordan array* $[e^{\frac{-x^2}{2}}, x]$ *is*

$$\begin{aligned} e^{-\beta x-\frac{x^2}{2}}\frac{d}{dx}\left(e^{\beta x+\frac{x^2}{2}}\frac{x^n}{n!}\right) &= e^{-\beta x-\frac{x^2}{2}}\left(\frac{e^{\frac{x^2}{2}+\beta x}x^{n+1}}{n!} + \frac{e^{\frac{x^2}{2}+\beta x}\beta x^n}{n!} + \frac{e^{\frac{x^2}{2}+\beta x}x^{n-1}}{(n-1)!}\right) \\ &= \frac{x^{n+1}}{n!} + \frac{\beta x^n}{n!} + \frac{x^{n-1}}{(n-1)!} \\ &= \frac{x^{n-1}}{(n-1)!} + \frac{\beta x^n}{n!} + \frac{(n+1)x^{n+1}}{(n+1)!}. \end{aligned}$$

Thus $\mathbf{L}^{-1}\overline{\mathbf{L}}$ *is the tridiagonal matrix*

$$\mathbf{S} = \begin{pmatrix} \beta & 1 & 0 & 0 & 0 & 0 \\ 1 & \beta & 1 & 0 & 0 & 0 \\ 0 & 2 & \beta & 1 & 0 & 0 \\ 0 & 0 & 3 & \beta & 1 & 0 \\ 0 & 0 & 0 & 4 & \beta & 1 \\ 0 & 0 & 0 & 0 & 5 & \ddots \end{pmatrix}.$$

Example.

$$e^{\beta x + \frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{\beta x^n}{n!} \sum_{m=0}^{\infty} \frac{x^{2m}}{2^m m!}.$$

With $\beta = 0$ *we have*

$$\mathbf{L} = \left[\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}, x\right] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 6 & 0 & 1 & 0 \\ 0 & 3 & 0 & 10 & 0 & 1 \end{pmatrix}, \quad \underline{(A001147)}$$

$$\mathbf{L^{-1}} = \left[\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}, x\right]^{-1} = \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}, x\right] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 0 \\ 3 & 0 & -6 & 0 & 1 & 0 \\ 0 & 3 & 0 & -10 & 0 & 1 \end{pmatrix}.$$

4.2 The associated subgroup

4.2.1 Ordinary associated subgroup

Riordan arrays $(1, xg(x))$ form the associated subgroup. They satisfy the group law where

$$(1, xg(x))(1, xf(x)) = (1, xf(xg(x)))$$

and have inverse Riordan arrays of the form $(1, \overline{xg}(x))$.

Proposition 4.2.1. *Let* $\mathbf{A} = (1, xg(x))$ *be a member of the associated subgroup of the Riordan group. Then its Stieltjes (production) matrix* P *is given by*

$$P = \begin{pmatrix} 0 & \alpha_0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \dots \\ 0 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \dots \\ 0 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots \\ 0 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \dots \\ 0 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$A(x) = \frac{x}{\overline{xg}(x)} \quad \text{and} \quad Z(x) = 0. \tag{4.5}$$

Proof. From [36] we have

$$A(x) = \frac{x}{\bar{f}(x)} = \frac{x}{\overline{xg}(x)}.$$

Also

$$\begin{aligned} Z(x) &= \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right) \\ &= \frac{1}{\overline{xg}(x)}\left(1 - \frac{1}{1}\right) \\ &= 0. \end{aligned}$$

□

Corollary 4.2.2. *Tridiagonal Stieltjes matrices relating to Riordan arrays from the ordinary associated subgroup have generating functions*

$$A(x) = 1 + \beta x + \alpha x^2, \quad Z(x) = 0.$$

Proof. Riordan arrays from the Appell subgroup with corresponding tridiagonal matrices have the form

$$\left(1, \frac{x}{1 - \beta x - \alpha x(xg(x))}\right)$$

so

$$\frac{x}{\overline{xg}(x)} = 1 + \beta x + \alpha x^2.$$

Applying eq. (4.5) gives the result. □

Example.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 5 & 5 & 3 & 1 & 0 & \dots \\ 0 & 15 & 14 & 9 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 1 & 0 & \dots \\ 0 & 2 & 1 & 1 & 1 & 1 & \dots \\ 0 & 6 & 2 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 5 & 5 & 3 & 1 & 0 & \dots \\ 0 & 15 & 14 & 9 & 4 & 1 & \dots \\ 0 & 52 & 44 & 28 & 14 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The sequence $A(x)$ counts the number of connected partitions of n (A099947) [124].

Example.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 4 & 5 & 3 & 1 & 0 & \dots \\ 0 & 10 & 12 & 9 & 4 & 1 & \dots \\ 0 & 26 & 32 & 25 & 14 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 0 & 1 & 1 & \dots \\ 0 & 4 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 4 & 5 & 3 & 1 & 0 & \dots \\ 0 & 10 & 12 & 9 & 4 & 1 & \dots \\ 0 & 26 & 32 & 25 & 14 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots, \end{pmatrix}.$$

The sequence $A(x)$ counts the number of irreducible diagrams with 2n nodes (A172395) [124].

4.2.2 Exponential associated subgroup

Proposition 4.2.3. *Let $A = [1, f(x)]$ be a member of the exponential associated subgroup of the exponential Riordan group. Then its Stieltjes (production) matrix P will have bivariate g.f.*

$$e^{xy}\left(f'(\bar{f}(x))y\right).$$

Proof. We have $g(x) = 1$ and hence $c(x) = 0$. Also, we have

$$r(x) = f'(\bar{f}(x)).$$

The result follows from eq. (4.2). ☐

The first few rows of tridiagonal Stieltjes matrices in this subgroup expand as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \beta & 1 & 0 & 0 & 0 & \dots \\ 0 & \alpha & 2\beta & 1 & 0 & 0 & \dots \\ 0 & 0 & 3\alpha & 3\beta & 1 & 0 & \dots \\ 0 & 0 & 0 & 6\alpha & 4\beta & 1 & \dots \\ 0 & 0 & 0 & 0 & 10\alpha & 5\beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general element

$$a_{(n,n)} = \frac{na_{n-1,n-1}}{n-1}, \quad a_{(n+1,n)} = \frac{(n+1)(a_{n,n-1})}{n-1}.$$

By expanding the second column of the matrix equation $\mathbf{LS} = \overline{\mathbf{L}}$ we see that generating functions from the associated subgroup with tridiagonal Stieltjes matrices satisfy the differential equation

$$\frac{d}{dx}\, g(x) = 1 + \beta g(x) + \frac{\alpha g(x)^2}{2}.$$

Solving the ordinary differential equation gives

$$g(x) = -\frac{-b + \sqrt{b^2-2c}\, tanh\left(\frac{1}{2}x\sqrt{b^2-2c} - Arctanh\left(\frac{b}{\sqrt{b^2-2c}}\right)\right)}{c}.$$

Example. *We take a member of the associated subgroup,* $[1, g(x)]$ *where the related Stieltjes matrix has first few rows*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & 3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 6 & 4 & 1 & \dots \\ 0 & 0 & 0 & 0 & 10 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Expanding the first column of the matrix equation $\bar{L} = LS$, the e.g.f. of the related Riordan matrix of the associated subgroup satisfies the equation

$$\frac{d}{dx}g(x) = 1 + g(x) + g(x)^2/2.$$

Solving gives

$$xg(x) = -1 + \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

Expanding for the first few terms we have $1, 1, 2, 5, 16, 61 \ldots$ (*A000111*).

Example. *Again, we take a member of the associated subgroup where $g(x)$ is an e.g.f. and the related Stieltjes matrix has first few rows*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 6 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 10 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Expanding the first column of the matrix equation $\bar{L} = LS$, the e.g.f. of the related Riordan matrix of the associated subgroup satisfies the equation

$$\frac{d}{dx}g(x) = 1 + g(x)^2/2,$$

and solving gives

$$xg(x) = \sqrt{2}\tan\left(\frac{x}{\sqrt{2}}\right).$$

Expanding for the first few terms we have $1, 1, 4, 34, 496, \ldots$ *which is the aerated sequence of reduced tangent numbers(A002105).*

4.3 The Bell subgroup

4.3.1 Ordinary Bell subgroup

Elements of the Bell subgroup have the form $(g(x), xg(x))$ and satisfy the group law as

$$(g(x), xg(x)) \cdot (f(x), xf(x)) = \Big(g(x)f(xg(x)), xf(xg(x)) \Big).$$

The Bell subgroup decomposes into the associated and Appell subgroups as

$$(g(x), xg(x)) = (g(x), x) \cdot (1, xg(x)).$$

Now, before we continue we introduce the following proposition which we will be of use to us in the section below.

Proposition 4.3.1.

$$\frac{x}{g(\overline{xg}(x))} = \overline{xg}(x) \tag{4.6}$$

Proof.

$$(g(x), xg(x))(1, x) = (g(x), xg(x))$$

so we have

$$\begin{aligned}
(1, x) &= (g(x), xg(x))^{-1}\big(g(x), xg(x)\big) \\
&= \left(\frac{1}{g(\overline{xg}(x))}, \overline{xg}(x) \right) (g(x), xg(x)) \\
&= \left(\frac{1}{g(\overline{xg}(x))} g(\overline{xg}(x)), \overline{xg}(x) g(\overline{xg}(x)) \right) \\
&= (1, \overline{xg}(x) g(\overline{xg}(x)))
\end{aligned}$$

so

$$\overline{xg}(x) = \frac{x}{g(\overline{xg}(x))}$$

□

Proposition 4.3.2. *Let $A = (g(x), xg(x))$ be a member of the Bell subgroup of the Riordan group. Then its Stieltjes (production) matrix P is given by*

$$P = \begin{pmatrix} \xi_0 & \alpha_0 & 0 & 0 & 0 & 0 & \ldots \\ \xi_1 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \ldots \\ \xi_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \ldots \\ \xi_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ldots \\ \xi_4 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \ldots \\ \xi_5 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$A(x) = \frac{x}{\overline{xg}(x)} \quad \text{and} \quad Z(x) = \frac{1}{\overline{xg}(x)} - 1. \tag{4.7}$$

Proof. We have $f(x) = xg(x)$ and hence $\bar{f}(x) = \overline{xg}(x)$. Thus

$$A(x) = \frac{x}{\overline{xg}(x)}.$$

Also

$$\begin{aligned} Z(x) &= \frac{1}{\overline{xg}(x)}\left(1 - \frac{1}{g(\overline{xg}(x))}\right) \\ &= \frac{1}{\overline{xg}(x)}\left(1 - \frac{\overline{xg}(x)}{x}\right) \\ &= \frac{1}{\overline{xg}(x)} - \frac{1}{x}. \end{aligned}$$

□

Corollary 4.3.3. *The tridiagonal Stieltjes matrices corresponding to Riordan arrays from the ordinary Bell subgroup have generating functions*

$$A(x) = 1 + \beta x + \alpha x^2, \quad Z(x) = \beta + \alpha x$$

Proof. Riordan arrays from the Bell subgroup with corresponding tridiagonal matrices have the form

$$\left(\frac{1}{1-\beta x-\alpha x^2 g(x)}, \frac{x}{1-\beta x-\alpha x^2 g(x)}\right)$$

with

$$\left(\frac{1}{1-\beta x-\alpha x^2 g(x)}, \frac{x}{1-\beta x-\alpha x^2 g(x)}\right)^{-1} = \left(\frac{1}{1+\beta x+\alpha x^2}, \frac{x}{1+\beta x+\alpha x^2}\right).$$

Thus

$$\overline{xg}(x) = \frac{x}{1+\beta x+\alpha x^2}.$$

Applying eq. (4.7) gives the result. □

4.3.2 Exponential Bell subgroup

For the exponential Bell subgroup we have Riordan arrays of the form $[g(x), \int g(x)]$, or alternatively $[\frac{d}{dx}h(x), h(x)]$.

Proposition 4.3.4. *Let $A = [g(x), \int g(x)]$ be a member of the exponential Bell subgroup of the exponential Riordan group. Then its Stieltjes (production) matrix P will have bivariate g.f.*

$$e^{xy}\left(\frac{g'(\overline{\int g(x)})}{g(\overline{\int g(x)})} + g(\overline{\int g(x)})y\right).$$

Proof.

$$c(x) = \frac{g'(\overline{\int g(x)})}{g(\overline{\int g(x)})},$$

and

$$r(x) = f'(\bar{f}(x)) = g(\overline{\int g(x)}).$$

The result follows from eq. (4.2). □

Now once again, we look at the form of the related tridiagonal Stieltjes matrices by equating columns of $\mathbf{LS} = \overline{\mathbf{L}}$, and we get the related tridiagonal Stieltjes matrices with first few elements

$$\begin{pmatrix} \gamma & 1 & 0 & 0 & 0 & \dots \\ \delta & \beta & 1 & 0 & 0 & \dots \\ 0 & \alpha & 2\beta-\gamma & 1 & 0 & \dots \\ 0 & 0 & 3\alpha-3\delta & 3\beta-2\gamma & 1 & \dots \\ 0 & 0 & 0 & 2(3\alpha-4\delta) & 4\beta-3\gamma & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with $(n,n)^{th}$ and $(n,n-1)^{th}$ entries

$$a_{nn} = \frac{na_{n-1,n-1}-\gamma}{n-1}, \quad a_{n+1,n} = \frac{(n+1)(a_{n,n-1}-\delta)}{(n-1)}.$$

Expanding the first column of the equation $LS = \overline{\mathbf{L}}$ we have the equation

$$\gamma g(x) + \delta \int g(x)\,dx = \frac{d}{dx}\,g(x) \tag{4.8}$$

Let us look at an example.

Example. *With γ, δ equal to one we have the e.g.f. of the form*

$$\frac{1}{1-\sin x}$$

with the sequence of coefficients having first few terms $1, 1, 2, 5, 16, 61\dots$ which is the sequence of the Euler numbers(A000111) and also counts the number of alternating permutations on n letters. We have

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 3 & 3 & 1 & 0 & \dots \\ 0 & 0 & 6 & 4 & 1 & \dots \\ 0 & 0 & 0 & 10 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example. *With* $\gamma = 1, \delta = 3$ *we have the e.g.f. of the form*

$$\frac{3}{\left(\sqrt{3}\cos\left(\frac{\sqrt{3}x}{2}\right) - \sin\left(\frac{\sqrt{3}x}{2}\right)\right)^2}$$

with the sequence of coefficients having first few terms $11, 3, 9, 39 \ldots$. *This is the sequence (A080635) which counts the number of permutations on n letters without double falls and without initial falls. We have*

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 1 & 0 & 0 & \ldots \\ 0 & 6 & 3 & 1 & 0 & \ldots \\ 0 & 0 & 12 & 4 & 1 & \ldots \\ 0 & 0 & 0 & 20 & 5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4.4 The Hitting time subgroup

4.4.1 Ordinary Hitting time subgroup

The *hitting-time* subgroup [26] of the Riordan group is comprised of matrices of the form

$$\left(\frac{xh'(x)}{h(x)}, h(x)\right).$$

We have the following Stieltjes matrix characterization of the hitting-time subgroup.

Proposition 4.4.1. *For a Riordan array* $(g(x), f(x))$ *to be an element of the hitting-time subgroup, it is necessary and sufficient that*

$$Z(x) = A'(x).$$

Proof. We show first that the condition is necessary. Thus let

$$A = (g(x), f(x)) = \left(\frac{xh'(x)}{h(x)}, h(x) \right).$$

Then

$$A(x) = \frac{x}{\bar{f}(x)} = \frac{x}{\bar{h}(x)}.$$

Thus

$$\bar{h}(x) = \frac{x}{A(x)}$$

and

$$\frac{1}{\bar{h}(x)} = \frac{A(x)}{x}.$$

Now

$$\begin{aligned} Z(x) &= \frac{1}{\bar{h}(x)} \left(1 - \frac{1}{g(\bar{h}(x))} \right) \\ &= \frac{1}{\bar{h}(x)} \left(1 - \frac{1}{\frac{\bar{h}(x)h'(\bar{h}(x))}{h(\bar{h}(x))}} \right) \\ &= \frac{1}{\bar{h}(x)} \left(1 - \frac{x}{\bar{h}(x)h'(\bar{h}(x))} \right) \\ &= \frac{1}{\bar{h}(x)} \left(1 - \frac{A(x)}{h'(\bar{h}(x))} \right) \\ &= \frac{A(x)}{x} \left(1 - \frac{A(x)}{h'(\bar{h}(x))} \right). \end{aligned}$$

Now differentiating the identity $h(\bar{f}(x)) = x$ with respect to x gives

$$h'(\bar{h}(x))(\bar{h})'(x) = 1$$

and so

$$h'(\bar{h}(x)) = \frac{1}{(\bar{h})'(x)}.$$

Now since

$$\bar{h}(x) = \frac{x}{A(x)}$$

we have

$$(\bar{h})'(x) = \frac{A(x) - xA'(x)}{A(x)^2},$$

and so we get

$$\begin{aligned} Z(x) &= \frac{A(x)}{x}\left(1 - A(x)\frac{A(x) - xA'(x)}{A(x)^2}\right) \\ &= \frac{A(x)}{x} - \frac{A(x)^2}{x}\frac{A(x) - xA'(x)}{A(x)^2} \\ &= \frac{A(x)}{x} - \frac{A(x)}{x} + \frac{xA'(x)}{x} \\ &= A'(x). \end{aligned}$$

Thus let

$$Z(x) = A'(x),$$

where

$$\bar{f}(x) = \frac{x}{A(x)} \quad \text{or} \quad A(x) = \frac{x}{\bar{f}(x)}.$$

Now

$$Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right).$$

Thus

$$g(\bar{f}(x)) = \frac{1}{1 - \bar{f}(x)Z(x)}$$

and hence

$$g(x) = \frac{1}{1 - xZ(f(x))}.$$

From this we infer that

$$\begin{aligned} g(x) &= \frac{1}{1-xZ(f(x))} \\ &= \frac{1}{1-xA'(f(x))} \\ &= \frac{1}{1-x\frac{\bar{f}(f(x))-f(x)(\bar{f})'(f(x))}{\bar{f}(f(x))^2}} \\ &= \frac{1}{1-x\frac{x-f(x)(\bar{f})'(f(x))}{x^2}} \\ &= \frac{1}{1-1+\frac{f(x)(\bar{f})'(f(x))}{x}} \\ &= \frac{x}{f(x)(\bar{f})'(f(x))} \\ &= \frac{xf'(\bar{f}(f(x)))}{f(x)} \\ &= \frac{xf'(x)}{f(x)}. \end{aligned}$$

□

Corollary 4.4.2. *Riordan arrays with tridiagonal Stieltjes matrices which are elements of the hitting time subgroup have the form*

$$\left(\frac{1}{1-\beta x-2\alpha g(x)}, \frac{x}{1-\beta x-\alpha g(x)}\right).$$

Proof. Let

$$g(x) = \frac{x}{1-\beta x-\alpha g(x)} = -\frac{1-\alpha x-\sqrt{x^2(\alpha^2-4\beta)-2\alpha x+1}}{2\beta x}$$

so

$$\frac{xg'(x)}{g(x)} = \frac{1}{\sqrt{x^2(\alpha^2-4\beta x)-2\alpha x+1}} = \frac{x}{1-\beta x-2\alpha g(x)}$$

□

Example.

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The related Riordan array has the form

$$(f(x), g(x)) = \left(\frac{1}{1 - x - 2xg(x)}, \frac{x}{1 - x - xg(x)} \right).$$

Chapter 5

Lattice paths and Riordan arrays

In this chapter we will study the well-known Motzkin and Łukasiewicz paths. Motzkin paths have well established links to orthogonal polynomials; as we have seen in [56], the entries of the tridiagonal Stieltjes matrix represent the weights of the possible steps in the Motzkin paths. We begin this chapter by introducing lattice paths, particularly Motzkin and Łukasiewicz paths which we study through the medium of Riordan arrays. We give a constructive proof of how Riordan arrays with non-tridiagonal Stieltjes matrices relate to Łukasiewicz paths. We then develop these paths through the medium of Riordan arrays.

Let us recall the definition of a lattice path which we introduced in section 2.5. A lattice path [79] is a sequence of points in the integer lattice $\mathbb{Z}^2$. A pair of consecutive points is called a step of the path. A valuation is a function on the set of possible steps of $\mathbb{Z}^2 \times \mathbb{Z}^2$. A valuation of a path is the product of the valuations of its steps. We concern ourselves with two types of paths, Motzkin paths and Łukasiewicz paths [151], which are defined as follows

Definition 5.0.1. *A Motzkin path [78]* $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$, *of length* n, *is a lattice path starting at* $(0, 0)$ *and ending at* $(n, 0)$ *that satisfies the following conditions*

1. *The elementary steps can be north-east(N-E), east(E) and south-east(S-E).*

2. Steps never go below the x axis.

Dyck paths are Motzkin paths without the possibility of an East(E) step.

Definition 5.0.2. *A Łukasiewicz path [78] $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$, of length n, is a lattice path starting at $(0,0)$ and ending at $(n,0)$ that satisfies the following conditions*

1. *The elementary steps can be north-east(N-E) and east(E) as those in Motzkin paths.*

2. *South-east(S-E) steps from level k can fall to any level above or on the x axis, and are denoted as $\alpha_{n,k}$, where n is the length of the south-east step and k is the level where the step ends.*

3. *Steps never go below the x axis.*

Finally, we introduce the Schröder paths, as we will encounter these paths at a later stage

Definition 5.0.3. *A Schröder path $\pi = (\pi(0), \pi(1), \ldots, \pi(2n))$, of semilength n, is a lattice path starting at $(0,0)$ and ending at $(2n,0)$ that satisfies the following conditions*

1. *The elementary steps can be north-east(N-E), east(E) and south-east(S-E) with easterly steps begin twice the length of the north-easterly and south-easterly steps.*

2. *Steps never go below the x axis.*

5.1 Motzkin, Schröder and Łukasiewicz paths

Firstly, let us illustrate the construction of the $(n+1)^{th}$ row of the Riordan array. The $(m,n)^{th}$ entry of the Riordan array is $l_{m,n}$ where m is the length of the path and n is the height of the final position of the last step. In [56], we see that the entries of the tridiagonal Stieltjes matrix represent the weights of the possible steps in the Motzkin paths. From the 0^{th} column calculated from the Stieltjes equation $\overline{\mathbf{L}} = \mathbf{LS}$, as

$$l_{n+1,0} = l_{n,0}\beta_{0,0} + l_{n,1}\alpha_{1,0}$$

so clearly, $l_{n+1,0}$ is calculated from $l_{n,0}$ with a level step at the zero level, and $l_{n,1}$, with an added south-east step, $\alpha_{1,0}$ so for $m > 0$,

$$l_{n+1,m} = l_{n,m-1} + l_{n,m}\beta_{m,m} + l_{n,m+1}\alpha_{m+1,m}. \tag{5.1}$$

From **(5.1)** we see the paths contributing to $l_{n+1,m}$ are as follows

- Paths of length n finishing at level $m-1(l_{n,m-1})$, adding one N-E step of weight 1.
- Paths of length n finishing at level $m(l_{n,m})$, adding one E step of weight $\beta_{m,m}$.
- Paths of length n finishing at level $m+1(l_{n,m+1})$, adding one S-E step of weight $\alpha_{m+1,m}$.

Now, consider a Riordan array with non-tridiagonl Stieltjes matrix,

$$\begin{pmatrix} \beta_{0,0} & 1 & 0 & 0 & 0 & 0 & \dots \\ \alpha_{1,0} & \beta_{1,1} & 1 & 0 & 0 & 0 & \dots \\ \alpha_{2,0} & \alpha_{2,1} & \beta_{2,2} & 1 & 0 & 0 & \dots \\ \alpha_{3,0} & \alpha_{3,1} & \alpha_{3,2} & \beta_{3,3} & 1 & 0 & \dots \\ \alpha_{4,0} & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \beta_{4,4} & 1 & \dots \\ \alpha_{5,0} & \alpha_{5,1} & \alpha_{5,2} & \alpha_{5,3} & \alpha_{5,4} & \beta_{5,5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.2}$$

Here we have the possibility of any length of south-easterly step, so for the 0^{th} column we have

$$l_{n+1,0} = l_{n,0}\beta_{0,0} + l_{n,1}\alpha_{1,0} + l_{n,2}\alpha_{2,0} + \cdots + l_{n,n}\alpha_{n,0},$$

and for any m we have

$$l_{n+1,m} = l_{n,m-1} + l_{n,m}\beta_{m,m} + l_{n,m+1}\alpha_{m+1,m} + l_{n,m+2}\alpha_{m+2,m} + \cdots + l_{n,n}\alpha_{n,m}. \tag{5.3}$$

From **(5.3)** we see the paths contributing to the $l^{th}_{n+1,m}$ Łukasiewicz path are as follows

- Paths of length n finishing at level $m-1(l_{n,m-1})$, adding one N-E step of weight 1.
- Paths of length n finishing at level $m(l_{n,m})$, adding one E step $\beta_{m,m}$.
- Paths of length n finishing at level $m+1(l_{n,m+1})$, adding an $\alpha_{m+1,m}$ Łukasiewicz step.
- Paths of length n finishing at level $m+1(l_{n,m+2})$, adding an $\alpha_{m+2,m}$ Łukasiewicz step.

$\vdots$

- Paths of length n finishing at level $m+1(l_{n,n})$, adding an $\alpha_{n,m}$ Łukasiewicz step.

Note that in this chapter we adopt the following notation for our Motzkin and Łukasiewicz paths. (α,β)-Motzkin path and (α,β)-Łukasiewicz paths can be viewed as coloured Motzkin/Łukasiewicz paths in the sense that there are β colours for each level step and α colours for each down step.

5.1.1 The binomial transform of lattice paths

The Binomial transform of generating functions has been of interest to us in previous chapters. We now look at the effect of the binomial transform of a Riordan array in terms of steps of Motzkin paths. Let us first look at Dyck paths, which have no level steps and then introduce the level steps via the Binomial transform.

The sequence that counts Dyck paths has g.f.

$$g(x)=\frac{1-\sqrt{(1-4\alpha x^2)}}{2\alpha x^2}.$$

The general Riordan array corresponding to Dyck paths has the form

$$\left(\frac{1-\sqrt{(1-4\alpha x^2)}}{2\alpha x^2},\frac{1-\sqrt{(1-4\alpha x^2)}}{2\alpha x}\right).$$

When $\alpha = 1$, this is the sequence of aerated Catalan numbers. The g.f. of the Dyck paths has continued fraction expansion

$$\cfrac{1}{1-\cfrac{\alpha x^2}{1-\cfrac{\alpha x^2}{1-\cfrac{\alpha x^2}{\ddots}}}}.$$

Now, the β^{th} binomial transform matrix has first few rows

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \beta & 1 & 0 & 0 & \dots \\ \beta^2 & 2\beta & 1 & 0 & \dots \\ \beta^3 & 3\beta^2 & 3\beta & 1 & \dots \\ \beta^4 & 4\beta^3 & 6\beta^2 & 4\beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(\frac{1}{1-\beta x}, \frac{x}{1-\beta x}\right).$$

Calculating the binomial transform of $(g(x), xg(x))$ we have

$$\left(\frac{1}{1-\beta x}, \frac{x}{1-\beta x}\right)\left(\frac{1-\sqrt{(1-4\alpha x^2)}}{2\alpha x^2}, \frac{1-\sqrt{(1-4\alpha x^2)}}{2\alpha x}\right)$$

resulting in the Riordan array with first column having the g.f.

$$\frac{1-\sqrt{(1-4\alpha\left(\frac{x}{1-\beta x}\right)^2)}}{2\alpha\left(\frac{x}{1-\beta x}\right)^2},$$

which has continued fraction expansion,

$$\cfrac{1}{1-\beta x-\cfrac{\alpha x^2}{1-\beta x-\cfrac{\alpha x^2}{1-\beta x-\cfrac{\alpha x^2}{\ddots}}}}.$$

From the continued fraction expansion we have seen in [56], the binomial transform has introduced the level steps in the Motzkin paths.

Now, let us look at the contribution of the binomial matrix to each path in our Riordan array. Let,

$$\mathbf{L} = \begin{pmatrix} a_{0,0} & 0 & 0 & 0 & \dots \\ a_{1,0} & a_{1,1} & 0 & 0 & \dots \\ a_{2,0} & a_{2,1} & a_{2,2} & 0 & \dots \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & \dots \\ a_{4,0} & a_{4,1} & a_{4,2} & a_{4,3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us now look at the paths in the Riordan array after multiplication by the Binomial matrix,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \beta & 1 & 0 & 0 & \dots \\ \beta^2 & 2\beta & 1 & 0 & \dots \\ \beta^3 & 3\beta^2 & 3\beta & 1 & \dots \\ \beta^4 & 4\beta^3 & 6\beta^2 & 4\beta & \dots \\ \beta^5 & 5\beta^4 & 10\beta^3 & 10\beta^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{0,0} & 0 & 0 & 0 & 0 & \dots \\ a_{1,0} & a_{1,1} & 0 & 0 & 0 & \dots \\ a_{2,0} & a_{2,1} & a_{2,2} & 0 & 0 & \dots \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & 0 & \dots \\ a_{4,0} & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \dots \\ a_{5,0} & a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We denote the elements in the Binomial transformed Riordan array, bl. We investigate the contribution to one such element, $bl_{5,0}$. From the matrix multiplication above we have

$$bl_{5,0} = \binom{5}{5}\beta^5 a_{0,m} + \binom{5}{4}\beta^4 a_{1,m} + \binom{5}{3}\beta^3 a_{2,m} + \binom{5}{2}\beta^3 a_{3,m} + \binom{5}{1}\beta a_{4,m} + \binom{5}{0}\beta a_{5,m}$$

We can now see the effect of the binomial transform on each of the level steps that contribute to the new step $bl_{5,0}$:

- $\binom{5}{5}\beta^5 a_{0,0}$, is the Dyck path of length 0, with a choice of 1 place for the 5 level steps,

$$\binom{5}{5} = \binom{1}{1}\binom{4}{4},$$

  with no choice of length of level paths as we are filling one position only.

- Now for $\binom{5}{4}\beta^4 a_{1,0}$, is the Dyck path of length 1, with a choice of 2 places for 4 level steps, and choice for arranging the level steps giving,

$$\binom{5}{4} = \binom{2}{1}\binom{3}{3} + \binom{2}{2}\binom{3}{2}.$$

- Now for $\binom{5}{3}\beta^3 a_{2,0}$, the path is length 2 with a choice of 3 places for 3 level steps, and choice for arranging the level steps giving,

$$\binom{5}{3} = \binom{3}{1}\binom{2}{2} + \binom{3}{2}\binom{2}{1} + \binom{3}{2}\binom{2}{0}.$$

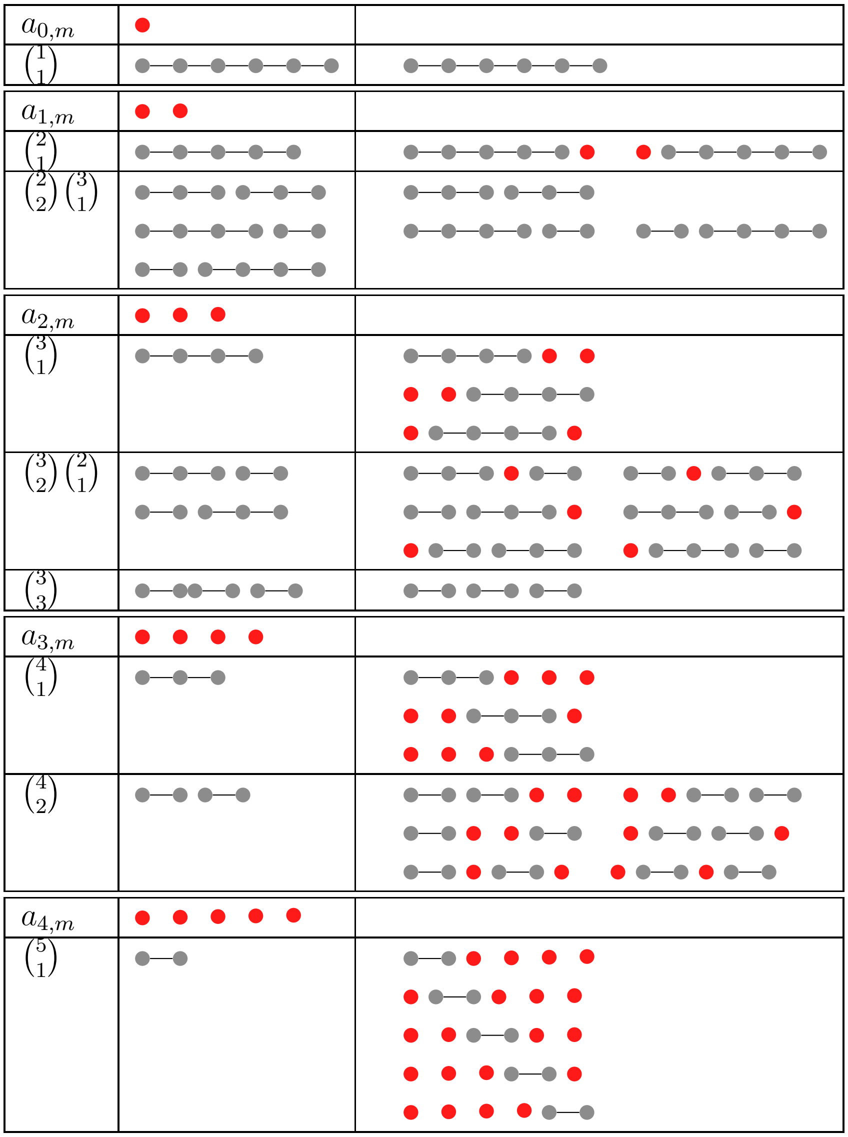
- Now for $\binom{5}{2}\beta^2 a_{3,0}$, the path is length 3 with a choice of 4 places for 2 level steps, and choice for arranging the level steps giving

$$\binom{5}{2} = \binom{4}{1}\binom{1}{1} + \binom{4}{2}\binom{1}{0}.$$

- Now for $\binom{5}{1}\beta a_{4,0}$, the path is length 4 with a choice of 5 places for 1 level steps,

$$\binom{5}{1} = \binom{5}{1}\binom{0}{0}.$$

We illustrate the effect of the binomial transform in the table below. The red dots represent the Motzkin path $a_{n,m}$ where n is the path length and m is the level of the last step. Note that as the paths may take on different forms, depending on m, the dots represent each of the N-E or S-E steps.



Now, for any lattice path step and the binomial transform we have

$$\binom{n}{m}\beta^m a_{n-m} = \sum_{q=1}^{min(m,n-m+1)} \binom{n-m+1}{q}\binom{m-1}{m-q}\beta^m a_{n-m}$$

- For the path a_{n-m} we have $n-m+1$ choices of positions for the m level steps, with $m \le n$.
- Now, we can choose q of these $n-m+1$ positions to place the m level steps.

- If we choose q of the $n-m+1$ positions to place the level steps, we now need to choose the number of level steps to put at each of the q positions.

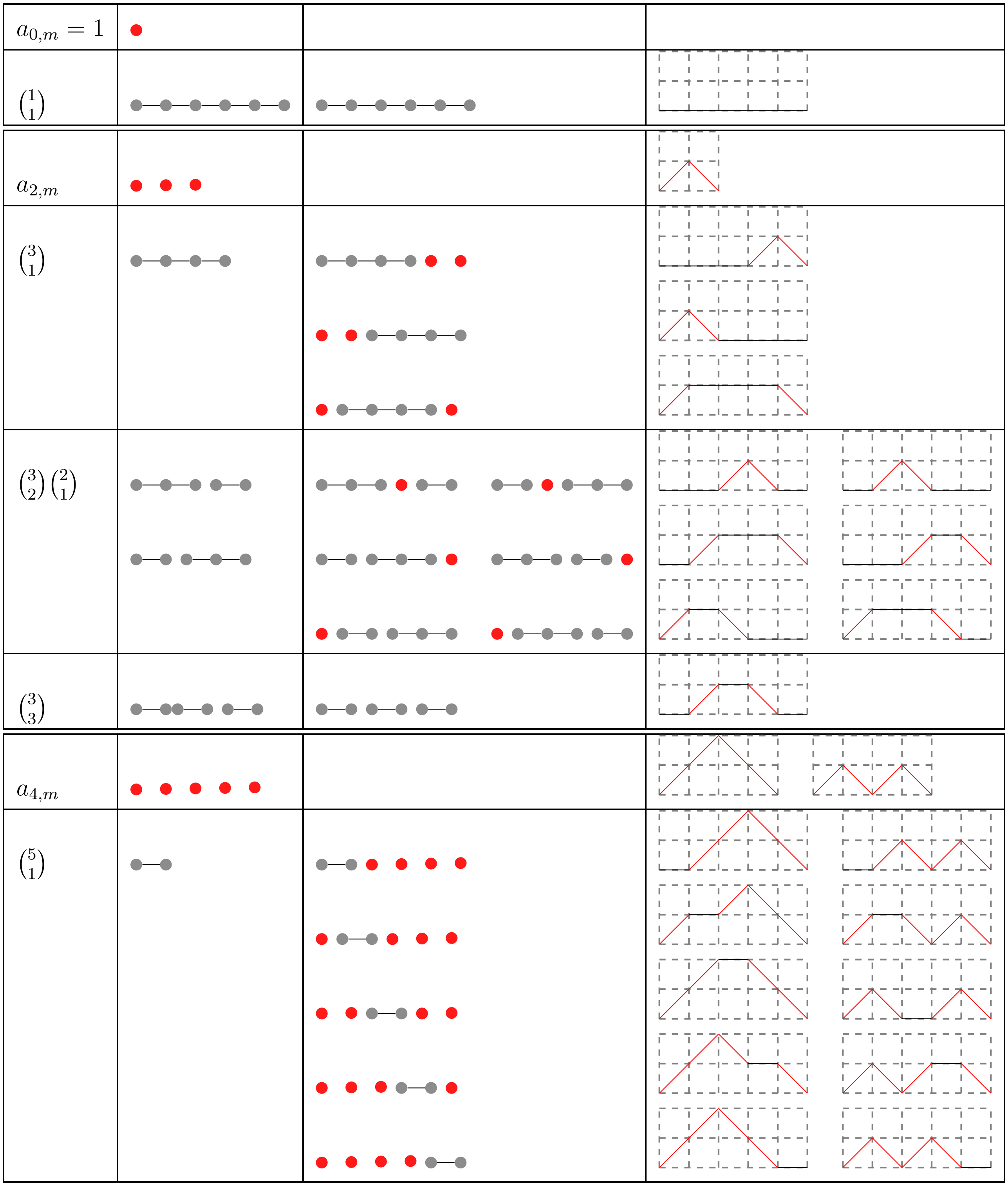
Example. *Let us look at the binomial transform of the step* $l_{5,0}$ *of the Dyck paths, which are counted by the aerated Catalan numbers. We have the following equation*

$$bl_{5,0} = \binom{5}{5}\beta^5 a_{0,m} + \binom{5}{4}\beta^4 a_{1,m} + \binom{5}{3}\beta^3 a_{2,m} + \binom{5}{2}\beta^3 a_{3,m} + \binom{5}{1}\beta a_{4,m} + \binom{5}{0}\beta a_{5,m}.$$

Since we count only paths of even length given that for Dyck paths since level steps are not permitted, we have

$$bl_{5,0} = \binom{5}{5}\beta^5 a_{0,m} + \binom{5}{3}\beta^3 \binom{5}{3}\beta^3 a_{2,m} + \binom{5}{1}\beta a_{4,m}.$$

We now illustrate each of the components of $bl_{5,0}$.


$a_{0,m} = 1$
$\binom{1}{1}$
$a_{2,m}$
$\binom{3}{1}$
$\binom{3}{2}\binom{2}{1}$
$\binom{3}{3}$
$a_{4,m}$
$\binom{5}{1}$

In the section above, we used Motzkin paths to illustrate the Binomial transform, however the Binomial transform of the Łukasiewicz paths follows the same construction. Thus from the construction of the Binomial transform above, we conclude that if a bijection exists between a Motzkin path and a Łukasiewicz path, the bijection is preserved under the binomial transform.

5.2 Some interesting Łukasiewicz paths

In this section we concern ourselves with Łukasiewicz paths where the S-E steps are restricted. We begin with the following proposition concerning Łukasiewicz paths with S-E steps which all have the same weight attached.

Proposition 5.2.1. *Łukasiewicz paths with steps weighted*

$$\begin{pmatrix} \beta & 1 & 0 & 0 & 0 & 0 & \dots \\ \alpha & \beta & 1 & 0 & 0 & 0 & \dots \\ \alpha & \alpha & \beta & 1 & 0 & 0 & \dots \\ \alpha & \alpha & \alpha & \beta & 1 & 0 & \dots \\ \alpha & \alpha & \alpha & \alpha & \beta & 1 & \dots \\ \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.4}$$

have o.g.f.

$$g(x) = \frac{1 + x(1-\beta) - \sqrt{((\beta+1)x-1)^2 - 4\alpha x^2}}{2x(1-\beta x + \alpha x)}. \tag{5.5}$$

Proof. From the Stieltjes equation we have the following

$$g(x) + \beta x g(x)^2 + \alpha x^2 g(x)^3 + \alpha x^3 g(x)^4 + \cdots = g(x)^2$$

so

$$\begin{aligned} g(x) &= 1+\beta x g(x)+\alpha(xg(x))^2(1+xg(x)+(xg(x))^2+\dots) \\ &= 1+\beta x g(x)+\frac{\alpha(xg(x))^2}{1-xg(x)} \end{aligned}$$

and solving for $g(x)$ we have

$$g(x)=\frac{1+x(1-\beta)-\sqrt{((\beta+1)x-1)^2-4\alpha x^2}}{2x(1-\beta x+\alpha x)}.$$

□

Now, we look at two particular Łukasiewicz paths, where the possible S-E steps have the same weight.

5.2.1 Łukasiewicz paths with no odd south-east steps

Let us look at the Łukasiewicz paths that have no odd south-east steps so we have related Stieltjes steps

$$\begin{pmatrix} \beta_{0,0} & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \beta_{1,1} & 1 & 0 & 0 & 0 & \dots \\ \alpha_{2,0} & 0 & \beta_{2,2} & 1 & 0 & 0 & \dots \\ 0 & \alpha_{3,1} & 0 & \alpha_{3,3} & 1 & 0 & \dots \\ \alpha_{4,0} & 0 & \alpha_{4,2} & 0 & \beta_{4,4} & 1 & \dots \\ 0 & \alpha_{5,1} & 0 & \alpha_{5,3} & 0 & \beta_{5,5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.6}$$

and from the Stieltjes equation we have

$$\beta f(x)+\alpha_{2,0}x^2(f(x))^3+\alpha_{4,0}x^4(f(x))^5+\dots=\frac{f(x)-1}{x}.$$

Rearranging gives

$$1+\beta x f(x)+\alpha_{2,0}x^3(f(x))^3+\alpha_{4,0}x^5(f(x))^5+\dots=f(x).$$

Now, let $y = xf(x)$,

$$1 + \beta y + \alpha y^3 + \alpha y^5 + \cdots = \frac{y}{x}.$$

Solving we have

$$1 + \beta y + \alpha y^3(1 + y^2 + \dots) = \frac{y}{x}$$
$$x(1 - y^2)(1 + \beta y) + x\alpha y^3 = y(1 - y^2)$$
$$x + x\beta y - xy^2 - x\beta y^3 + x\alpha y^3 = y - y^3.$$

Solving for x we have

$$x + x\beta y - xy^2 - x\beta y^3 + x\alpha y^3 = y - y^3$$
$$x = \frac{y - y^3}{1 + \beta y - y^2 + y^3(\alpha - \beta)}.$$

and

$$y^3(x\alpha - x\beta + 1) - xy^2 + y(x\beta - 1) + x = 0.$$

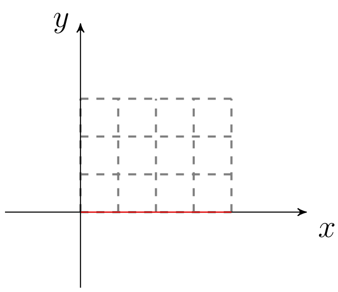
Solving for y, the first few terms of the g.f. expansion are

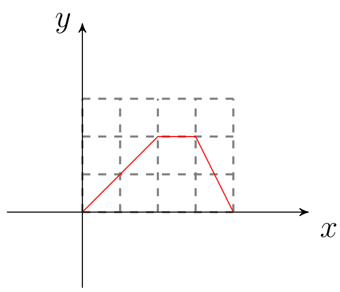
$$x + \beta x^2 + \beta^2 x^3 + \dots$$

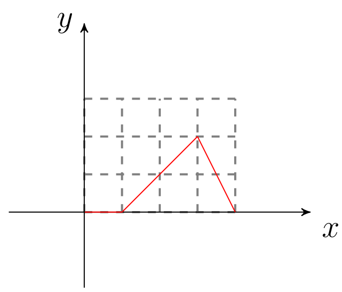
Example. *For the* $(1,1)$*-Łukasiewicz paths of the form above we have the g.f.*

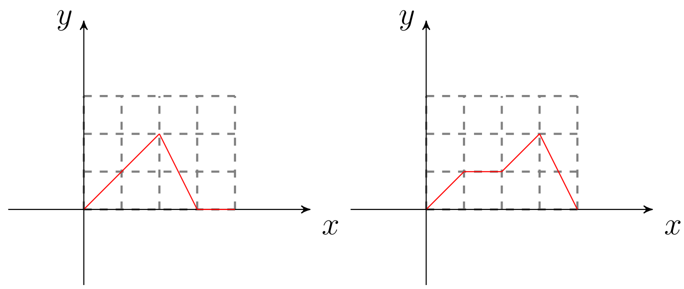
$$\frac{x}{3} - \frac{2\sqrt{x^2 - 3x + 3}\sin\left(\frac{\arcsin\frac{x(2x^2 - 9x - 18)}{2(x^2 - 3x + 3)^{3/2}}}{3}\right)}{3}$$

of the sequence $1, 1, 1, 2, 5, \dots$ *(A101785). The paths corresponding to length* 4 *are illustrated below.*









According to [124], (A101785) also counts the number of ordered trees with n edges in which every non-leaf vertex has an odd number of children. The corresponding Riordan array is

$$\left(\frac{1-x^2}{1+x-x^2}, \frac{x(1-x^2)}{1+x-x^2}\right)^{-1}.$$

5.2.2 Łukasiewicz paths with no even south-east steps

Let us look at the Łukasiewicz Paths that have no even south-east steps so we have related Stieltjes steps

$$\begin{pmatrix} \beta_{0,0} & 1 & 0 & 0 & 0 & 0 & \dots \\ \alpha_{1,0} & \beta_{1,1} & 1 & 0 & 0 & 0 & \dots \\ 0 & \alpha_{2,1} & \beta_{2,2} & 1 & 0 & 0 & \dots \\ \alpha_{3,0} & 0 & \alpha_{3,2} & \beta_{3,3} & 1 & 0 & \dots \\ 0 & \alpha_{4,1} & 0 & \alpha_{4,3} & \beta_{4,4} & 1 & \dots \\ \alpha_{5,0} & 0 & \alpha_{5,2} & 0 & \alpha_{5,4} & \beta_{5,5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.7}$$

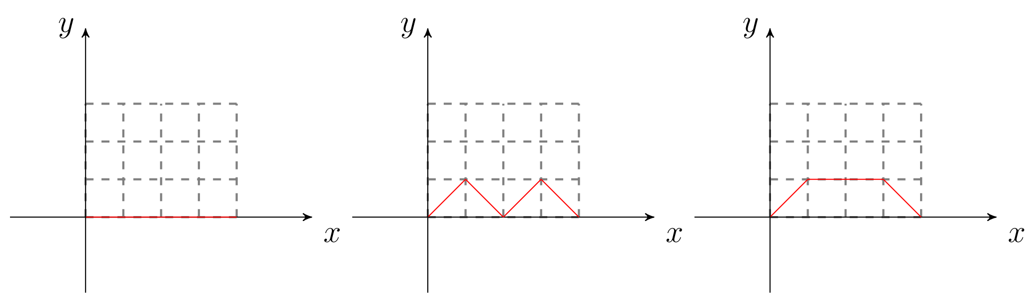
and again from the Stieltjes equation we have

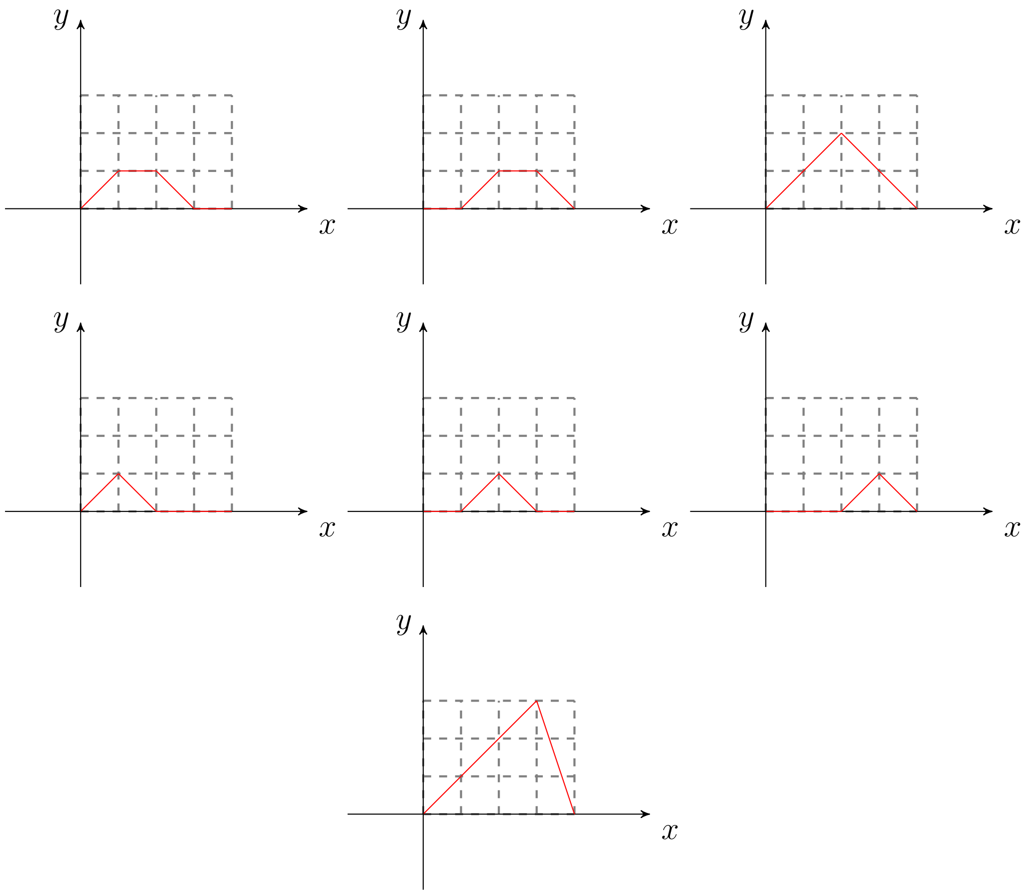
$$\beta f(x) + \alpha_{1,0}x(f(x))^2 + \alpha_{3,0}x^3(f(x))^4 + \dots = \frac{f(x)-1}{x}.$$

Rearranging we have

$$1 + \beta x f(x) + \alpha x^2(f(x))^2 + \alpha_{4,0}x^4(f(x))^4 + \dots = f(x).$$

Let $y = xf(x)$, then
$$1 + \beta y + \alpha y^2 + \alpha y^4 + \cdots = \frac{y}{x}.$$
Solving we have
$$1 + \beta y + \alpha y^2(1 + y^2 + \dots) = \frac{y}{x}$$
$$x(1 - y^2)(1 + \beta y) + x\alpha y^2 = y(1 - y^2)$$
$$x + x\beta y - xy^2 - x\beta y^3 + x\alpha y^2 = y - y^3$$
and solving for x we have
$$x = \frac{y - y^3}{1 + \beta y + y^2(\alpha - 1) - \beta y^3}$$
and
$$x + y(x\beta - 1) + xy^2(\alpha - 1) + y^3(1 - x\beta) = 0.$$
Solving for y above we obtain the first few terms of the g.f. expansion as
$$x + \beta x^2 + x^3(\alpha + \beta^2) + \beta x^4(3\alpha + \beta^2) + \dots.$$
The corresponding Riordan array is
$$\left(\frac{1 - x^2}{1 + x - x^2}, \frac{x(1 - x^2)}{1 + x - x^2}\right)^{-1}.$$
Example. *The* $(1, 1)$*-Łukasiewicz paths of the form above have g.f.*
$$\frac{2\sqrt{3}\sin\left(\frac{\arcsin\frac{3\sqrt{3}x}{2|x-1|}}{3}\right)}{3}$$
of the sequence with first few terms $1, 1, 2, 4, 10, 26, 73, \dots$ *(A049130). We illustrate the paths of length* 4 *below.*





5.3 A (β, β)-Łukasiewicz path

From (5.5) above, if $\alpha = \beta$ we have the g.f.

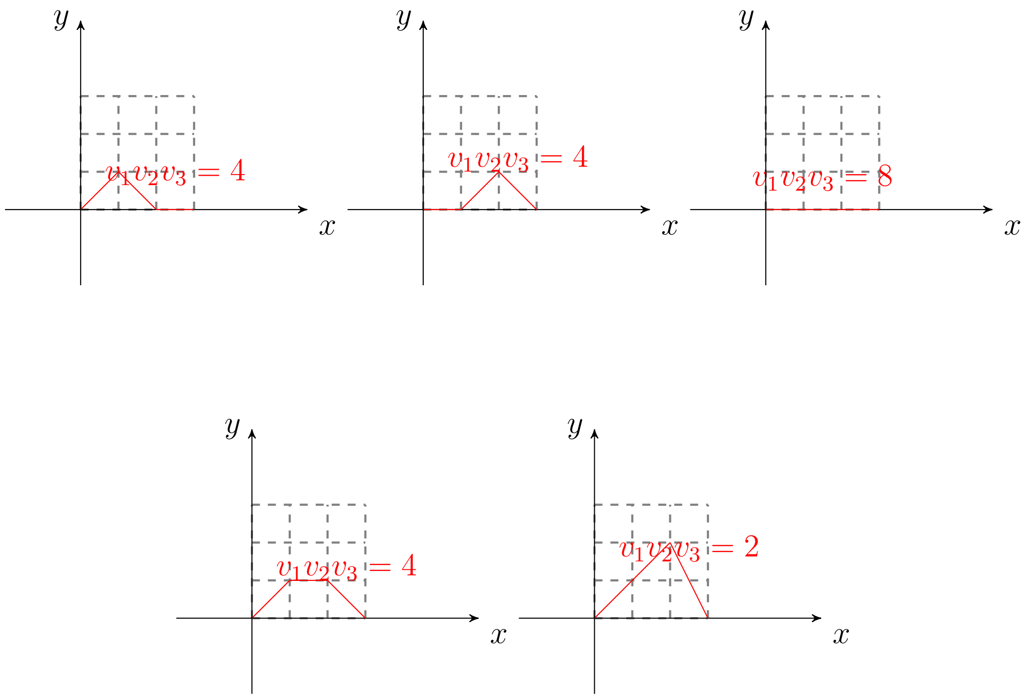
$$xg(x) = \frac{1 + x(1-\beta) - \sqrt{((\beta-1)x)^2 - 2x(\beta+1)}}{2}.$$

Let us look at an interesting example.

Example. *The* $(2,2)$*-Łukasiewicz paths have g.f.*

$$xg(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2},$$

which is the g.f. of the sequence with first few terms $1, 2, 6, 22, \dots$ (A006318). *The corresponding paths for* $n = 3$ *are shown below.*



The above example leads us to the next section where we provide a bijection between certain Łukasiewicz paths and Schröder paths.

5.3.1 A bijection between the (2,2)-Łukasiewicz and Schröder paths

In this section we give a constructive proof of a bijection between the (2,2)-Łukasiewicz paths and the Schröder paths. Firstly we introduce the different steps possible in the (2,2)-Łukasiewicz paths and the Schröder paths. u is the the N-E step $(1, 1)$, and d the S-E step $(1, -1)$, in both the Łukasiewicz and the Schröder paths. E steps possible are $b = (1, 0)$ in the Łukasiewicz paths and $b^+ = (2, 0)$ in the Schröder paths. We denote b_1 and b_2 the two choice of colours for the E steps in the Łukasiewicz paths. Łukasiewicz steps are denoted $l^n = (1, -n)$. We denote d_1 and d_2 the two choice of colours for the

S-E steps in the Łukasiewicz paths, and similarly l_1^n and l_2^n the choice of Łukasiewicz steps.

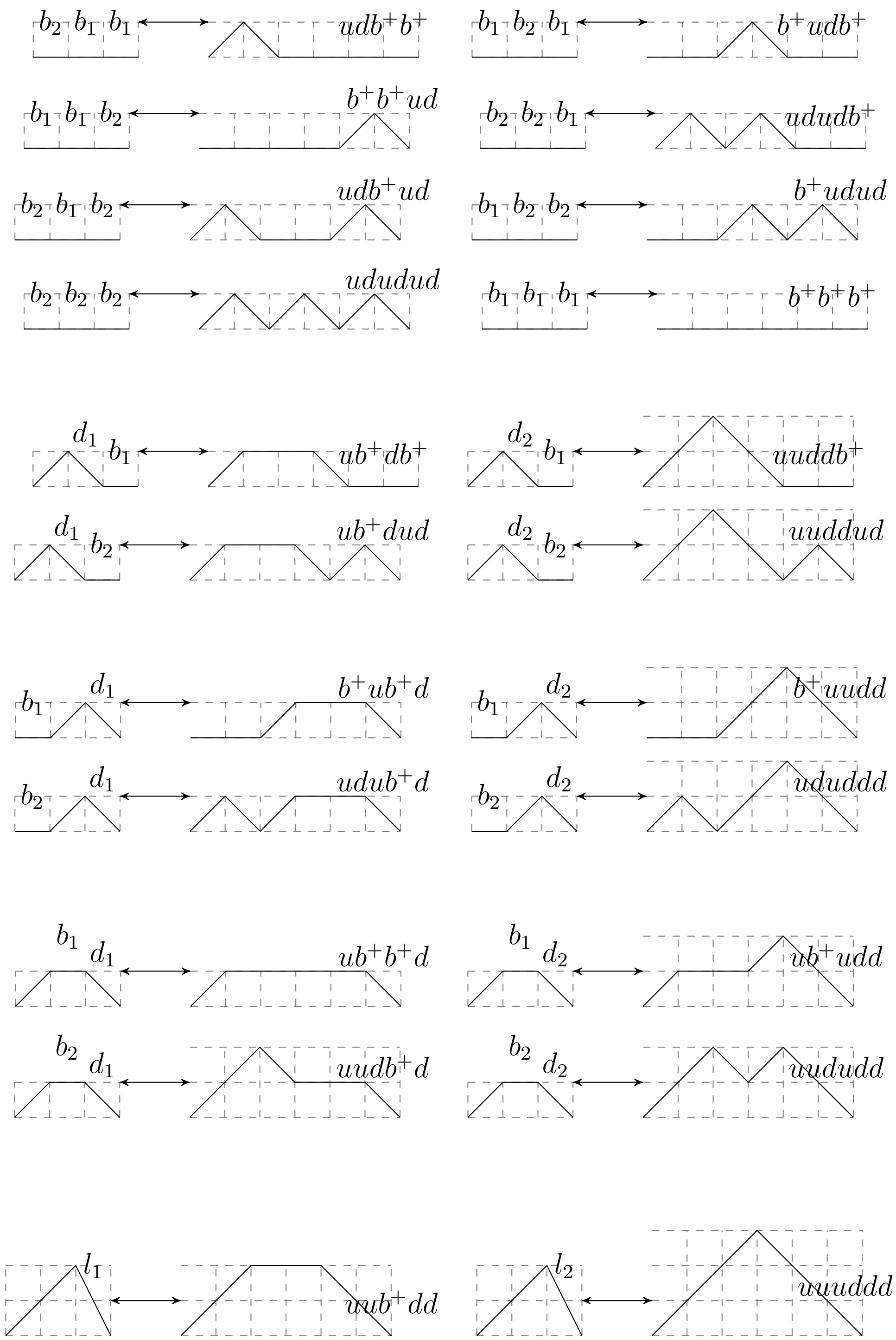
Denote $\mathcal{L}_n$ the set of $(2,2)$-Łukasiewicz paths of length n and $\mathcal{S}_{2n}$ the set of Schröder paths of length $2n$. Now, we construct a map $\phi : \mathcal{L}_n \to \mathcal{S}_{2n}$. Given a $(2,2)$-Łukasiewicz path P of length n we can obtain a lattice path $\phi(P)$ of length $2n$ by the following procedure,

1. u remains unchanged
2. Replace b_1 with b^+, and b_2 with a ud step.
3. Replace d_1 with b^+d, and d_2 with a udd step.
4. Replace l_1^n with b^+d^n, and l_2^n with ud^{n+1}

Conversely, we can obtain the $(2,2)$-Łukasiewicz paths of length n from the Schröder paths of length $2n$ by the following procedure,

1. u remains unchanged
2. Replace b^+ with b_1 and ud with a b_2 step.
3. Replace b^+d with d_1, and udd with a d_2 step.
4. Replace b^+d^n with l_1^n, and ud^{n+1} with l_2^n.

Let us look at the paths for $n = 4$



$b_2\ b_1\ b_1$
udb^+b^+
$b_1\ b_2\ b_1$
b^+udb^+
$b_1\ b_1\ b_2$
b^+b^+ud
$b_2\ b_2\ b_1$
$udud b^+$
$b_2\ b_1\ b_2$
udb^+ud
$b_1\ b_2\ b_2$
b^+udud
$b_2\ b_2\ b_2$
$ududud$
$b_1\ b_1\ b_1$
$b^+b^+b^+$
$d_1\ b_1$
ub^+db^+
$d_2\ b_1$
$uuddb^+$
$d_1\ b_2$
ub^+dud
$d_2\ b_2$
$uuddud$
$b_1\ d_1$
b^+ub^+d
$b_1\ d_2$
b^+uudd
$b_2\ d_1$
$udub^+d$
$b_2\ d_2$
$ududdd$
$b_1\ d_1$
ub^+b^+d
$b_1\ d_2$
ub^+udd
$b_2\ d_1$
$uudb^+d$
$b_2\ d_2$
$uududd$
l_1
uub^+dd
l_2
$uuuddd$

5.4 A bijection between certain Lukasiewicz and Motzkin paths

The g.f. of interest here is that of the inverse binomial transform of the Catalan numbers, also known as the Motzkin sums [124]. The tridiagonal Stieltjes matrix corresponding to the Motzkin sums, which count the Motzkin paths of length n with no horizontal steps at level 0 has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0\ldots \\ 1 & 1 & 1 & 0\ldots \\ 0 & 1 & 1 & 1\ldots \\ 0 & 0 & 1 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

the first column of the related Riordan array having g.f. in continued fraction form of

$$\cfrac{1}{1-\cfrac{x^2}{1-x-\cfrac{x^2}{1-x-\cfrac{x^2}{\cdots}}}} \qquad \text{(A005043)}.$$

The Stieltjes matrix corresponding to the Łukasiewicz steps has first few entries

$$\begin{pmatrix} 0 & 1 & 0 & 0\ldots \\ 1 & 0 & 1 & 0\ldots \\ 1 & 1 & 0 & 1\ldots \\ 1 & 1 & 1 & 0\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with the first column of the related Riordan array satisfying the equation

$$(g(x))^2(x+x^2) - g(x)(1+x) + 1 = 0.$$

Solving the equation above gives

$$g(x) = \frac{\sqrt{3x-1}\sqrt{-x-1}+x+1}{2x(x+1)},$$

which again counts the sequence of Motzkin paths (A005043).

Here we give a constructive proof of a bijection between the (1,0)-Łukasiewicz paths and the Motzkin paths, without the possibility of a level steps on the x axis. Again, we recall that u represents N-E step $(1, 1)$, d the S-E step $(1, -1)$. E steps are $b = (1, 0)$ and Łukasiewicz steps are $l^n = (1, -n)$,

Denote $\mathcal{M}_n$ the set of Motzkin paths of length n with no level steps on the x-axis and $\mathcal{L}_n$ the set of $(1, 0)$-Łukasiewicz paths of length n. Now, we construct a map $\phi : \mathcal{M}_n \to \mathcal{L}_n$. Given a $(1, 1)$-Motzkin path P of length n with no level steps on the x-axis, we can obtain a lattice path $\phi(P)$ of length n by the following procedure,

1. We move along the path until we find the first S-E(d) step, we then move to the step before the S-E step.
   - If this is a ud step move onto the next S-E step
   - if this is a bd step, it now becomes ul ($\ldots ubbbbudbubbduddd \cdots \to \ldots ubbbbudbubuluddd \ldots$).
2. Now, we move to the next step left of the ul,
   - if this is ud step we stop and move onto the next S-E step($\ldots ubbbbudbuduluddd \cdots \to \ldots ubbbbudbuuduluddd \ldots$).
   - If this is a b step bul becomes uul^2($\ldots ubbbbudbubuluddd \cdots \to \ldots ubbbbudbuuul^2uddd \ldots$).
   - If the step is a u, we now have uul giving N-E and a Łukasiewicz step in succession of the same length. Since these are the same length we now proceed to the next S-E step. ($\ldots ubbbbudbuuuludddd \cdots \to \ldots ubbbbudbuuuludddd \ldots$).
3. Repeat 1 and 2 until their are no remaining E steps.

Conversely, we can obtain the $(1, 1)$-Motzkin paths of length n with no level steps on the x-axis from the $(1, 0)$-Łukasiewicz paths of length n by the following procedure,

1. Start at the right most l or d step. If the next step to the left of this is u, stop and move on to the next l or d step.

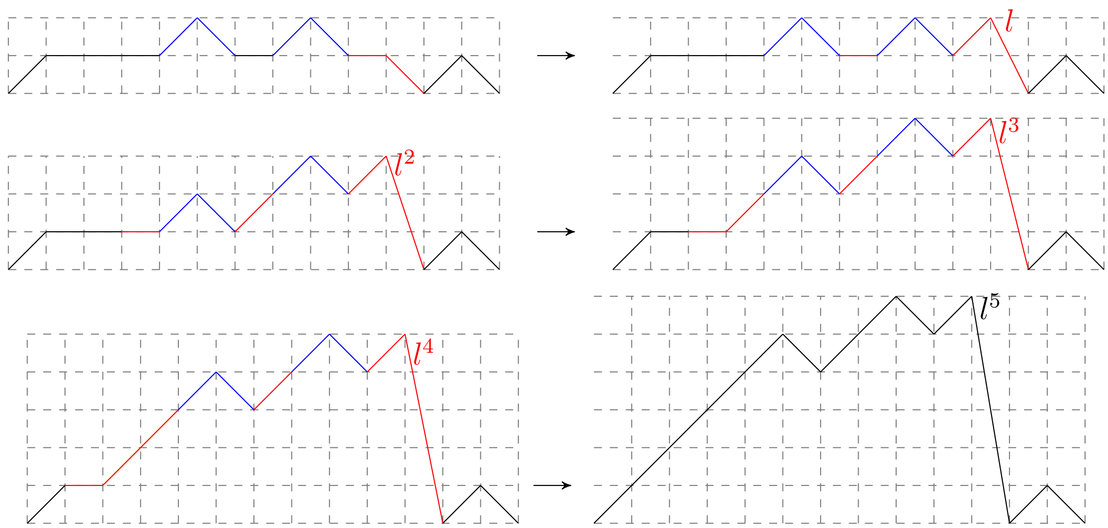
2. If the next step is a d step, starting with the right most u step, we find the corresponding u step at the level of the d step. We now leave both these unchanged and move to the next left d or l step.

3. If the next step is a step l^n, starting with the u step at the corresponding level, we count $n+1$ corresponding u steps, ignoring any ud steps between the u steps. The $u^{n+1}l^n$ step now becomes ub^nd.

4. We move onto the next d or l step and repeat. Repeat until all l steps have been removed.

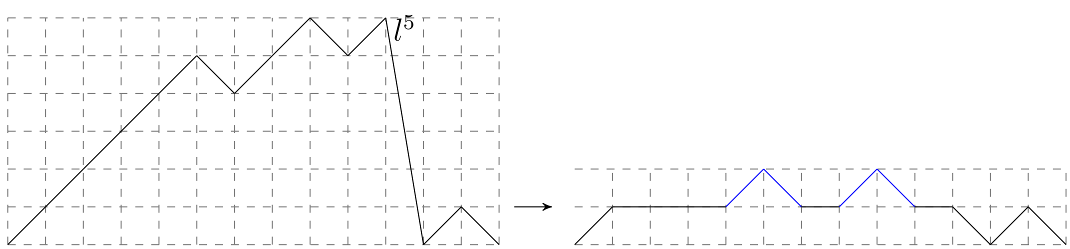
Let us illustrate the moves in two paths below

$$\dots ubbbudbudbdud \cdots \to \dots ubbbudbudulud \cdots \to \dots ubbbuduudul^2ud \cdots \to$$
$$\dots ubbuuduudul^3ud \cdots \to \dots ubuuduudul^4ud \cdots \to \dots uuuuduudul^5ud \dots$$

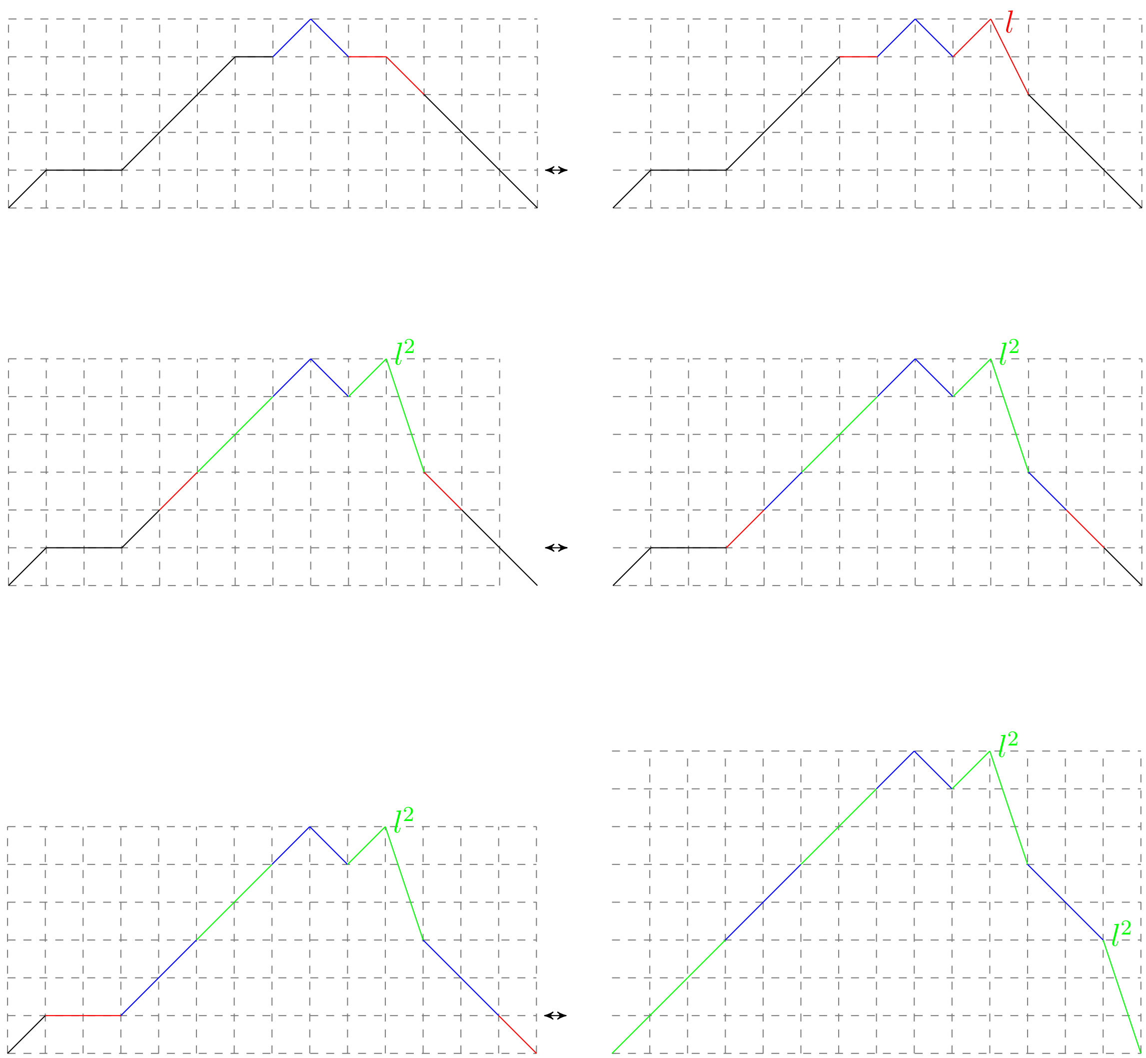


Conversely we have

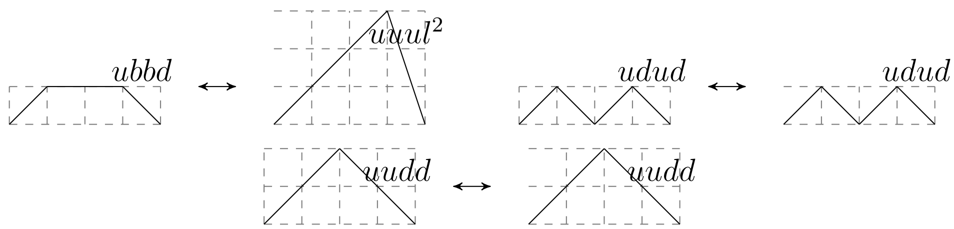
$$uuuuduudul^5ud \cdots \to \dots ubbbudbudbdud \dots$$



$$\ldots ubbuuubudbdddd \cdots \leftrightarrow \ldots ubbuuubudulddd \cdots \leftrightarrow \ldots ubbuuuuudul^2ddd \cdots \leftrightarrow \ldots ubbuuuuudul^2ddd \cdots \leftrightarrow \ldots ubuuuuuudul^2ddl \cdots \leftrightarrow \ldots uuuuuuuudul^2ddl^2 \ldots$$

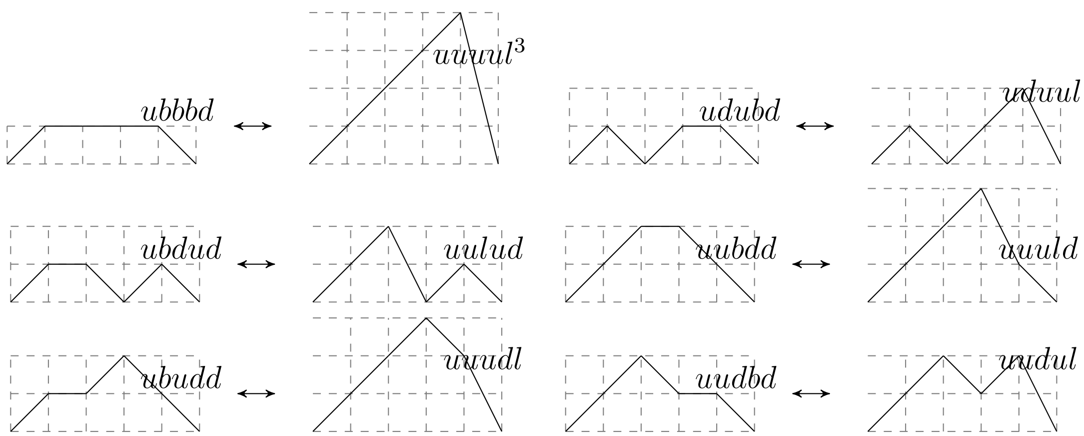


Example. *We look at the paths for* $n = 3$



We look at the paths for $n = 4$

Example.



5.5 Lattice paths and exponential generating functions

In this section we study certain paths that are enumerated by e.g.f.'s. Firstly, we introduce the following proposition

Proposition 5.5.1. *Łukasiewicz paths with level and north east steps of all the same weight which are enumerated by e.g.f.'s satisfy the ordinary differential equation*

$$\frac{d}{dx}f(x) = f(x)(\alpha e^x + \beta - \alpha). \tag{5.8}$$

Proof. We begin by looking at exponential Riordan arrays of the form $\mathbf{L} = [f(x), x]$. Expanding the first column of the matrix equation $\overline{\mathbf{L}} = \mathbf{LS}$ where

$$\mathbf{L} = \begin{pmatrix} \beta_{0,0} & 1 & 0 & 0 & 0 & 0 & \dots \\ \alpha_{1,0} & \beta_{1,1} & 1 & 0 & 0 & 0 & \dots \\ \alpha_{2,0} & \alpha_{2,1} & \beta_{2,2} & 1 & 0 & 0 & \dots \\ \alpha_{3,0} & \alpha_{3,1} & \alpha_{3,2} & \beta_{3,3} & 1 & 0 & \dots \\ \alpha_{4,0} & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \beta_{4,4} & 1 & \dots \\ \alpha_{5,0} & \alpha_{5,1} & \alpha_{5,2} & \alpha_{5,3} & \alpha_{5,4} & \beta_{5,5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.9}$$

we have

$$\begin{aligned} \frac{d}{dx}f(x) &= \beta_{0,0}f(x) + \alpha_{1,0}f(x)x + \alpha_{2,0}f(x)\frac{x^2}{2!} + \alpha_{3,0}f(x)\frac{x^3}{3!} + \dots \\ &= f(x)(\beta_{0,0} + \alpha_{1,0}x + \alpha_{2,0}\frac{x^2}{2!} + \alpha_{3,0}\frac{x^3}{3!} + \dots). \end{aligned}$$

Now let $\alpha(x) = \beta_{0,0} + \alpha_{1,0}x + \alpha_{2,0}\frac{x^2}{2!} + \alpha_{3,0}\frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \alpha_{n,0}\frac{x^n}{n!}, \quad \alpha_{0,0} = \beta_{0,0}$

$$\frac{d}{dx}f(x) = f(x)\alpha(x).$$

If $\alpha = \alpha_{1,0} = \alpha_{2,0} \dots, \quad \beta_{0,0} = \beta$, then

$$\begin{aligned} \frac{d}{dx}f(x) &= f(x)\beta + f(x)(\alpha x + \alpha\frac{x^2}{2!} + \alpha\frac{x^3}{3!} + \dots) \\ &= f(x)\beta + \alpha f(x)(e^x - 1) \\ &= f(x)(\alpha e^x + \beta - \alpha). \end{aligned}$$

□

Example. *Łukasiewicz paths enumerated by e.g.f.'s with no level step satisfy the ODE*

$$f(x)(-\alpha+\alpha e^x)=\frac{d}{dx}f(x).$$

Solving for $f(x)$ we get the e.g.f. of the sequence with first few terms which expand as $1,\alpha,\alpha,(3\alpha^2+\alpha),(10\alpha^2+\alpha),(15\alpha^3+25\alpha^2+\alpha),\ldots$ (A000296).

Corollary 5.5.2. *Łukasiewicz paths with $\alpha,\beta=1$ are enumerated by the e.g.f. of the Bell numbers*

Proof. Solving the differential equation

$$\frac{d}{dx}f(x)=f(x)(\alpha e^x+\beta-\alpha),$$

results in e.g.f.'s of the form

$$e^{\alpha e^x-\alpha+\beta x-\alpha x}.$$

For $\alpha,\beta=1$ we have

$$f(x)=e^{e^x-1}$$

which is the e.g.f. of the Bell numbers. □

Now, we expand $e^{\alpha e^x-\alpha+\beta x-\alpha x}$ to study the Łukasiewicz steps which are enumerated by the Bell numbers, the first few terms in the series expand as

$$1+\beta x+(\alpha+\beta^2)\frac{x^2}{2!}+(3\alpha\beta+\alpha+\beta^3)\frac{x^3}{3!}+(3\alpha^2+6\alpha\beta^2+4\alpha\beta+\alpha+\beta^4)\frac{x^4}{4!}+$$

$$(15\alpha^2\beta+10\alpha^2+10\alpha\beta^3+10\alpha\beta^2+5\alpha\beta+\alpha+\beta^5)\frac{x^5}{5!}+$$

$$(15\alpha^3+45\alpha^2\beta^2+60\alpha^2\beta+25\alpha^2+15\alpha\beta^4+20\alpha\beta^3+15\alpha\beta^2+6\alpha\beta+\alpha+\beta^6)\frac{x^6}{6!}+\ldots$$

The corresponding Stieltjes matrix related to the Łukasiewicz path expansion of the Bell numbers begins

$$\left(\begin{array}{cccccc} \beta & 1 & 0 & 0 & 0 & \dots \\ \alpha & \beta & 1 & 0 & 0 & \dots \\ \alpha & 2\alpha & \beta & 1 & 0 & \dots \\ \alpha & 3\alpha & 3\alpha & \beta & 1 & \dots \\ \alpha & 4\alpha & 6\alpha & 4\alpha & \beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The related Riordan array has the form

$$[e^{\alpha(e^{\beta x}-1)+x(\beta-\alpha)}, x].$$

We note that

$$n![x^n]e^{\alpha(e^{\beta x}-1)+x(\beta-\alpha)} = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{i=0}^{k} \frac{1}{i!} \sum_{j=0}^{i} \frac{(-1)^j \binom{i}{j} (i-j)^k \alpha^i (\beta-\alpha)^{n-k}}{(n-k)!}.$$

We see from the above that we now have a Stieljtes matrix of Łukasiewicz steps that also enumerates the Bell numbers. We note that we have previously encountered the Bell numbers in the form of the continued fraction expansion [56]

$$\cfrac{1}{1-\beta x-\cfrac{\alpha x^2}{1-2\beta x-\cfrac{2\alpha x^2}{1-3\beta x-\cfrac{3\alpha x^2}{1-4\beta x-\cfrac{4\alpha x^2}{\dots}}}}}.$$

The first few terms of this power series expands as

$$1+\beta x+(\alpha+\beta^2)x^2+(4\alpha\beta+\beta^3)x^3+(3\alpha^2+11\alpha\beta^2+\beta^4)x^4+$$
$$(25\alpha^2\beta+26\alpha\beta^3+\beta^5)x^5+(15\alpha^3+130\alpha^2\beta^2+57\alpha\beta^4+\beta^6)x^6+$$
$$(210\alpha^3\beta+546\alpha^2\beta^3+120\alpha\beta^5+\beta^7)x^7\dots$$

The related tridiagonal Stieltjes matrix has first few rows

$$\begin{pmatrix} \beta & 1 & 0 & 0 & 0 & \dots \\ \alpha & 2\beta & 1 & 0 & 0 & \dots \\ 0 & 2\alpha & 3\beta & 1 & 0 & \dots \\ 0 & 0 & 3\alpha & 4\beta & 1 & \dots \\ 0 & 0 & 0 & 4\alpha & 5\beta & \dots \\ 0 & 0 & 0 & 0 & 5\alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

and related Riordan array has the form

$$\left[e^{\frac{\alpha}{\beta^2}(e^{\beta x}-1)+x\left(\frac{\beta^2-\alpha}{\beta}\right)}, \frac{e^{\beta x}-1}{\beta}\right].$$

We note that

$$n![x^n]e^{\frac{\alpha}{\beta^2}(e^{\beta x}-1)+x\left(\frac{\beta^2-\alpha}{\beta}\right)} = \sum_{k=0}^{n}\frac{n!}{k!}\sum_{i=0}^{k}\frac{1}{i!}\sum_{j=0}^{i}\frac{(-1)^j\binom{i}{j}(\beta(i-j))^k(\frac{\alpha}{\beta^2})^i\left(\beta-\frac{\alpha}{\beta}\right)^{(n-k)}}{(n-k)!}.$$

From the above we see that a bijection between the Łukasiewicz and Motzkin paths that enumerate the Bell numbers exists.

5.6 Lattice paths and reciprocal sequences

In this section we will look at lattice paths that relate to certain set partitions. Before we proceed we take this opportunity to define set partitions and the subsets of set partitions that will be of interest to us in this chapter. In [56], Flajolet shows a bijection between weighted lattice paths and set partitions both counted by the Bell numbers. We use this bijection to study the subsets of partitions which relate to certain paths.

Definition 5.6.1. *[81] Let $\mathbb{X}$ be a set. A* set partition *of $\mathbb{X}$ is a collection $\mathbb{P}$ on nonempty, pairwise disjoint subsets of $\mathbb{X}$ whose union is $\mathbb{X}$. Each element of $\mathbb{P}$ is called a block of the partition. The cardinality of $\mathbb{P}$ (which may be infinite) is called the number of blocks of the partition.*

For example if $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then $P = \{1, 6\}\{3, 4\}\{2, 5, 7, 8\}$.

Definition 5.6.2. *[81] Let $S(n, k)$ be the number of set partitions of $\{1, 2, \ldots, n\}$ into exactly k blocks. $S(n, k)$ is called a* Stirling number of the second kind*. Let $B(n)$ be the total number of set partitions of $\{1, 2, \ldots, n\}$. The sequence $B(n)$ is called the sequence of* Bell numbers.

Before we proceed we note that some generating functions which we will encounter in the chapter below, and that we have previously encountered in Chapter 4, count certain subsets of set partitions, so we take this opportunity to define the partitions that will be of interest to us here.

Definition 5.6.3. *[80] A partition of $[n] = \{1, 2, \ldots, n\}$ is called* connected *if no proper subinterval of $[n]$ is a union of blocks (sometimes referred to as irreducible diagrams).*

Definition 5.6.4. *[80]* irreducible partition *are partitions which cannot be "factored" into sub partitions, i.e. partitions of $[n]$ for which 1 and n are in the same connected component.*

connected irreducible

Figure 5.1: A representation of partitions [80]

In the section below we study paths related to g.f.'s of the form

$$z(x) = \frac{1}{x}\big(1 - \frac{1}{g(x)}\big). \tag{5.10}$$

As this relates closely to the reciprocal sequence of the g.f. $g(x)$, we will refer to the sequence $z(x)$ as the *adjusted reciprocal sequence* throughout this chapter. Let us recall the definition of reciprocal function from [162] which we will use to calculate the *adjusted reciprocal sequence.*

Definition 5.6.5. *A reciprocal series $g(x) = \sum_{n=0} a_n x^n$ with $a_0 = 1$, of a series $f(x) = \sum_{n=0} b_n x^n$ with $b_0 = 1$, is a power series where $g(x)f(x) = 1$, which can be calculated as follows*

$$\sum_{n=0}^{\infty} a_n x^n = -\sum_{n=0}^{\infty}\sum_{i=1}^{n} b_i a_{n-i} x^n, \qquad a_0 = 1 \tag{5.11}$$

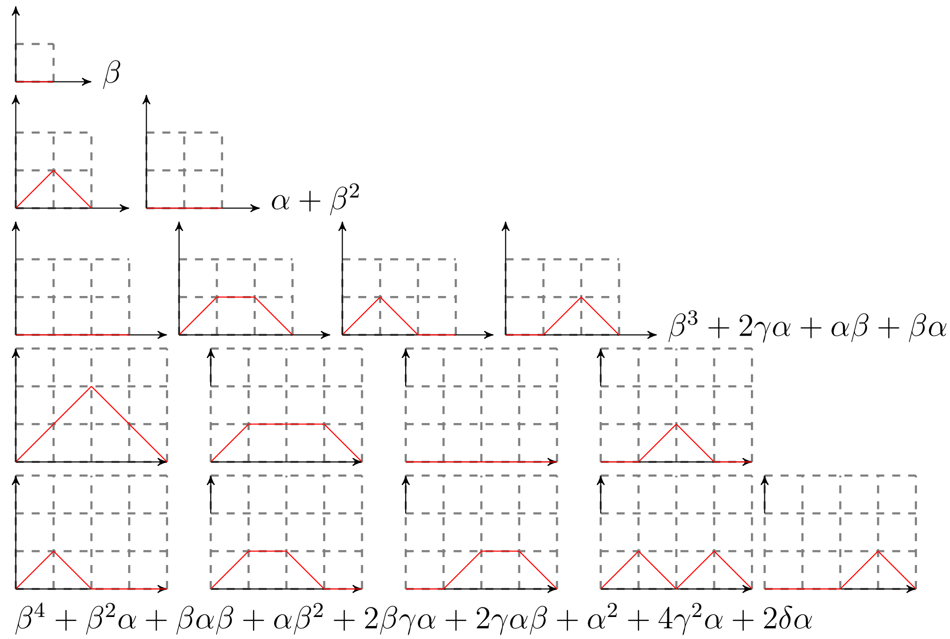
Using the definition above, we can calculate the paths corresponding to the adjusted reciprocal sequence. As we are interested in studying subsets of partitions relating to paths, we will begin by studying the Bell numbers which count all possible set partitions. We start by taking the continued fraction expansion of the " weighted" Bell numbers where we can distinguish the weights for the steps at each level

$$\cfrac{1}{1-\beta x-\cfrac{\alpha x^2}{1-2\gamma x-\cfrac{2\delta x^2}{1-3\omega x-\cfrac{3\epsilon x^2}{\cdots}}}}$$

Now, expanding the first few members of the sequence relating to the g.f. above, we have

$$1, \beta, \beta^2+\alpha, \beta^3+\beta\alpha+\alpha\beta+2\gamma\alpha, \beta^4+\beta^2\alpha+\beta\alpha\beta+\alpha\beta^2+2\beta\gamma\alpha+2\gamma\alpha\beta+\alpha^2+4\gamma^2\alpha+2\delta\alpha.$$

Let us look at the corresponding paths



We note that the reciprocal sequence is calculated recursively, so we now proceed by using the paths above to calculate the paths of the adjusted reciprocal sequence recursively. Let us expand the first few paths of this recursive procedure. Firstly expanding (5.11) for the first few elements we have

- $a_1 x = -(b_1 a_0)x$,
  - a_1 consists of the paths of b_1.
- $a_2 x^2 = -(b_1 a_1 + b_2 a_0)x^2$,
  - a_2 consists of the paths of b_2 which can not be formed from the union of paths $b_1 a_1 = b_1^2$.
- $a_3 x^3 = -(b_1 a_2 + b_2 a_1 + b_3 a_0)x^3$,
  - a_3 consists of the paths of b_3 which can not be formed from the union of paths of $b_1 a_2 = b_1(b_2 - b_1^2)$ and $b_2 a_1 = b_2 b_1$.
- $a_4 x^4 = -(b_1 a_3 + b_2 a_2 + b_3 a_1 + b_4 a_0)x^4$,
  - a_4 consists of the paths of b_4 which can not be formed from the union of paths of $b_1 a_3 = b_1(b_3 - b_1(b_2 - b_1^2))$, $b_2 a_2 = b_2(b_2 - b_1^2)$,$b_3 b_1$.

and so on until

- a_n consists of the paths of b_n which can not be formed from the union of paths of b_1a_{n-1}, b_2a_{n-2} and so on until $b_{n-1}a_1$. That is, a_n consists of the paths in b_n that can not be constructed by some union of paths from $b_0 \to b_n$.

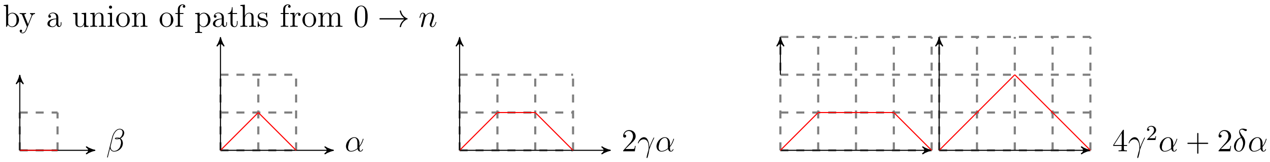
Now replacing the weighted lattice path steps in the expansion above we have first few terms

$$\begin{aligned}
a_1 &= -\big\{b_1a_0\big\} = -\beta \\
a_2 &= -\big\{b_1a_1 + b_2a_0\big\} = -\big\{-\beta\beta + (\beta^2+\alpha)\big\} = -\alpha \\
a_3 &= -\big\{b_1a_2 + b_2a_1 + b_3a_0\big\} = -\big\{-\beta\alpha - \beta^3 - \alpha\beta + \beta^3 + \beta\alpha + \alpha\beta + 2\gamma\alpha\big\} = -\big\{2\gamma\alpha\big\} \\
a_4 &= -\big\{b_1a_3 + b_2a_2 + b_3a_1 + b_4a_0\big\} \\
&= -\Bigg\{-\Big((\beta(\beta^3+2\gamma\alpha)) + (\beta^2+\alpha)\alpha + (\beta^3+\beta\alpha+\alpha\beta+2\gamma\alpha)\beta\Big) \\
&\quad +\Big(\beta^4 + \beta^2\alpha + \beta\alpha\beta + \alpha\beta^2 + 2\beta\gamma\alpha + 2\gamma\alpha\beta + \alpha^2 + 4\gamma^2\alpha + 2\delta\alpha\Big)\Bigg\} \\
&= -\{4\gamma^2\alpha + 2\delta\alpha\}
\end{aligned}$$

so we can describe the first few terms of the adjusted reciprocal sequence in terms of weights of lattice paths as

$$\beta, \alpha, 2\gamma\alpha, 4\gamma^2\alpha + 2\delta\alpha \ldots$$

with corresponding paths below, which are the paths for each n which cannot be formed by a union of paths from $0 \to n$



Now, similarly, for partitions corresponding to these paths, since no union of paths creates these paths, no subsets of these partitions exists. That is, we are counting the number of set partitions of n which do not have a proper subset of parts with a union equal to a subset $1, 2, ..., j$ with $j < n$. Now, let us look at some sequences of interest.

Example. *Let $g(x)$ be the g.f. for the Bell numbers, so we have the adjusted reciprocal sequence*

$$z(x) = \frac{1}{x}\left(1 - \frac{1}{g(x)}\right),$$

with first few elements $1, 1, 2, 6, 22, 92, 426, 2146, \ldots$ *(A074664). This sequence counts the number of set partitions of* n *which do not have a proper subset of parts with a union equal to a subset* $1, 2, ..., j$ *with* $j < n$*. Fig. (5.2) shows the first few paths and corresponding partitions*

Example. *We consider the Young Tableaux numbers, which count all the partitions for each* n *which have less than two element in each partition. The g.f. of the Young Tableaux numbers has a continued fraction expansion of the form*

$$\cfrac{1}{1-x-\cfrac{x^2}{1-x-\cfrac{2x^2}{1-x-\cfrac{3x^2}{1-x-4x^2}\cdots}}}.$$

We look at the adjusted reciprocal sequence for the Young Tableaux. The first few coefficients of $z(x)$ *expand as* $1, 1, 1, 3, 7, 23, 71, 255, \ldots$ *(A140456), so this sequence counts the number of set partitions of* n*, considering only sets of partitions of size less than or equal to two, which do not have a proper subset of parts with a union equal to a subset* $1, 2, ..., j$ *with* $j < n$*. Fig. (5.3) shows the first few paths and corresponding partitions.*

Example. *We consider the Bessel numbers, which count the number of non-overlapping partitions. The Bessel numbers have a continued fraction expansion of the form*

$$\cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-3x-\cfrac{x^2}{1-4x-x^2}\cdots}}}.$$

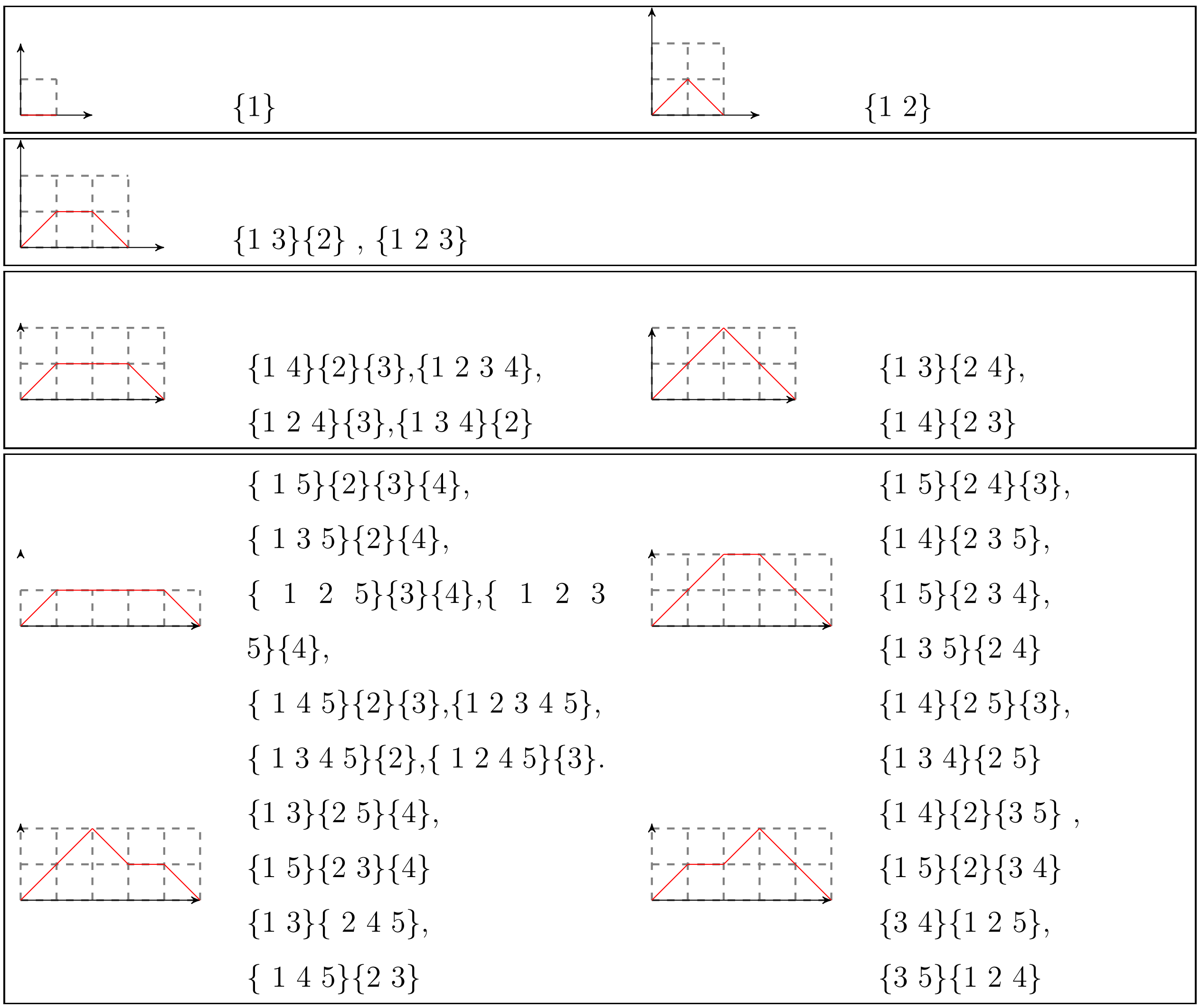


Figure 5.2: Paths and partitions corresponding to the adjusted reciprocal sequence of the Bell numbers

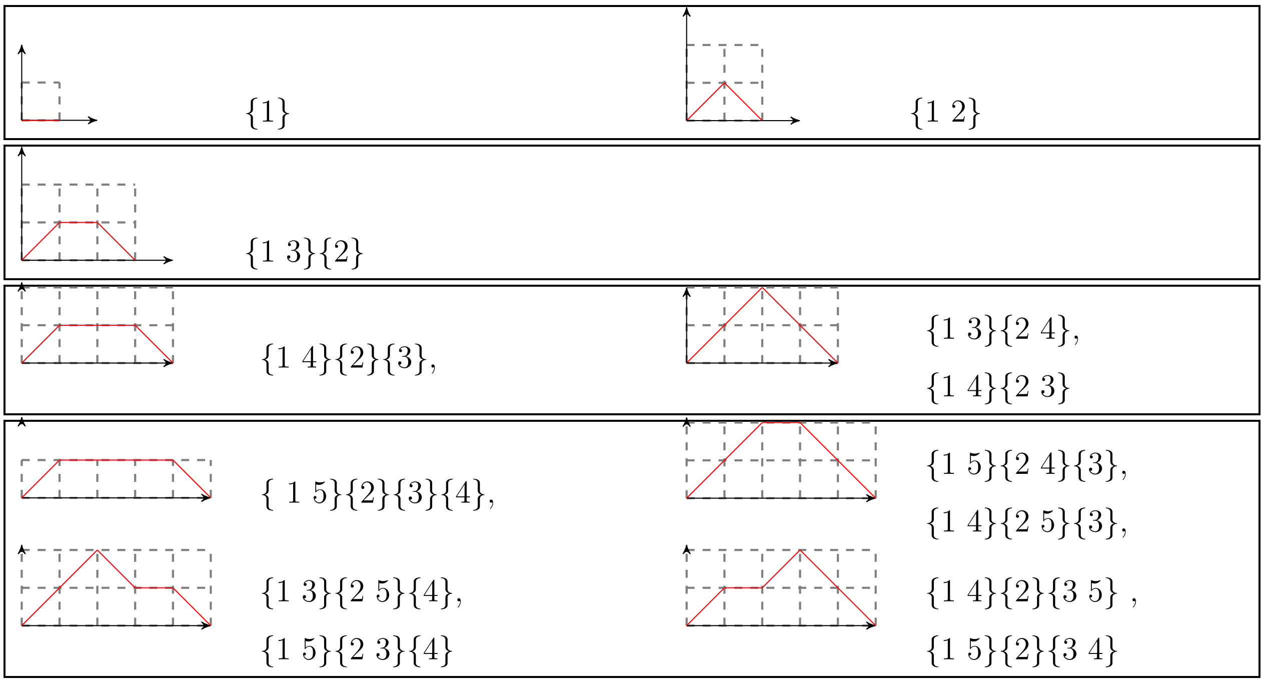


Figure 5.3: Paths and partitions corresponding to the adjusted reciprocal sequence of the Young numbers

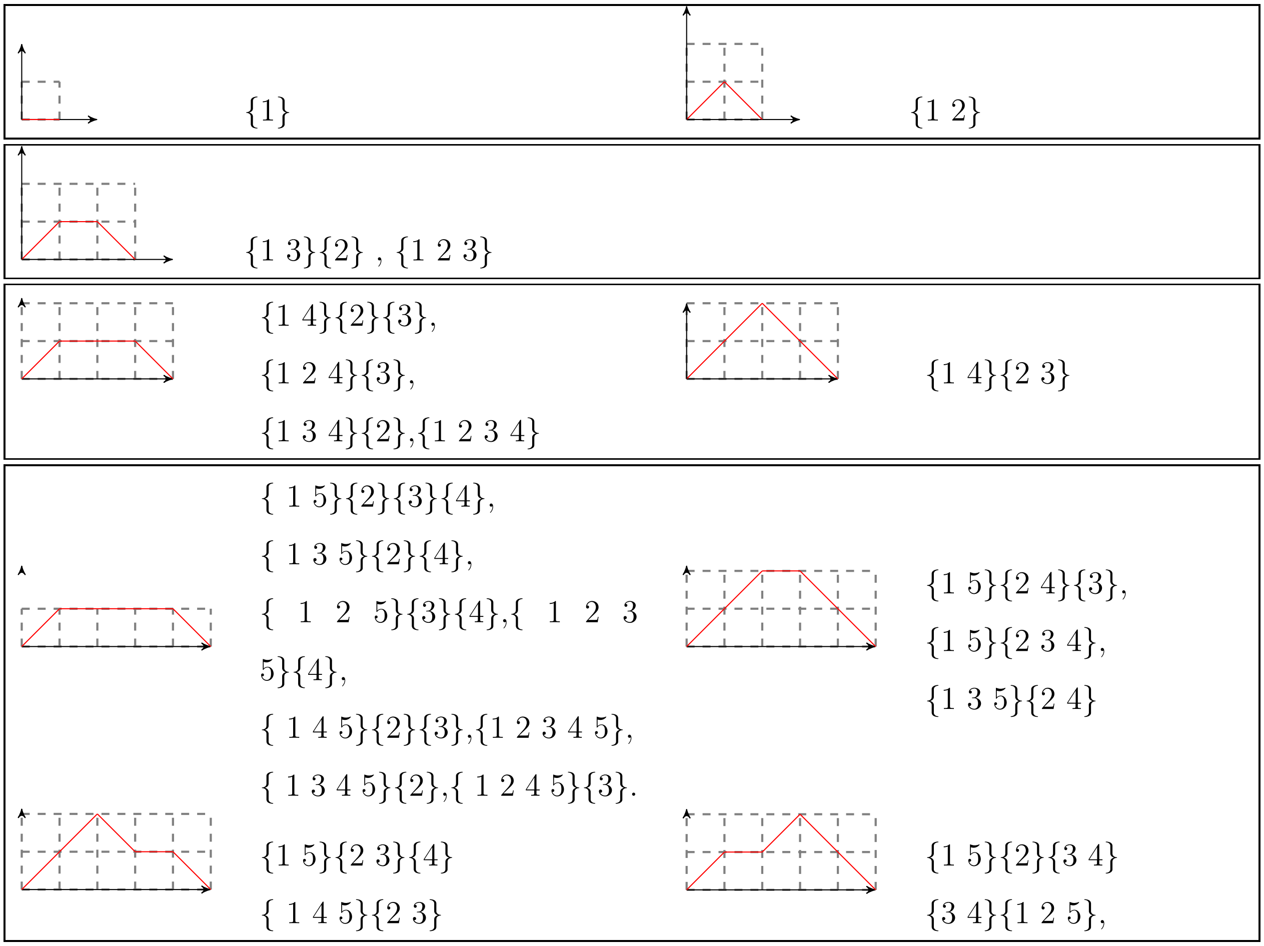


Figure 5.4: Paths and partitions corresponding to the adjusted reciprocal sequence of the Bessel numbers

Then the adjusted reciprocal sequence $z(x)$ for the Bessel numbers has first few coefficients of $z(x)$ given by $1, 1, 2, 5, 15, 51, 189, 748, 3128, 13731 \ldots$ *(A153197), so this sequence counts the number of set partitions of n, only considering sets of non - overlapping partitions, which do not have a proper subset of parts with a union equal to a subset $1, 2, ..., j$ with $j < n$. Fig. (5.4) shows the first few paths and corresponding partitions.*

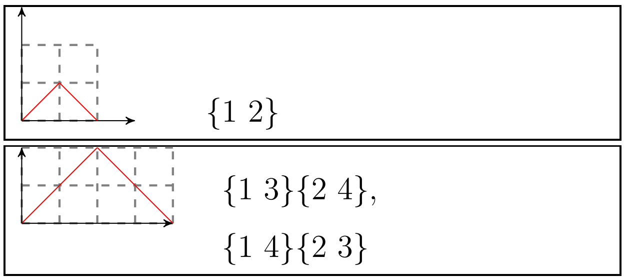


Figure 5.5: Paths and partitions corresponding to the adjusted reciprocal sequence of the double factorial numbers

Example. *Our last example is the double factorial numbers, which count all partitions for each n which have only an even number of elements in each partition. The double factorial numbers have a continued fraction expansion of the form*

$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{2x^2}{1-\cfrac{3x^2}{1-4x^2}\cdots}}}.$$

We look at the adjusted reciprocal sequence for the double factorial numbers. The first few coefficients of $z(x)$ expand as $1, 0, 1, 0, 2, 0, 10, 0, 74, 0, 706 \ldots$ Aerated (A000698), so this sequence counts the number of set partitions of n, considering only sets of even partitions, which do not have a proper subset of parts with a union equal to a subset $1, 2, ..., j$ with $j < n$. Fig. (5.5) shows the first few paths and corresponding partitions.

5.7 Bijections between Motzkin paths and constrained Łukasiewicz paths

In section 5.6 we have looked at the *adjusted reciprocal sequence* relating to Motzkin paths. Let us now look at the *adjusted reciprocal sequence* for the Łukasiewicz paths. From section 5.2 we have the following continued fraction expansion for Łukasiewicz paths with all Łukasiewicz steps of weight α and level steps of weight β,

$$\cfrac{1}{1+x(1-\beta)+\cfrac{x(x(\beta-\alpha)-1)}{1+x(1-\beta)+\cfrac{x(x(\beta-\alpha)-1)}{\ddots}}}$$

Similarly, as for the Motzkin paths shown in section 5.6, the *adjusted reciprocal sequence* counts the Łukasiewicz paths that do not return to the x axis. Looking at one such example of these paths, with $\beta = 0$ and $\alpha = 1$ we have the

$$g(x) = \cfrac{1}{1+x(1)+\cfrac{x(x(-1)-1)}{1+x(1)+\cfrac{x(x(-1)-1)}{\ddots}}}$$

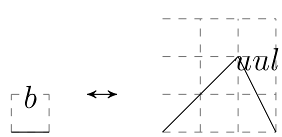
so the *adjusted reciprocal sequence*

$$z(x) = \frac{1}{x}(1 - \frac{1}{g(x)})$$

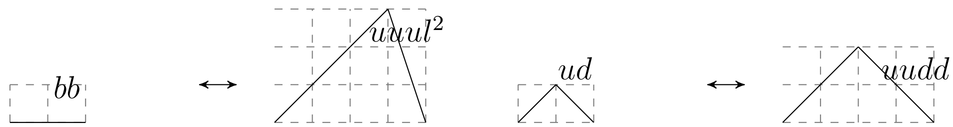
is the sequence of Motzkin numbers (A001006).

From above, we introduce the following bijection between the $(1,1)$-Motzkin paths of length n and the $(1,0)$-Łukasiewicz paths of length $n+2$ which do not return to the x-axis until the final step.

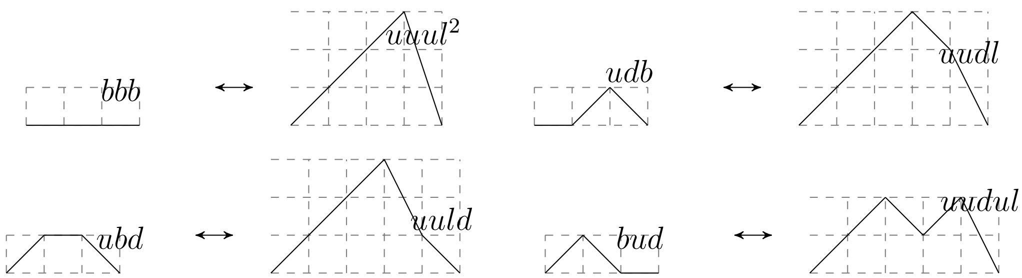
Before we give the bijection, let us look at the paths corresponding to $n = 1, 2, 3$ and 4. For $n = 1$ we have the following path



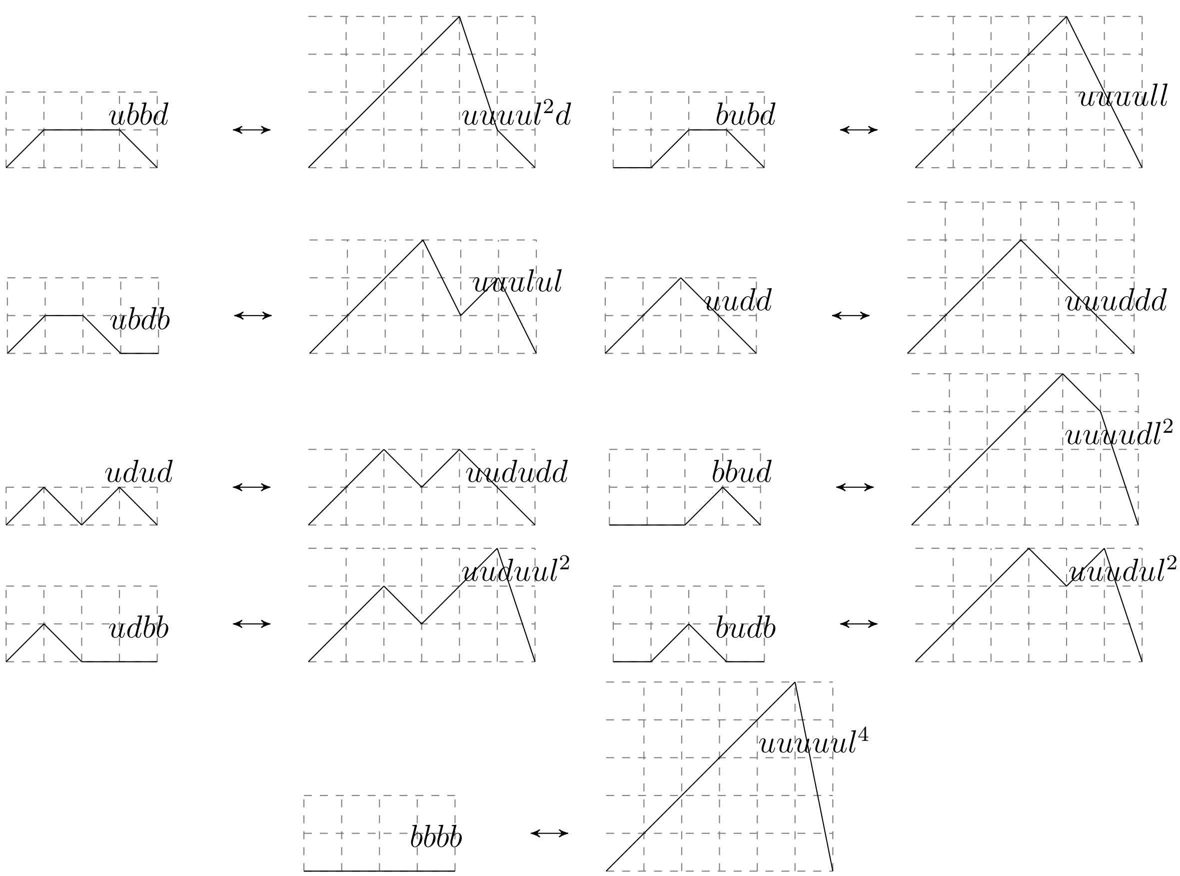
For $n = 2$ we have the following two paths



For $n = 3$ we have the following four paths



For $n = 4$ we have the following nine paths



Denote $\mathcal{M}_n$ the set of $(1,1)$-Motzkin paths of length n and $\mathcal{L}_{n+2}$ the set of $(1,0)$-Łukasiewicz paths of length $n+2$. Now, we construct a map $\phi : \mathcal{M}_n \to \mathcal{L}_{n+2}$. Given a $(1,1)$-Motzkin path P of length n we can obtain a lattice path $\phi(P)$ of length $n+2$ by the following procedure,

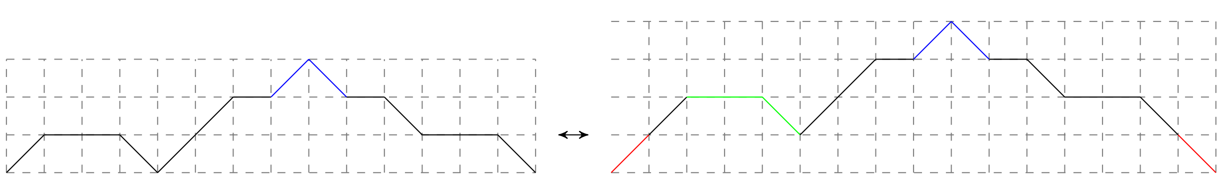
1. Firstly, before the first step of the $(1,1)$-Motzkin path add a N-E step u, and at the end of the $(1,1)$-Motzkin path, add a step S-E step, d. We now have a path of length $n+2$ which does not return to the x-axis.
2. All ud steps remain unchanged
3. For each E step b find the corresponding d step at that level, the E step b becomes a N-E step and the d step becomes l. If there is another E step at the same level, leaving the ud steps between them unchanged, this becomes a u step and the l becomes l^2
4. Repeat the last step until there are no E steps in the path. In general, if we have r E steps at the same level, ignoring all ud steps, the r E steps become u steps and the corresponding Łukasiewicz step becomes l^{r+1}

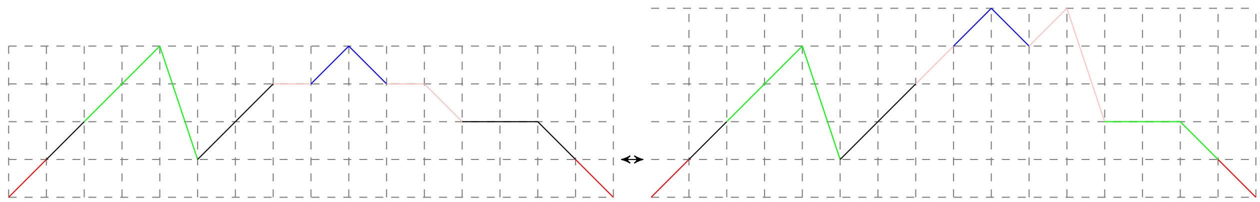
Conversely, we can obtain the $(1,1)$-Motzkin paths of length n from the $(1,0)$-Łukasiewicz paths of length $n+2$ by the following procedure,

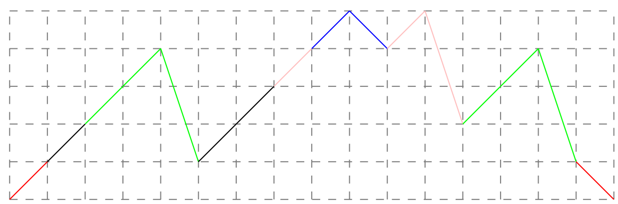
1. For each Łukasiewicz step, l^r of length r, find the corresponding $r+1$ N-E steps, u, ignoring all ud steps. The first u step remains the same, change the following r N-E steps to E steps, b. The l^r step now becomes a N-E step d.
2. Remove the first u and last d step to create the $(1,1)$-Motzkin path of length n.

Example. $\ldots ubbduubudbdbbd \cdots \leftrightarrow \ldots uuuul^2uubudbdbbdd \ldots$

$$\leftrightarrow \ldots uuuul^2uubudbdbbdd \cdots \leftrightarrow \ldots uuuul^2uuuudul^2bbdd \cdots \leftrightarrow \ldots uuuul^2uuuudul^2bbdd \ldots$$

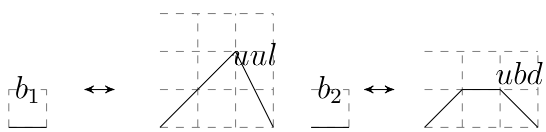




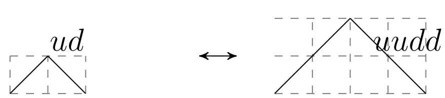


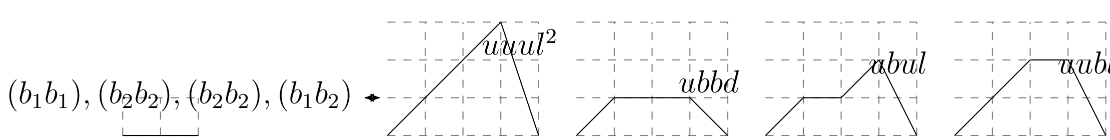
We note that as the binomial transform preserves the bijection, so following from above we can show a similar bijection between the paths of length n counted by the Catalan numbers and the $(1,1)$-Łukasiewicz paths of length $n+2$ which do not return to the x-axis until the final step. The paths counted by the Catalan numbers are $(1,2)$-Motzkin paths. Let us denote the two choices for the level step as b_1 and b_2.

Let us look at the paths corresponding to $n=1$ and 2. For $n=1$ we have the following paths



For $n=2$ we have the following two paths





To show the bijection between the $(1,2)$-Motzkin paths and the $(1,1)$-Łukasiewicz paths, we have the following procedure:

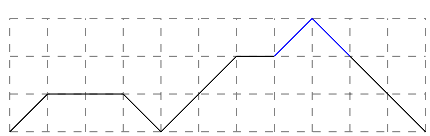
Denote $\mathcal{C}_n$ the set of $(1,2)$-Motzkin paths of length n, where the level steps have two choices namely, b_1 and b_2, and $\mathcal{W}_{n+2}$ the set of Łukasiewicz paths of length $n+2$, which do not return to the x-axis until the last step. Now, we construct a map $\phi : \mathcal{C}_n \to \mathcal{W}_{n+2}$. Given a path P of length n we can obtain a lattice path $\phi(P)$ of length $n+2$ by the following procedure,

1. Firstly, before the first step of the $(1,2)$-Motzkin path add a N-E step u, and at the end of the $(1,2)$-Motzkin path, add a step S-E step, d. We now have a path of length $n+2$ which does not return to the x axis.
2. All ud steps remain unchanged
3. For each E step b_1 find the corresponding d step at that level, the E step b_1 becomes a N-E step and the d step becomes l. If there is another E step at the same level, leaving the ud steps between them unchanged, this becomes a u step and the l becomes l^2. E steps b_2 remain unchanged
4. Repeat the last step until there are no E steps b_1 remaining in the path.

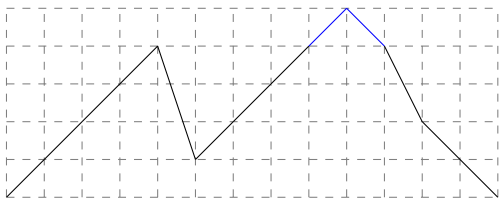
Conversely, we can obtain the $(1,2)$-Motzkin paths of length n from the $(1,1)$-Łukasiewicz paths of length $n+2$ by the following procedure,

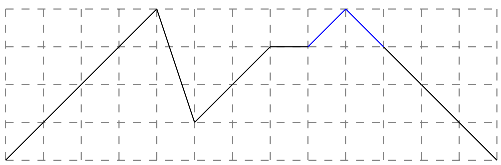
1. For each Łukasiewicz step, l^r of length r, find the corresponding $r+1$ N-E steps u, ignoring all ud steps. The first u step remains the same, change the following r N-E steps to E steps, b. The l^r step now becomes a N-E step d.
2. b_2 steps remain unchanged, as do ud steps.
3. Remove the first u and last d step to create the $(1,2)$-Motzkin paths of length n.

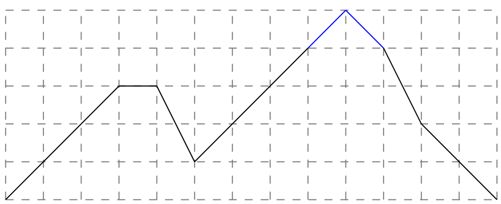
Example.

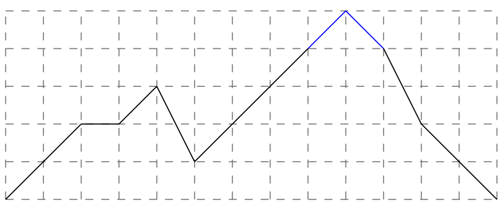


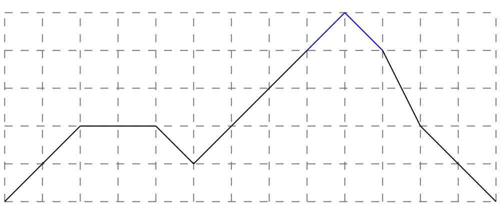
gives the eight paths

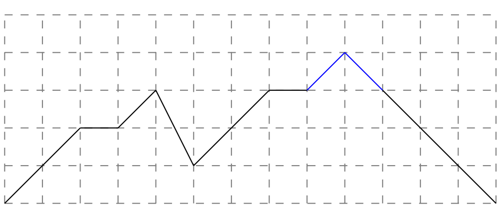

$\dots ub_1b_1duub_1uddd\cdots \leftrightarrow \dots uuuul^2uuuudldd\dots$

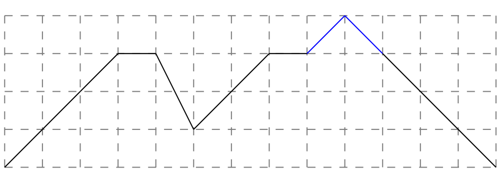

$\dots ub_1b_1duub_2uddd\cdots \leftrightarrow \dots uuuul^2uubudddd\dots$

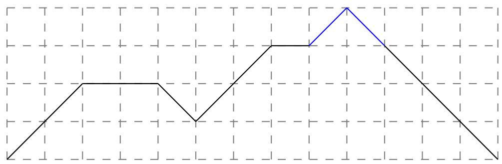

$\dots ub_1b_2duub_1uddd\cdots \leftrightarrow \dots uuubluuuudldd\dots$


$\dots ub_2b_1duub_1uddd\cdots \leftrightarrow \dots uubuluuuudldd\dots$


$\dots ub_2b_2duub_1uddd\cdots \leftrightarrow \dots uubbduuuudldd\dots$


$\dots ub_2b_1duub_2uddd\cdots \leftrightarrow \dots uubuluubudddd\dots$


$\dots ub_1b_2duub_2uddd\cdots \leftrightarrow \dots uuubluubudddd\dots$

 $\ldots ub_2b_2duub_2uddd \cdots \leftrightarrow \ldots uubbduubudddd \ldots$

Chapter 6

Hankel decompositions using Riordan arrays

This chapter is inspired by Peart and Woan [103], who studied Hankel matrices using a Riordan array decomposition, where the Riordan array has an associated tridiagonal Stieltjes matrices. As we saw in the last chapter, these Riordan arrays relate to Motzkin paths. In this chapter, we study these Hankel matrices, decomposing them in an alternative manner, where the related Riordan array in the decomposition has a non-tridiagonal Stieltjes matrix. The Stieltjes matrices are those that relate to Łukasiewicz paths. In the previous chapter we studied Łukasiewicz paths. In eq. (5.5) from proposition 5.2.1 we showed that Łukasiewicz paths with steps weighted

$$\begin{pmatrix} \beta & 1 & 0 & 0 & 0 & 0 & \dots \\ \alpha & \beta & 1 & 0 & 0 & 0 & \dots \\ \alpha & \alpha & \beta & 1 & 0 & 0 & \dots \\ \alpha & \alpha & \alpha & \beta & 1 & 0 & \dots \\ \alpha & \alpha & \alpha & \alpha & \beta & 1 & \dots \\ \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{6.1}$$

have o.g.f.

$$xg(x) = \frac{1 + x(1-\beta) - \sqrt{((\beta+1)x-1)^2 - 4\alpha x^2}}{2(1-\beta x + \alpha x)}.$$

In this chapter we look at Hankel matrix decompositions involving Riordan arrays with generating functions of this form. From studying this alternative decomposition we study the relationship between Motzkin and Łukasiewicz paths through the medium of Riordan arrays.

In the first section we will introduce and develop on results relating to Riordan arrays of g.f.'s of the form in [103]. We then introduce results relating to Hankel matrices decompomposed into Riordan arrays relating to certain Łukasiewicz paths. As the Hankel transform is invariant under the Binomial transform, we concern ourselves with g.f.'s of the Binomial transform of certain functions. Finally, before we leave this chapter we look at a second decomposition of a Hankel matrix. Generating functions relating to Hankel matrices in this section are unrelated to paths, and are simply inspired by earlier work in the chapter.

6.1 Hankel decompositions with associated tridiagonal Stieltjes matrices

Firstly, let us recall some relevant results relating to Hankel matrices of the form studied in [103].

Theorem 6.1.1. *[103, Theorem 1] Let $\mathbf{H} = (a_{nk})_{n,k\geq 0}$ be the Hankel matrix generated by the sequence $1, a_1, a_2, a_3, \ldots$ Assume that $\mathbf{H} = \mathbf{LDU}$ where*

$$\mathbf{L} = (l_{nk})_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ l_{1,0} & 1 & 0 & 0 & \ldots \\ l_{2,0} & l_{2,1} & 1 & 0 & \ldots \\ l_{3,0} & l_{3,1} & l_{3,2} & 1 & \ldots \\ l_{4,0} & l_{4,1} & l_{4,2} & l_{4,3} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} d_0 & 0 & 0 & 0 & \dots \\ 0 & d_1 & 0 & 0 & \dots \\ 0 & 0 & d_2 & 0 & \dots \\ 0 & 0 & 0 & d_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad d_i \neq 0, \quad \mathbf{U} = \mathbf{L}^T.$$

Then the Stieltjes matrix $\mathbf{S_L}$ *is tridiagonal, with the form*

$$\begin{pmatrix} \beta_0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & \beta_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & \beta_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\beta_0 = a_1, \quad \alpha_1 = d_1, \quad \beta_k = l_{k+1,k} - l_{k,k+1}, \quad \alpha_k = \frac{d_{k+1}}{d_k}, \quad k \geq 1.$$

Following from this result, theorems 2.4.4 and 2.4.5 which we introduced in section 2.4 define the Riordan arrays that satisfy theorem 6.1.1. Here, for o.g.f.'s, drawing from the continued fraction expansion of the related g.f.'s we offer the following proof,

Proposition 6.1.2. *If the Stieltjes matrix* $\mathbf{S}$ *has the form*

$$\mathbf{S} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & 0 & \dots \\ \alpha_1 & \beta_1 & 1 & 0 & 0 & \dots \\ 0 & \alpha_2 & \beta_1 & 1 & 0 & \dots \\ 0 & 0 & \alpha_2 & \beta_1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

then the related Riordan array $(g(x), xf(x))$ *has the form*

$$\left(\frac{1}{1-\beta_0 x-\alpha_1 x(xf(x))}, \frac{x}{1-\beta_1 x-\alpha_2 x(xf(x))}\right) \tag{6.2}$$

where

$$xf(x)=\frac{-(\beta_1 x-1)-\sqrt{(\beta_1 x-1)^2-4\alpha_2 x^2}}{2\alpha_2 x}. \tag{6.3}$$

Proof. Let $g(x)=\sum_{n=0}^{\infty} a_n x_n$, with $a_0=1$. A continued fraction expansion relating to the Stieltjes matrix $\mathbf{S}$ above has the form [76]

$$g(x) = \cfrac{1}{1-\beta_0 x-\cfrac{\alpha_1 x^2}{1-\beta_1 x-\cfrac{\alpha_2 x^2}{1-\beta_1 x-\frac{\alpha_2 x^2}{\cdots}}}} = \frac{1}{1-\beta_0 x-\alpha_1 x(xf(x))},$$

where $xf(x)$ is

$$xf(x) = \cfrac{x}{1-\beta_1 x-\cfrac{\alpha_2 x^2}{1-\beta_1 x-\cfrac{\alpha_2 x^2}{1-\beta_1 x-\frac{\alpha_2 x^2}{\cdots}}}} = \frac{1}{1-\beta_1 x-\alpha_2 x(xf(x))}.$$

Solving for $xf(x)$ gives

$$xf(x)=\frac{-(\beta_1 x-1)-\sqrt{(\beta_1 x-1)^2-4\alpha_1 x^2}}{2\alpha_1 x}$$

resulting in eq. (6.3). □

Note that if $\beta_1=\beta_0$ and $\alpha_1=\alpha_2$ we obtain the Riordan array

$$(g(x), xg(x))=\left(\frac{1}{1-\beta_0 x-\alpha_1 x(xg(x))}, \frac{x}{1-\beta_0 x-\alpha_1 x(xg(x))}\right).$$

From [76], the Hankel determinant h_n of order n of $a_n = [x^n]g(x)$ is

$$h_n = a_0^n \alpha_1^{n-1} \alpha_2^{n-2} \dots \alpha_{n-1}^2 \alpha_{n-1}.$$

From the form of the continued fraction expansion of g.f.'s satisfying theorem 6.1.1, the Hankel determinant h_n of order n of $a_n = [x^n]g(x)$, when the Riordan array $\mathbf{L}$ satisfying theorem 6.1.1 has the form $(g(x), xf(x))$, with $a_0 = 1$, is

$$h_n = a_0^n \alpha_1^{n-1} \alpha_2^{n-2} \dots \alpha_2^2 \alpha_{n-1} = \alpha_1^{n-1} \alpha_2^{\sum_{i=i}^{n-2} i} = \alpha_1^{n-1} \alpha_2^{\frac{n^2-3n+2}{2}},$$

and when the Riordan array $\mathbf{L}$ satisfying theorem 6.1.1 has the form $(g(x), xg(x))$, the Hankel determinant is,

$$h_n = a_0^n \alpha_1^{n-1} \alpha_2^{n-2} \dots \alpha_2^2 \alpha_{n-1} = \alpha_1^{\sum_{i=i}^{n-1} i} = \alpha^{\frac{n^2-n}{2}}.$$

In this chapter we concern ourselves with the Binomial transform of certain functions. We take this opportunity to present the following result.

Proposition 6.1.3. *Let* $\mathbf{LDL}^T = \mathbf{H}$*, with* $\mathbf{H}$ *a Hankel matrix,* D *a diagonal matrix and* $\mathbf{L}$ *be a Riordan matrix (g(x),f(x))([(g(x),f(x)]) with related Stieltjes matrix* $\mathbf{S}$*, so that* $\mathbf{L}^{-1}\overline{\mathbf{L}} = \mathbf{S}$*. Then, the Riordan array related to the binomial transform of the Hankel matrix* $\mathbf{H}$ *has Stieltjes matrix* $\mathbf{S} + \mathbf{I}$*.*

Proof. The Binomial transform of $\mathbf{H}$ in matrix form is $\mathbf{QHQ}^T$ where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\dots \\ 1 & 1 & 0 & 0 & 0\dots \\ 1 & 2 & 1 & 0 & 0\dots \\ 1 & 3 & 3 & 1 & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(\frac{1}{1-x}, \frac{1}{1-x}\right) = [e^x, x].$$

Now if $\mathbf{LL}^T = \mathbf{H}$ we have the binomial transform of $\mathbf{H}$

$$\begin{aligned} \mathbf{QHQ}^T &= \mathbf{QLL}^T\mathbf{Q}^T \\ &= \mathbf{QL}(\mathbf{QL})^T \end{aligned}$$

The Stieltjes matrix $\mathbf{S}$ of the Riordan array $\mathbf{L}$ satisfies the equation $\mathbf{L}^{-1}\overline{\mathbf{L}} = \mathbf{S}$. Thus we wish to show that the Riordan array $\mathbf{QL}$ satisfies the equation

$$(\mathbf{QL})^{-1}\overline{\mathbf{QL}} = \mathbf{L}^{-1}\mathbf{Q}^{-1}\overline{\mathbf{QL}} = \mathbf{I} + \mathbf{S}. \tag{6.4}$$

Firstly we show that

$$\mathbf{Q}^{-1}\overline{\mathbf{QL}} = \mathbf{L} + \overline{\mathbf{L}}.$$

For o.g.f.'s we have

$$\left(\frac{1}{1+x}, \frac{x}{1+x}\right)\overline{\left(\frac{1}{1-x}, \frac{x}{1-x}\right)(g(x), f(x))} = \left(\frac{1}{1+x}, \frac{x}{1+x}\right)\overline{\left(\frac{1}{1-x}g(\frac{x}{1-x}), f(\frac{x}{1-x})\right)}.$$

The first column of the "beheaded" Riordan array above is

$$\frac{\frac{1}{1-x}g\left(\frac{x}{1-x}\right) - 1}{x}$$

and the n^{th} column

$$\frac{\frac{1}{1-x}g\left(\frac{x}{1-x}\right)f\left(\frac{x}{1-x}\right)^n}{x}.$$

Now, left multiplication of the matrix $\mathbf{Q}^{-1}$ gives the resulting first column

$$\frac{1}{1+x}\frac{\frac{1}{1-\frac{x}{1+x}}g(x) - 1}{\frac{x}{1+x}} = \frac{g(x)-1}{x} + g(x)$$

and n^{th} column

$$\frac{1}{1+x}\frac{\frac{1}{1-\frac{x}{1+x}}g(x)f(x)^n}{\frac{x}{1+x}} = \frac{g(x)f(x)^n}{x} + g(x)f(x)^n.$$

These columns form the matrix $\mathbf{L} + \overline{\mathbf{L}}$. Similarly, for e.g.f.'s we have

$$[e^{-x}, x]\frac{d}{dx}[e^x g(x), f(x)].$$

Expanding the first column gives

$$[e^{-x}, x]\frac{d}{dx}\left(e^x g(x)\right) = [e^{-x}, x](e^x \frac{d}{dx} g(x) + e^x g(x)) = \frac{d}{dx} g(x) + g(x),$$

and for the n^{th} column we have

$$[e^{-x}, x]\left(e^x \frac{d}{dx}\left(g(x)\frac{f(x)^n}{n!}\right) + e^x g(x)\frac{f(x)^n}{n!}\right) = \frac{d}{dx}\left(g(x)\frac{f(x)^n}{n!}\right) + g(x)\frac{f(x)^n}{n!}.$$

These columns form the matrix $\mathbf{L} + \overline{\mathbf{L}}$.

Now we have shown for both e.g.f.'s and o.g.f.'s that

$$\mathbf{Q}^{-1}\overline{\mathbf{QL}} = \mathbf{L} + \overline{\mathbf{L}}.$$

Pre-multiplying by the matrix $\mathbf{L}^{-1}$ gives eq. (6.4). □

Corollary 6.1.4.

$$\mathbf{L}^{-1}(\mathbf{Q}^n\mathbf{L})^{-1}\overline{\mathbf{Q}^n\mathbf{L}} = n\mathbf{I} + \mathbf{S}.$$

Similarly to above, we can show that

$$(\mathbf{Q}^n\mathbf{L})^{-1}\overline{\mathbf{Q}^n\mathbf{L}} = n\mathbf{L} + \bar{\mathbf{L}},$$

with the binomial Riordan array

$$\mathbf{Q}^n = \left(\frac{1}{1-nx}, \frac{1}{1-nx}\right)$$

for o.g.f.'s and $[e^{nx}, x]$ *for e.g.f.'s.*

The Hankel matrices formed from the series $f(x)$ listed in the table below decompose according to Theorem 6.1.1. That is, $\mathbf{H} = \mathbf{LDL}$. For the power series in the table below, $\mathbf{D}$ is the diagonal matrix and $\mathbf{L}$ is the Riordan array $(f(x), xf(x))$.

Power series	A-number
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}0^{n-2k}x^n$	$\frac{1-\sqrt{1-4x^2}}{2x^2}$ *Aerated* (A126120)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}x^n$	$\frac{(1-x)-\sqrt{(1-x)^2-4x^2}}{2x^2}$ (A001006)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}2^{n-2k}x^n$	$\frac{(1-2x)-\sqrt{(1-2x)^2-4x^2}}{2x^2}$ (A000108)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}3^{n-2k}x^n$	$\frac{(1-3x)-\sqrt{(1-3x)^2-4x^2}}{2x^2}$ (A000108)
$\vdots$	$\vdots$
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}h^{n-2k}x^n$	$\frac{(1-hx)-\sqrt{(1-hx)^2-4x^2}}{2x^2}$

In table above, when $h = 0$ we have the sequence

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}0^{n-2k}x^n,$$

which has a non zero term when $n = 2k$. Thus we have

$$\sum_{n=0}^{\infty}\frac{1}{n/2+1}\binom{n}{n/2}x^n = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{2n},$$

the sequence of aerated Catalan numbers. We now offer the following proof by induction.

Proposition 6.1.5. *The binomial transform of the sequence*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}h^{n-2k}x^n$$

is the sequence

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}(h+1)^{n-2k}x^n$$

Proof. We assume true for n and by induction we prove true for $n+1$, that is we prove the following,

$$\begin{aligned}\sum_{r=0}^{n}\binom{n}{r}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{r}{2k}h^{r-2k} &= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}(h+1)^{n-2k}\\ &= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}\sum_{m=0}^{n-2k}\binom{n-2k}{m}h^m.\end{aligned}$$

Firstly, when $n=0$,

$$\sum_{r=0}^{0}\binom{0}{r}\sum_{k=0}^{\lfloor\frac{0}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{r}{2k}h^{r-2k} = \sum_{k=0}^{\lfloor\frac{0}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{0}{2k}\sum_{m=0}^{0-2k}\binom{0-2k}{m}h^m = 1.$$

By a change of variable

$$\sum_{r=0}^{n}\binom{n}{r}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{r}{2k}h^{r-2k} = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\sum_{m=0}^{n-2k}\binom{n}{2k+m}\binom{2k+m}{2k}h^m.$$

By the cross product of Binomial coefficients we have

$$\begin{aligned}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k+1}\binom{2k}{k}\sum_{m=0}^{n-2k}\binom{n}{2k+m}\binom{2k+m}{2k}h^m &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k+1}\binom{2k}{k}\sum_{m=0}^{n-2k}\binom{n}{2k}\binom{n-2k}{m}h^m \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}\sum_{m=0}^{n-2k}\binom{n-2k}{m}h^m \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}(h+1)^{n-2k}.\end{aligned}$$

□

Riordan arrays formed from the power series $f(x)$ in the table below, once again satisfy the decomposition $\mathbf{LDL}^T = \mathbf{H}$. For the power series in the table below, $\mathbf{D}$ is the diagonal matrix where the n^{th} element of $\mathbf{D}$, $D_{n,n}$ is m^n, for some $m > 0$ and $\mathbf{L}$ is the Riordan array $(f(x), xf(x))$.

Power series	A-number
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}0^{n-2k}2^kx^n$	$\frac{1-\sqrt{1-8x^2}}{4x^2}$ *Aerated* (A052701)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}1^{n-2k}2^kx^n$	$\frac{(1-x)-\sqrt{(1-x)^2-8x^2}}{4x^2}$ (A025235)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}2^{n-2k}2^kx^n$	$\frac{(1-2x)-\sqrt{(1-2x)^2-8x^2}}{4x^2}$ (A071356)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{n-2k}3^{n-2k}2^kx^n$	$\frac{(1-3x)-\sqrt{(1-3x)^2-8x^2}}{4x^2}$ (A001002)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{n-2k}0^{n-2k}3^kx^n$	$\frac{1-\sqrt{1-12x^2}}{6x^2}$ Aerated (A005159)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{n-2k}3^kx^n$	$\frac{(1-x)-\sqrt{(1-x)^2-12x^2}}{6x^2}$ (A025237)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{n-2k}2^{n-2k}3^kx^n$	$\frac{(1-2x)-\sqrt{(1-2x)^2-12x^2}}{6x^2}$ (A122871)
$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{n-2k}3^{n-2k}3^kx^n$ ⋮ $\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{n-2k}h^{n-2k}m^kx^n$	$\frac{(1-3x)-\sqrt{(1-3x)^2-12x^2}}{6x^2}$ (A107264) ⋮ $\frac{(1-hx)-\sqrt{(1-hx)^2-4mx^2}}{2mx^2}$

6.2 Hankel matrices and non-tridiagonal Stieltjes matrices

Now, as shown in the last section, we looked at Hankel matrices which decomposed into $\mathbf{L_1 D L_1}^T$, where the Riordan array $\mathbf{L_1}$ has an associated tridiagonal Stieltjes matrix. Here, we decompose the Hankel matrix $H_f(x)$, where $\mathbf{L} = (f(x), xf(x))$, into

$$\mathbf{LB} \cdot \mathbf{D} \cdot (\mathbf{LB})^T \tag{6.5}$$

where the Riordan array, $\mathbf{L_1}$ from the previous section has been decomposed further into the product of two Riordan arrays, the second array being defined throughout this section as the matrix $\mathbf{B}$, which is the "shifted" Binomial matrix with the first column 0^n. The first few rows have the form

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 1 & 3 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(1, \frac{x}{1-x}\right). \tag{6.6}$$

The generating function of the first column of the Riordan array $\mathbf{L}$, $f(x)$, is the generating function in eq. (5.5) with $\alpha = \beta = \beta_0$. That is, $f(x)$ is the g.f. for Łukasiewicz paths with east(E) and south-easterly(S-E) steps of the same weight, thus the related Stieltjes matrix has the form

$$\begin{pmatrix} \beta_0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta_0 & \beta_0 & 1 & 0 & 0 & 0 & \dots \\ \beta_0 & \beta_0 & \beta_0 & 1 & 0 & 0 & \dots \\ \beta_0 & \beta_0 & \beta_0 & \beta_0 & 1 & 0 & \dots \\ \beta_0 & \beta_0 & \beta_0 & \beta_0 & \beta_0 & 1 & \dots \\ \beta_0 & \beta_0 & \beta_0 & \beta_0 & \beta_0 & \beta_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6.7}$$

Thus $\mathbf{H}_f$ is the Hankel matrix where $a_n = [x^n]f(x)$, to which we associate the g.f. $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where

$$\begin{aligned} f(x) &= \frac{1}{(1-(\beta_0-1)x) - xf(x)} \\ &= \frac{1-(\beta_0-1)x - \sqrt{(1-(\beta_0-1)x)^2 - 4x}}{2x} \end{aligned}$$

and $f(x)$ satisfies the equation

$$xf(x)^2 - f(x)(1-(\beta_0-1)x)+1=0.$$

$\mathbf{L}$ is the Riordan array $(f(x), xf(x))$ and $\mathbf{D}$ is the diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \beta_0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \beta_0^2 & 0 & 0 & \dots \\ 0 & 0 & 0 & \beta_0^3 & 0 & \dots \\ 0 & 0 & 0 & 0 & \beta_0^4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proposition 6.2.1.

$$\mathbf{H}_f(x) = \mathbf{LB}\cdot\mathbf{D}\cdot\mathbf{LB}^T. \tag{6.8}$$

where

$$f(x) = \frac{1-(\beta_0-1)x-\sqrt{(1-(\beta_0-1)x)^2-4x}}{2x},$$

with

$$\mathbf{B} = \left(1, \frac{x}{1-x}\right)$$

and $\mathbf{D}$ *is the diagonal matrix with* $D_{n,n} = \beta_0^n$.

Proof. Let the Riordan arrays $\mathbf{L} = (f(x), xf(x))$ and $\mathbf{LB} = (f(x), xf_1(x))$. Thus

$$\begin{aligned} \mathbf{LB} &= \mathbf{L}\left(1, \frac{x}{1-x}\right) \\ &= \left(\frac{1-(\beta_0-1)x-\sqrt{(1-(\beta_0-1)x)^2-4x}}{2x}, \frac{1-(1+\beta_0)x-\sqrt{(1-(\beta_0-1)x)^2-4x}}{2\beta_0 x}\right) \\ &= (1+\beta_0 x f_1(x), xf_1(x)), \end{aligned}$$

with

$$xf_1(x) = \frac{1-(1+\beta_0)x-\sqrt{(1-(\beta_0-1)x)^2-4x}}{2\beta_0 x},$$

which satisfies the equation

$$\beta_0 x(xf_1(x))^2 + (xf_1(x))(x(1+\beta_0)-1) + x = 0.$$

Now, we can rewrite **LB** as

$$\left(\frac{1}{1-\beta_0 x-\beta_0 x(xf_1(x))}, \frac{x}{1-(\beta_0+1)x-\beta_0 x(xf_1(x))}\right).$$

This satisfies Theorem 6.1.1 with associated Stieltjes matrix

$$\mathbf{S_{LB}} = \begin{pmatrix} \beta_0 & 1 & 0 & 0\ldots \\ \beta_0 & \beta_0+1 & 1 & 0\ldots \\ 0 & \beta_0 & \beta_0+1 & 1\ldots \\ 0 & 0 & \beta_0 & \beta_0+1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

thus eq. (6.8) is also satisfied. □

Corollary 6.2.2. *The n^{th} Hankel transform of $a_n = [x^n]f(x)$ where $\mathbf{H}_f$ satisfies eq. (6.8) is $\beta_0^{\frac{n(n+1)}{2}}$.*

Investigating the form of the Hankel matrix formed from the sequence of coefficients of the series reversion of

$$xf(x) = \frac{1-(\beta_0-1)x-\sqrt{(1-(\beta_0-1)x)^2-4x}}{2},$$

we let

$$xg(x) = \text{Rev}(xf(x)) = \frac{x(1-x)}{1+(\beta_0-1)x}$$

and find for the form of the Hankel matrix $\mathbf{H}_g$, that we have the decomposition

$$\mathbf{H}_g = \mathbf{L}^{-1}\mathbf{M}\cdot\mathbf{D}\cdot(\mathbf{L}^{-1}\mathbf{M})^T \tag{6.9}$$

where **M** is the matrix with first column 0^n and second column with g.f.

$$xf(x) = \frac{1-(1+\beta_0)x-\sqrt{(1-(\beta_0-1)x)^2-4x}}{2\beta_0 x}.$$

All other columns are zero.

Now, let us look at some examples of g.f.'s that satisfy Theorem 6.8 above.

Example. *Let* $\mathbf{L} = (f(x), xf(x))$, *where*

$$f(x) = \frac{1}{1 - xf(x)},$$

is the g.f. of the sequence of Catalan numbers (A000108). Two continued fraction forms of the g.f. of the Catalan numbers are

$$f(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}}} = \cfrac{1}{1 - x - \cfrac{x^2}{1 - 2x - \cfrac{x^2}{1 - 2x - \cfrac{x^2}{\ddots}}}}.$$

Now, we have $\mathbf{H}_f = \mathbf{LB} \cdot (\mathbf{LB})^T$, *the first few rows of this matrix equation expand as*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 1 & 1 & 0 & 0 \dots \\ 2 & 2 & 1 & 0 \dots \\ 5 & 5 & 3 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 0 & 1 & 0 & 0 \dots \\ 0 & 1 & 1 & 0 \dots \\ 0 & 1 & 2 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 0 & 1 & 0 & 0 \dots \\ 0 & 1 & 1 & 0 \dots \\ 0 & 1 & 2 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 1 & 1 & 0 & 0 \dots \\ 2 & 2 & 1 & 0 \dots \\ 5 & 5 & 3 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T.$$

The associated Stieltjes matrices of $\mathbf{L}$ *and* $\mathbf{LB}$ *are*

$$\mathbf{S_L} = \begin{pmatrix} 1 & 1 & 0 & 0\ldots \\ 1 & 1 & 1 & 0\ldots \\ 1 & 1 & 1 & 1\ldots \\ 1 & 1 & 1 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which are the Łukasiewicz paths with weight of one for all E and S-E steps, and

$$\mathbf{S_{LB}} = \begin{pmatrix} 1 & 1 & 0 & 0\ldots \\ 1 & 2 & 1 & 0\ldots \\ 0 & 1 & 2 & 1\ldots \\ 0 & 0 & 1 & 2\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which are the Motzkin paths with weight of one for all S-E steps and weight of two for all E steps, except the x-axis E step, which has weight one.

We note that $\mathbf{L}^{-1} = (g(x), xg(x)) = \big(1-x, x(1-x)\big)$ *and* $\mathbf{H}_g = \mathbf{L}^{-1}\mathbf{M}\cdot\mathbf{D}\cdot(\mathbf{L}^{-1}\mathbf{M})^T$ *expands as*

$$\begin{pmatrix} 1 & 0 & 0\ldots \\ -1 & 1 & 0\ldots \\ 0 & -2 & 1\ldots \\ 0 & 1 & -3\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & 1 & 0\ldots \\ 0 & 2 & 0\ldots \\ 0 & 5 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & -1 & 0\ldots \\ 0 & 0 & -1\ldots \\ 0 & 0 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & 1 & 0\ldots \\ 0 & 2 & 0\ldots \\ 0 & 5 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0\ldots \\ -1 & 1 & 0\ldots \\ 0 & -2 & 1\ldots \\ 0 & 1 & -3\ldots \\ \vdots & \vdots & \ddots \end{pmatrix}^T.$$

Example. *Let* $\mathbf{L} = (f(x), xf(x))$ *where* $f(x)$ *satisfies the equation*

$$f(x)^2x - f(x)(1-x) + 1 = 0,$$

with

$$f(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x},$$

which is the g.f. of the sequence of the large Schröder numbers,(A006318). Two forms of the continued fraction expansion of $f(x)$ are given by

$$f(x) = \cfrac{1}{1 - x - \cfrac{x}{1 - x - \cfrac{x}{1 - x - \cfrac{x}{\ddots}}}} = \cfrac{1}{1 - 2x - \cfrac{2x^2}{1 - 3x - \cfrac{2x^2}{1 - 3x - \cfrac{2x^2}{\ddots}}}}.$$

Now, $\mathbf{H}_f = \mathbf{LB} \cdot \mathbf{D} \cdot (\mathbf{LB})^T$, *which expands as*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 2 & 1 & 0 & 0 \dots \\ 6 & 4 & 1 & 0 \dots \\ 22 & 16 & 6 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 0 & 1 & 0 & 0 \dots \\ 0 & 1 & 1 & 0 \dots \\ 0 & 1 & 2 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 4 & 0 & \dots \\ 0 & 0 & 0 & 8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 0 & 1 & 0 & 0 \dots \\ 0 & 1 & 1 & 0 \dots \\ 0 & 1 & 2 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 2 & 1 & 0 & 0 \dots \\ 6 & 4 & 1 & 0 \dots \\ 22 & 16 & 6 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T.$$

The associated Stieltjes matrices of $\mathbf{L}$ *and* $\mathbf{LB}$ *are,*

$$\mathbf{S_L} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \dots \\ 2 & 2 & 1 & 0 & 0 \dots \\ 2 & 2 & 2 & 1 & 0 \dots \\ 2 & 2 & 2 & 2 & 1 \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which are the Łukasiewicz paths with weight of two for all E and S-E steps, and

$$\mathbf{S_{LB}} = \begin{pmatrix} 2 & 1 & 0 & 0\ldots \\ 2 & 3 & 1 & 0\ldots \\ 0 & 2 & 3 & 1\ldots \\ 0 & 0 & 2 & 3\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which are the Motzkin paths with weight of two for all S-E steps and weight of three for all E steps, except the x-axis E step which has weight two.

We note that $\mathbf{L}^{-1} = (g(x), xg(x)) = \left(\frac{1-x}{1+x}, \frac{x(1-x)}{1+x}\right)$. $\mathbf{H}_g = \mathbf{L}^{-1}\mathbf{M} \cdot \mathbf{D} \cdot (\mathbf{L}^{-1}\mathbf{M})^T$ *expands as*

$$\begin{pmatrix} 1 & 0 & 0\ldots \\ -2 & 1 & 0\ldots \\ 2 & -4 & 1\ldots \\ -2 & 8 & -6\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & 1 & 0\ldots \\ 0 & 3 & 0\ldots \\ 0 & 11 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & -2 & 0\ldots \\ 0 & 0 & -2\ldots \\ 0 & 0 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & 1 & 0\ldots \\ 0 & 3 & 0\ldots \\ 0 & 11 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0\ldots \\ -2 & 1 & 0\ldots \\ 2 & -4 & 1\ldots \\ -2 & 8 & -6\ldots \\ \vdots & \vdots & \ddots \end{pmatrix}^T.$$

Example. *Let* $\mathbf{L} = (f(x), xf(x))$, *where* $f(x)$ *satisfies the equation*

$$f(x)^2x - f(x)(1-2x) + 1 = 0$$

with

$$f(x) = \frac{1-2x-\sqrt{4x^2-8x+1}}{2x},$$

which is the g.f. of the sequence of planar rooted trees with n *nodes and tricolored end* *nodes (A047891). Two continued fraction expansions of the g.f. have the form,*

$$f(x) = \cfrac{1}{1-2x-\cfrac{x}{1-2x-\cfrac{x}{1-2x-\cfrac{x}{\ddots}}}} = \cfrac{1}{1-3x-\cfrac{3x^2}{1-4x-\cfrac{3x^2}{1-4x-\cfrac{3x^2}{\ddots}}}}.$$

We have $\mathbf{H}_f = \mathbf{LB}\cdot\mathbf{D}\cdot(\mathbf{LB})^T$ *which expands as*

$$\begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 3 & 1 & 0 & 0\ldots \\ 12 & 6 & 1 & 0\ldots \\ 57 & 33 & 9 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 1 & 1 & 0\ldots \\ 0 & 1 & 2 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 0 & \ldots \\ 0 & 0 & 9 & 0 & \ldots \\ 0 & 0 & 0 & 27 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 1 & 1 & 0\ldots \\ 0 & 1 & 2 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 3 & 1 & 0 & 0\ldots \\ 12 & 6 & 1 & 0\ldots \\ 57 & 33 & 9 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T.$$

The associated Stieltjes matrices of $\mathbf{L}$ *and* $\mathbf{LB}$ *are*

$$\mathbf{S_L} = \begin{pmatrix} 3 & 1 & 0 & 0 & 0\ldots \\ 3 & 3 & 1 & 0 & 0\ldots \\ 3 & 3 & 3 & 1 & 0\ldots \\ 3 & 3 & 3 & 3 & 1\ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which are the Łukasiewicz paths with weight three for all E and S-E steps, and

$$\mathbf{S_{LB}} = \begin{pmatrix} 3 & 1 & 0 & 0\ldots \\ 3 & 4 & 1 & 0\ldots \\ 0 & 3 & 4 & 1\ldots \\ 0 & 0 & 3 & 4\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which are the Motzkin paths with weight of three for all S-E steps and weight of four for all E steps, except the x-axis E step which has weight three.

We note that $\mathbf{L}^{-1} = (g(x), xg(x)) = \left(\frac{1-x}{1+2x}, \frac{x(1-x)}{1+2x}\right)$. $\mathbf{H}_g = \mathbf{L}^{-1}\mathbf{M}\cdot\mathbf{D}\cdot(\mathbf{L}^{-1}\mathbf{M})^T$ *expands as*

$$\begin{pmatrix} 1 & 0 & 0\ldots \\ -3 & 1 & 0\ldots \\ 6 & -6 & 1\ldots \\ -12 & 21 & -9\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & 1 & 0\ldots \\ 0 & 2 & 0\ldots \\ 0 & 4 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & -3 & 0\ldots \\ 0 & 0 & -3\ldots \\ 0 & 0 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & 1 & 0\ldots \\ 0 & 2 & 0\ldots \\ 0 & 4 & 0\ldots \\ \vdots & \vdots & \ddots \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0\ldots \\ -3 & 1 & 0\ldots \\ 6 & -6 & 1\ldots \\ -12 & 21 & -9\ldots \\ \vdots & \vdots & \ddots \end{pmatrix}^T.$$

Corollary 6.2.3. *A bijection exists between Łukasiewicz paths with weight α for all E and S-E steps, where the first few elements of the associated Stieltjes matrix expand as*

$$\begin{pmatrix} \alpha & 1 & 0 & 0 & 0\ldots \\ \alpha & \alpha & 1 & 0 & 0\ldots \\ \alpha & \alpha & \alpha & 1 & 0\ldots \\ \alpha & \alpha & \alpha & \alpha & 1\ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and Motzkin paths with weight of α for all S-E steps and weight of $\alpha+1$ for all E steps, except the x-axis E step which has weight α. The first few elements of the associated Stieltjes expand as

$$\begin{pmatrix} \alpha & 1 & 0 & 0\ldots \\ \alpha & \alpha+1 & 1 & 0\ldots \\ 0 & \alpha & \alpha+1 & 1\ldots \\ 0 & 0 & \alpha & \alpha+1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. Proof follows from eq. (6.8). □

We now turn our attention to Riordan arrays $\mathbf{L} = (g(x), xf(x))$ with $g(x) \neq f(x)$ and we introduce the following proposition. In terms of paths, we will now look at paths where the weights of the steps returning to the x-axis, or level steps on the x-axis differ from the other possible steps of the paths considered.

Proposition 6.2.4.

$$\mathbf{H}_g(x) = \mathbf{LB} \cdot \mathbf{D} \cdot \mathbf{LB}^T,$$

where the Riordan array

$$\mathbf{L} = (g(x), xf(x)) = \left(\frac{1}{1 - (\beta_0 - \frac{\alpha_0}{\beta_1})x - \frac{\alpha_0}{\beta_1} xf(x)}, \frac{x}{(1 - (\beta_1 - 1)x) - xf(x)} \right) \quad (6.10)$$

with

$$xf(x) = \frac{1 - (\beta_1 - 1)x - \sqrt{(1 - (\beta_1 - 1)x)^2 - 4x}}{2},$$

and $\mathbf{L}$ *has the associated Stieltjes matrix*

$$\mathbf{S_L} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 \ldots \\ \alpha_0 & \beta_1 & 1 & 0 \ldots \\ \alpha_0 & \beta_1 & \beta_1 & 1 \ldots \\ \alpha_0 & \beta_1 & \beta_1 & \beta_1 \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that following the notation of the Stieltjes matrix above, S-E steps returning to the axis have the same weight, α_0 with E steps on the x-axis having weight β_0. All other S-E and E steps have weight β_1.

Proof. Now for $\mathbf{LB} = (g(x), xf_1(x))$, with $\mathbf{L} = (g(x), xf(x))$ we have the Riordan array

$$\mathbf{L}\left(1, \frac{x}{1-x}\right) = \left(\frac{1}{1 - (\beta_0 - \frac{\alpha_0}{\beta_1})x - \frac{\alpha_0}{\beta_1} xf(x)}, \frac{1 - (1 + \beta_1)x - \sqrt{(1 - (\beta_1 - 1)x)^2 - 4x}}{2\beta_1} \right).$$

As $xf_1(x) = \frac{1-(1+\beta_1)x-\sqrt{(1-(\beta_1-1)x)^2-4x}}{2\beta_1}$ we can rewrite $\mathbf{LB}$ as

$$\left(\frac{1}{1 - \beta_0 x - \alpha_0 x^2 f_1(x)}, xf_1(x) \right),$$

which has associated Stieltjes matrix

$$\mathbf{S_{LB}} = \begin{pmatrix} \beta_0 & 1 & 0 & 0\dots \\ \alpha_0 & \beta_1+1 & 1 & 0\dots \\ 0 & \beta_1 & \beta_1+1 & 1\dots \\ 0 & 0 & \beta_1 & \beta_1+1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can rewrite

$$\mathbf{LB} = \left(\frac{1}{1-\beta_0 x-\alpha_0 x(xf_1(x))}, \frac{x}{1-(\beta_1+1)x-\beta_1 x(xf_1(x))}\right).$$

The Riordan array above satisfies Theorem 6.1.1, thus

$$\mathbf{H}_g(x) = \mathbf{LB}\cdot\mathbf{D}\cdot\mathbf{LB}^T.$$

□

Example. *We consider the g.f. with continued fraction expansion*

$$\cfrac{1}{1-x-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{x}{\ddots}}}},$$

which is the g.f. of the sequence (A026671).

The associated Stieltjes matrices of **L** *and* **LB** *are*

$$\mathbf{S_L} = \begin{pmatrix} 3 & 1 & 0 & 0\dots \\ 2 & 1 & 1 & 0\dots \\ 2 & 1 & 1 & 1\dots \\ 2 & 1 & 1 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which are the Lukasiewicz paths with weight of one for all E and S-E steps except those returning to the x-axis, when the E step has weight three and the S-E steps have weight two, and

$$\mathbf{S_{LB}} = \begin{pmatrix} 3 & 1 & 0 & 0\dots \\ 2 & 2 & 1 & 0\dots \\ 0 & 1 & 2 & 1\dots \\ 0 & 0 & 1 & 2\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which are the Motzkin paths with weight of one for all S-E steps and weight of two for all E steps, except the x-axis E step which has weight three and the S-E step returning to the x-axis has weight of two.

6.2.1 Binomial transforms

Now, as the Hankel transform is invariant under the Binomial transform [78], we are interested in the form of the Binomial transform of the generating functions above. Firstly let us look at generating functions of the form in eq. (6.8), that satisfy the equation

$$xf(x)^2 - f(x)(1-(\beta_1-1)x)+1=0.$$

We have the following proposition

Proposition 6.2.5. *An o.g.f. which satisfies*

$$x(1-\beta x)f(x)^2 - f(x)(1-(\beta_1-1+\beta)x)+1=0 \tag{6.11}$$

with continued fraction expansion of $f(x)$

$$f(x) = \cfrac{1}{1-(\beta_1-1+\beta)x-\cfrac{x(1-\beta x)}{1-(\beta_1-1+\beta)x-\cfrac{x(1-\beta x)}{1-(\beta_1-1+\beta)x-\cfrac{x(1-\beta x)}{\ddots}}}}$$

has related Stieltjes matrix

$$\begin{pmatrix} \beta+\beta_1 & 1 & 0 & 0\ldots \\ \beta_1 & \beta+\beta_1 & 1 & 0\ldots \\ \beta_1 & \beta_1 & \beta+\beta_1 & 1\ldots \\ \beta_1 & \beta_1 & \beta_1 & \beta+\beta_1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. Applying proposition 2.4.8 to the Riordan array $(f(x), xf(x))$, where $xf(x)$ is

$$xf(x) = \frac{x}{1-(\beta_1-1+\beta)x-(1-\beta x)xf(x)},$$

gives the required result. ☐

Corollary 6.2.6. *The β^{th} binomial transform of the Riordan array $(f_\beta(x), xf_\beta(x))$ with $f(x)$ satisfying*

$$xf(x)^2 - f(x)(1-(\beta_1-1)x)+1=0$$

satisfies

$$x(1-\beta x)f_\beta(x)^2 - f_\beta(x)(1-(\beta_1-1+\beta)x)+1.$$

Example. *For $\beta_1 = 2$ we have*

$$xf(x)^2 - f(x)(1-x)+1=0$$

with

$$f(x) = \frac{x-1+\sqrt{x^2-6x+1}}{-2x}.$$

The g.f. $f(x)$ has a continued fraction expansion

$$\cfrac{1}{(1-x)-\cfrac{x}{(1-x)-\cfrac{x}{(1-x)-\cfrac{x}{\ddots}}}},$$

which is the g.f. for the sequence of the large Schröeder numbers (A006318). The Hankel matrix decomposes as

$$\begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 2 & 1 & 0 & 0\ldots \\ 6 & 4 & 1 & 0\ldots \\ 22 & 16 & 6 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 1 & 1 & 0\ldots \\ 0 & 1 & 2 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 0 & 2 & 0 & 0\ldots \\ 0 & 0 & 4 & 0\ldots \\ 0 & 0 & 0 & 8\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 1 & 1 & 0\ldots \\ 0 & 1 & 2 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 2 & 1 & 0 & 0\ldots \\ 6 & 4 & 1 & 0\ldots \\ 22 & 16 & 6 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T$$

so $h_n = 2^{\frac{n(n+1)}{2}}$ is the n^{th} Hankel transform with first few elements

$$1, 2, 8, 64, 1024, \ldots (\underline{A006125}).$$

Now, the binomial transform, $(g_1(x), xf_1(x))$, with $\beta = 1$ satisfies,

$$x(1-x)f_1(x)^2 - f_1(x)(1-2x) + 1 = 0$$

with

$$f_1(x) = \frac{2x-1+\sqrt{8x^2-8x+1}}{2x(x-1)},$$

and $f_1(x)$ has a continued fraction expansion

$$\cfrac{1}{(1-2x)-\cfrac{x(1-x)}{(1-2x)-\cfrac{x(1-x)}{(1-2x)-\cfrac{x(1-x)}{\ddots}}}},$$

which is the g.f. for the sequence (A174347).

Now we look at g.f.'s of the form where $g(x) \neq f(x)$, and the Riordan array $\mathbf{L} = \big(g(x), xf(x)\big)$ has

$$g(x) = \frac{1}{1-(\beta+\beta_1)x - x(1-\beta x)f(x)}, \quad f(x) = \frac{1}{1-(\beta+\beta_0)x - x(1-\beta x)f(x)}.$$

In terms of paths, all E and S-E steps have the same weighting, except for that of the E step on the x-axis.

The associated Stieltjes matrix has the form

$$\begin{pmatrix} \beta+\beta_1 & 1 & 0 & 0\ldots \\ \beta_0 & \beta+\beta_0 & 1 & 0\ldots \\ \beta_0 & \beta_0 & \beta+\beta_0 & 1\ldots \\ \beta_0 & \beta_0 & \beta_0 & \beta+\beta_0\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The g.f. of $g_1(x)$, the g.f. of the first row of the Riordan array after multiplication of the Binomial array, has a continued fraction expansion

$$\cfrac{1}{1-(\beta+\beta_1-1)x-\cfrac{x(1-\beta x)}{1-(\beta_0-1+\beta)x-\cfrac{x(1-\beta x)}{1-(\beta_0-1+\beta)x-\cfrac{x(1-\beta x)}{\ddots}}}}.$$

We note that

$$\mathbf{L^{-1}} = \left(\frac{x^2(\beta_0-\beta_1)-x(\beta_0-\beta_1-1)-1}{x^2\beta-x(\beta+\beta_0)-1}, \frac{x(x-1)}{\beta x^2-x(\beta+\beta_0)-1}\right).$$

Let us look at some examples of Riordan arrays of the above form.

Example. *For* $\beta_0=0, \beta_1=2$ *with* $\beta=0,1,2$,

$$g(x)=\frac{1}{1-2x-xf(x)}$$

and $g(x)$ *has a continued fraction expansion*

$$\cfrac{1}{1-2x-\cfrac{x}{1-\cfrac{x}{1-x\ldots}}}.$$

The coefficients of the related power series form the sequence (A001700). Note that for $\mathbf{L}=(g(x), xf(x))$ *then*

$$\mathbf{L}^{-1}=(2x^2-3x+1, x(1-x)).$$

The Binomial transform of $g(x)$ *is*

$$g_1(x)=\frac{1}{1-3x-x(1-x)f(x)}$$

and $g_1(x)$ *has a continued fraction expansion*

$$\cfrac{1}{1-3x-\cfrac{x(1-x)}{1-x-\cfrac{x(1-x)}{1-x-x(1-x)\ldots}}}.$$

The coefficients of the related power series form the sequence (A026378). If $\mathbf{L_1} = (g_1(x), xf_1(x))$ *then the Riordan array*

$$\mathbf{L_1}^{-1} = \left(\frac{-(2x^2-3x+1)}{x^2-x-1}, \frac{x(x-1)}{x^2-x-1}\right).$$

$\mathrm{Rev}(xf_1(x))$ *are the alternating signed Fibonacci numbers. The second Binomial transform* $(g_2(x), xf_2(x))$, *with* $\beta = 2$, *has*

$$g_2(x) = \frac{1}{1-4x-x(1-2x)f(x)},$$

and $g_2(x)$ *has a continued fraction expansion*

$$\cfrac{1}{1-4x-\cfrac{(1-2x)x}{1-2x-\cfrac{(1-2x)x}{1-2x-x(1-2x)\ldots}}}.$$

The coefficients of the related power series form the sequence (A005573).

$$\mathbf{L_2}^{-1} = ((g_2(x), xf_2(x)))^{-1} = \left(\frac{-(2x^2-3x+1)}{2x^2-2x-1}, \frac{x(x-1)}{2x^2-2x-1}\right)$$

where $\mathrm{Rev}(xf_2(x))$ *is the sequence A028859. In general, the related Stieltjes matrices for each of the* β *values above is*

$$\begin{pmatrix} 2+\beta & 1 & 0 & 0\ldots \\ 1 & 1+\beta & 1 & 0\ldots \\ 1 & 1 & 1+\beta & 1\ldots \\ 1 & 1 & 1 & 1+\beta\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the related Łukasiewicz paths have weight one for all the S-E steps, and weight β *for all E steps, except those E steps on the x-axis, where they have weight* $\beta + 2$.

Now finally we turn our attention to the binomial transforms of g.f.'s of the form in eq. (6.12) where $g(x) \neq f(x)$, and the Riordan array $\mathbf{L} = \big(g(x), xf(x)\big)$ has the form

$$\left(\frac{1}{1-(\beta_0-\frac{\alpha_0}{\beta_1})x-\frac{\alpha_0}{\beta_1}xf(x)}, \frac{x}{(1-(\beta_1-1)x)-xf(x)}\right).$$

Here, we have the binomial transform

$$\left(\frac{1}{1-(\beta_0-\frac{\alpha_0}{\beta_1}+\beta)x-\frac{\alpha_0}{\beta_1}(1-\beta)xf(x)}, \frac{x}{(1-(\beta_1-1)x)-x(1-\beta)f(x)}\right).$$

Example. *We look at the first binomial transform of the g.f.*

$$\cfrac{1}{1-x-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{x}{\ddots}}}},$$

which is the g.f. for the sequence(A026671). The associated Stieltjes matrix is

$$\begin{pmatrix} 3 & 1 & 0 & 0\dots \\ 2 & 1 & 1 & 0\dots \\ 2 & 1 & 1 & 1\dots \\ 2 & 1 & 1 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with related Łukasiewicz paths with E and S-E steps all of weight one except for E steps on the x-axis having weight three and S-E steps returning to the x-axis having weight *two. We have a continued fraction expansion for the binomial transform*

$$\cfrac{1}{1-2x-\cfrac{2x(1-x)}{1-x-\cfrac{x(1-x)}{1-x-\cfrac{x(1-x)}{\ddots}}}},$$

which is the g.f. of the sequence (A026671), with corresponding Stieltjes matrix

$$\begin{pmatrix} 4 & 1 & 0 & 0\dots \\ 2 & 2 & 1 & 0\dots \\ 2 & 1 & 2 & 1\dots \\ 2 & 1 & 1 & 2\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which has the same steps as the Stieltjes above, excepts for an increase of the weight of one for each of the level steps.

As we noted in section 5.1.1 where we constructed the binomial transform of a Motzkin path, a bijection between a Motzkin and Łukasiewicz path is preserved under the binomial transform. We refer the reader to section 5.4, where we have constructed one such bijection satisfying the g.f.'s from this section, where the related Stieltjes matrices have the form

$$\mathbf{S_L} = \begin{pmatrix} 0 & 1 & 0 & 0\dots \\ 1 & 0 & 1 & 0\dots \\ 1 & 1 & 0 & 1\dots \\ 1 & 1 & 1 & 0\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which are the Łukasiewicz paths with weight of one for all S-E steps and no E steps permitted, and

$$\mathbf{S_{LB}} = \begin{pmatrix} 0 & 1 & 0 & 0 \dots \\ 1 & 1 & 1 & 0 \dots \\ 0 & 1 & 1 & 1 \dots \\ 0 & 0 & 1 & 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which are the Motzkin paths with weight of one for all steps except no E steps permitted on the x-axis. We note that this is the inverse binomial transform of the g.f. from example 6.2.

6.3 A second Hankel matrix decomposition

Finally before we leave this chapter we naturally lead ourselves to consider a similar decomposition as in eq. (6.8), where we look at the Riordan array decomposition involving the inverse of the "shifted binomial" matrix **B**, as defined in the last section. This inspires the last section in this chapter. We note that we do not consider related lattice paths in this last section as the form of the Stieltjes matrix does not correspond to any paths that we have previously encountered. Once again, we will also look at the binomial transforms of Riordan arrays satisfying our new Riordan array decomposition, and again we will consider the form of their associated Stieltjes matrices.

We now introduce the following decomposition of a Hankel matrix H_g :

$$H_g = \mathbf{L}^{-1}\mathbf{M} \cdot \mathbf{D} \cdot (\mathbf{L}^{-1}\mathbf{M})^T, \tag{6.12}$$

where the Hankel matrix H_g is the matrix where $a_n = [x^n]g(x)$ and $\mathbf{L}$ is the Riordan array $(g(x), xg(x))$, where

$$g(x) = \frac{(1+nx)}{1+(2n+\beta)x+(n^2+\beta n+1)x^2}, \tag{6.13}$$

with $\beta > 0$, $\mathbf{D}$ a diagonal matrix, and $\mathbf{M}$ the matrix with first column 0^n and second column equal to $\text{Rev}(xg(x))$, where

$$\begin{aligned} \text{Rev}(xg(x)) = xf(x) &= \frac{x}{1-(2n+\beta)x+(n-(n^2+\beta n+1)x)xf(x)} && (6.14) \\ &= \frac{\sqrt{x^2(\beta^2-4)-2x\beta+1}+x(\beta+2n)-1}{2(n-x(\beta n+n^2+1))}. && (6.15) \end{aligned}$$

We will see in the examples below that the Hankel matrix H_f, with $a_n = [x^n]f(x)$ where $f(x)$ is the g.f. of the series reversion of $g(x)$, decomposes into $\mathbf{LB}^{-m} \cdot \mathbf{LB}^{-m^T}$, for some m, where $\mathbf{L} = (f(x), xf(x))$ and $\mathbf{B} = (1, \frac{1}{1+x})$, so so we have the following,

$$\begin{aligned}(f(x), xf(x))\left(1, \frac{x}{1+nx}\right) &= \left(1, \frac{x}{1+nx}\right) \\ &= \left(f(x), \frac{xf(x)}{1+n(xf(x))}\right).\end{aligned}$$

Now $\frac{xf(x)}{1+n(xf(x))}$ simplifies to

$$\frac{1-\beta x-\sqrt{x^2(\beta^2-4)-2x\beta+1}}{2x},$$

giving the Riordan array $(g(x), xf(x))$:

$$\left(\frac{\sqrt{x^2(\beta^2-4-2x\beta+1)+x(\beta+2n)-1}}{2(n-x(\beta n+n^2+1))}, \frac{1-\beta x-\sqrt{x^2(\beta^2-4)-2x\beta+1}}{2x}\right)$$

which is the Riordan array

$$\left(\frac{1}{1-(\beta+n)x-x(xf(x))}, xf(x)\right),$$

which has a corresponding tridiagonal Stieltjes matrices.

Let us look at some examples. Note that as the g.f.'s of the Riordan arrays depend on two variables, we will use the notation $\mathbf{L} = \mathbf{L}_{n,\beta}$ for the rest of this section.

Example. *For $\beta = 1$, **MDM** has the form*

$$\begin{pmatrix}1 & 0 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 2 & 0 & 0\ldots \\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\ldots \\ 0 & -1 & 0 & 0\ldots \\ 0 & 0 & -1 & 0\ldots \\ 0 & 0 & 0 & -1\ldots \\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 2 & 0 & 0\ldots \\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}^T$$

The second column of $\mathbf{M}$ *is the g.f. for the sequence of the Motzkin numbers. Now,* $\beta = 1$, *we have* $\mathbf{L}_{n,0}$, *let us now look at the Riordan arrays generated when* $n = 0, 1$ *and* 2.

For $n = 0$ *we have,*

$$\mathbf{L}_{n,0}^{-1} = \left(\frac{1}{1+x+x^2}, \frac{x}{1+x+x^2}\right)$$

with first few rows of the Riordan array

$$\mathbf{L}_{n,0}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0\dots \\ -1 & 1 & 0 & 0\dots \\ 0 & -2 & 1 & 0\dots \\ 1 & 1 & -3 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and associated Stieltjes matrix

$$\mathbf{S}_{\mathbf{L}_{n,0}^{-1}} = \begin{pmatrix} -1 & 1 & 0 & 0\dots \\ -1 & -1 & 1 & 0\dots \\ -2 & -1 & -1 & 1\dots \\ -4 & -2 & -1 & -1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\mathbf{L}_{n,0} = (f(x), xf(x))$, *where*

$$xf(x) = \frac{x}{1 - x - x(xf(x))},$$

which is the g.f. of the sequence of Motzkin numbers (A001006). The first few rows *of the Riordan array expand as*

$$\mathbf{L}_{n,0} = \begin{pmatrix} 1 & 0 & 0 & 0\dots \\ 1 & 1 & 0 & 0\dots \\ 2 & 2 & 1 & 0\dots \\ 4 & 5 & 3 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with associated Stieltjes matrix

$$\mathbf{S}_{\mathbf{L}_{n,0}} = \begin{pmatrix} 1 & 1 & 0 & 0\dots \\ 1 & 1 & 1 & 0\dots \\ 0 & 1 & 1 & 1\dots \\ 0 & 0 & 1 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that $(\mathbf{L}_{n,0}) \cdot (\mathbf{L}_{n,0}^T) = \mathbf{H}$.

For $n = 1$ *we have*

$$\mathbf{L}_{n,1}^{-1} = \left(\frac{1+x}{1+3x+3x^2}, \frac{x(1+x)}{1+3x+3x^2}\right),$$

with first few rows of the Riordan array

$$\mathbf{L}_{n,1}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0\dots \\ -2 & 1 & 0 & 0\dots \\ 3 & -4 & 1 & 0\dots \\ -3 & 10 & -6 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with associated Stieltjes matrix

$$\mathbf{S}_{\mathbf{L}_{n,1}^{-1}} = \begin{pmatrix} -2 & 1 & 0 & 0\ldots \\ -1 & -2 & 1 & 0\ldots \\ -2 & -1 & -2 & 1\ldots \\ -4 & -2 & -1 & -2\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\mathbf{L}_{n,1} = (f(x), xf(x))$, *where*

$$xf(x) = \frac{x}{1-3x+(xf(x))(1-3x)},$$

which is the g.f. of the sequence of numbers of directed animals of size n (A005773), with first few rows of the Riordan array

$$\mathbf{L}_{n,1} = \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 2 & 1 & 0 & 0\ldots \\ 5 & 4 & 1 & 0\ldots \\ 13 & 14 & 6 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and associated Stieltjes matrix,

$$\mathbf{S}_{\mathbf{L}_{n,1}} = \begin{pmatrix} 2 & 1 & 0 & 0\ldots \\ 1 & 2 & 1 & 0\ldots \\ -1 & 1 & 2 & 1\ldots \\ 1 & -1 & 1 & 2\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that $\mathbf{L}_{n,1}(\mathbf{B}^{-1}) \cdot (\mathbf{L}_{n,1}(\mathbf{B}^{-1}))^T = \mathbf{H}_f$. *The first few rows of* $\mathbf{L}_{n,1}(\mathbf{B}^{-1})$ *expand* *as,*

$$\begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 2 & 1 & 0 & 0\ldots \\ 5 & 4 & 1 & 0\ldots \\ 13 & 14 & 6 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & -1 & 1 & 0\ldots \\ 0 & 1 & -2 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ 2 & 1 & 0 & 0\ldots \\ 5 & 3 & 1 & 0\ldots \\ 13 & 9 & 4 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with associated Stieltjes matrix,

$$\mathbf{S}_{\mathbf{L}_{n,1}(\mathbf{B}^{-1})} = \begin{pmatrix} 2 & 1 & 0 & 0\ldots \\ 1 & 1 & 1 & 0\ldots \\ 0 & 1 & 1 & 1\ldots \\ 0 & 0 & 1 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Finally, for $n = 2$ we have

$$\mathbf{L}_{n,2}^{-1} = \left(\frac{1+2x}{1+5x+7x^2}, \frac{x(1+2x)}{1+5x+7x^2}\right),$$

with first few rows of the Riordan array

$$\mathbf{L}_{n,2}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0\ldots \\ -3 & 1 & 0 & 0\ldots \\ 8 & -6 & 1 & 0\ldots \\ -19 & 25 & -9 & 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and associated Stieltjes matrix,

$$\mathbf{S}_{L_{n,2}^{-1}} = \begin{pmatrix} -3 & 1 & 0 & 0\dots \\ -1 & -3 & 1 & 0\dots \\ -2 & -1 & -3 & 1\dots \\ -4 & -2 & -1 & -3\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\mathbf{L}_{n,2} = (f(x), xf(x))$, *where*

$$xf(x) = \frac{x}{1-5x+(xf(x))(2-7x)},$$

which is the g.f. for the sequence (A059738). The Riordan array has first few rows

$$\mathbf{L}_{n,2} = \begin{pmatrix} 1 & 0 & 0 & 0\dots \\ 3 & 1 & 0 & 0\dots \\ 10 & 6 & 1 & 0\dots \\ 34 & 29 & 9 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and associated Stieltjes matrix,

$$\mathbf{S}_{\mathbf{L}_{n,2}} = \begin{pmatrix} 3 & 1 & 0 & 0\dots \\ 1 & 3 & 1 & 0\dots \\ -2 & 1 & 3 & 1\dots \\ 4 & -2 & 1 & 3\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have $\mathbf{L}_{n,2}(\mathbf{B}^{-2})\cdot(\mathbf{L}_{n,2}(\mathbf{B}^{-2}))^T = \mathbf{H}_f$

$$\mathbf{L}_{n,2}(\mathbf{B}^{-2}) = \begin{pmatrix} 1 & 0 & 0 & 0\dots \\ 3 & 1 & 0 & 0\dots \\ 10 & 6 & 1 & 0\dots \\ 34 & 29 & 9 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0\dots \\ 0 & 1 & 0 & 0\dots \\ 0 & -1 & 1 & 0\dots \\ 0 & 1 & -2 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 & 0\dots \\ 3 & 1 & 0 & 0\dots \\ 10 & 4 & 1 & 0\dots \\ 34 & 15 & 5 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with associated Stieltjes matrix,

$$\mathbf{S}_{\mathbf{L}_{n,2}\mathbf{B}^{-2}} = \begin{pmatrix} 3 & 1 & 0 & 0\dots \\ 1 & 1 & 1 & 0\dots \\ 0 & 1 & 1 & 1\dots \\ 0 & 0 & 1 & 1\dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, we have seen some examples above of Stieltjes matrices related to generating functions of the form in eq. (6.15). From eq. (4.7) in proposition 4.3.2 we saw that the rows of the Stieltjes matrix corresponding to the Riordan array $(g(x), xg(x))$ of the Bell subgroup have the form

$$A(x) = \frac{x}{\overline{xg}(x)} \quad \text{and} \quad Z(x) = \frac{1}{\overline{xg}(x)} - \frac{1}{x}.$$

Using this result, we have the following

Corollary 6.3.1. *The Stieltjes matrix of the Riordan array* $(f(x), xf(x))$ *where*

$$xf(x) = \frac{x}{1-(2n+\beta)x+(n-(n^2+\beta n+1)x)f(x)}$$

is formed by the power series

$$A(x) = 1+(\beta+n)x+x^2\sum_{m=0}^{\infty}(-n)^m x^{m+2}$$

and

$$Z(x) = (\beta+n)x + x^2 \sum_{m=0}^{\infty} (-n)^m x^{m+2}$$

Proof. The rows of the Stieltjes matrix corresponding to the Riordan array $(f(x), xf(x))$ of the Bell subgroup have the form

$$A(x) = \frac{x}{\overline{xf}(x)} \quad \text{and} \quad Z(x) = \frac{1}{\overline{xf}(x)} - \frac{1}{x}.$$

From eq. (6.13) we have the series reversion of $xf(x)$

$$\text{rev}(xf(x)) = xg(x) = \frac{(1+nx)x}{(1+(2n+\beta)x+(n^2+\beta n+1)x^2)}.$$

We have

$$A(x) = \frac{(1+(2n+\beta)x+(n^2+\beta n+1)x^2)}{(1+nx)} = 1+(\beta+n)x + x^2 \sum_{m=0}^{\infty} (-n)^m x^{m+2},$$

and

$$Z(x) = \frac{(1+(2n+\beta)x+(n^2+\beta n+1)x^2)}{(1+nx)x} - \frac{1}{x} = (\beta+n)x + x^2 \sum_{m=0}^{\infty} (-n)^m x^{m+2}$$

□

Now, we note that the g.f. from eq. (6.12)

$$g(x) = \frac{(1+nx)}{1+(2n+\beta)x+(n^2+\beta n+1)x^2}$$

can be rewritten in the form

$$\frac{1}{1+(n+\beta)x+\frac{x^2}{1+nx}}.$$

In the table in Fig. (6.3) below we see that for each β, the functions

$$g(x) = \frac{(1+nx)}{1+(2n+\beta)x+(n^2+\beta n+1)x^2}$$

are successive inverse transforms for increasing n and for the series reversion,

$$\begin{aligned}\text{Rev}(xg(x)) = xf(x) &= \frac{x}{1-(2n+\beta)x+(n-(n^2+\beta n+1)x)xf(x)} \\ &= \frac{\sqrt{x^2(b^2-4)-2xb+1}+x(b+2n)-1}{2(n-x(\beta n+n^2+1))}\end{aligned}$$

for each value of n the functions $xf(x)$ are successive binomial transforms for increasing values of β. This leads us to the final proposition in this chapter:

Proposition 6.3.2. *The inverse binomial transform of*

$$g(x) = \frac{(1+nx)}{1+(2n+\beta)x+(n^2+\beta n+1)x^2} = \frac{1}{1+(n+\beta)x+\frac{x^2}{1+nx}},$$

is

$$\cfrac{1}{1+(n+1+\beta)x+\cfrac{x^2}{1+(n+1)x}},$$

and the binomial tranform of

$$\text{Rev}(xg(x)) = xf(x) = \frac{x}{1-(2n+\beta)x+(n-(n^2+\beta n+1)x)xf(x)}$$

is

$$\frac{x}{1-(2n+(\beta+1))x+(n-(n^2+(\beta+1)n+1)x)xf(x)}$$

Proof. Using Riordan arrays we proceed by showing that left multiplication of $g(x)$ by the inverse of the Binomial Riordan array gives the required g.f. That is

$$\begin{aligned}
\left(\frac{1}{1+x},\frac{x}{1+x}\right)\frac{1}{1+(n+\beta)x+\dfrac{x^2}{1+nx}} &= \frac{1}{1+x}\frac{1}{1+(n+\beta)\left(\frac{x}{1+x}\right)+\dfrac{\left(\frac{x}{1+x}\right)^2}{1+n\left(\frac{x}{1+x}\right)}}\\
&= \frac{1}{1+x}\frac{1+\dfrac{nx}{1+x}}{1+\dfrac{nx}{1+x}+\dfrac{x(n+\beta)}{1+x}+\dfrac{nx^2(n+\beta)}{(1+x)^2}+\dfrac{x}{(1+x)^2}}\\
&= \frac{1+x(1+n)}{1+x(1+n)+(1+x(1+n))(n+1+\beta)x+x^2}\\
&= \frac{1}{1+(n+1+\beta)x+\dfrac{x^2}{1+(n+1)x}}.
\end{aligned}$$

For the series reversion $f(x)$, we have

$$\begin{aligned}
&\left(\frac{1}{1-x},\frac{x}{1-x}\right)\frac{\sqrt{x^2(\beta^2-4)-2x\beta+1}+x(\beta+2n)-1}{2(n-x(\beta n+n^2+1))}\\
&= \frac{1}{1-x}\frac{\sqrt{\left(\frac{x}{1-x}\right)^2(\beta^2-4)-2(\frac{x}{1-x})\beta+1}+(\frac{x}{1-x})(\beta+2n)-1}{2(\frac{x}{1-x})(n-(\frac{x}{1-x})(\beta n+n^2+1))}\\
&= \frac{\sqrt{x^2(\beta^2-4)-2(x(1-x))\beta+(1-x)^2}+x(\beta+2n)-(1-x)}{2x(n(1-x)-x(\beta n+n^2+1))}\\
&= \frac{\sqrt{x^2((\beta+1)^2-4)-2x(\beta+1)+1}+x((\beta+1)+2n)-1}{2(n-x(\beta n+n^2+1))}
\end{aligned}$$

□

β	$n=-1$	$n=0$	$n=1$
1	$g(x)=\frac{1}{1+\frac{x^2}{1-x}}$ $1,0,-1,-1,0,1,1,\ldots$ $f(x)=\frac{1}{1+x+(-1-x)f(x)}$ $1,0,1,1,3,6,15,\ldots$ (A005043)	$g(x)=\frac{1}{1+x+x^2}$ $1,-1,0,1,-1,0,1,-1,\ldots$ $f(x)=\frac{1}{1-x-xf(x)}$ $1,1,2,4,9,21,\ldots$ (A001006)	$\frac{1}{1+2x+\frac{x^2}{1+x}}$ $1,-2,3,-3,9,\ldots$ $f(x)=\frac{1}{1-3x+(1-3x)f(x)}$ $1,2,5,13,35,96,\ldots$ (A005773)
2	$g(x)=\frac{1}{1+1x+\frac{x^2}{1-x}}$ $1,-1,0,0,0,\ldots$ $f(x)=\frac{1}{1-f(x)}$ $1,1,2,5,14,42,\ldots$ (A000108)	$g(x)=\frac{1}{1+2x+x^2}$ $1,-2,3,-4,5,\ldots$ $f(x)=\frac{1}{1-2x-xf(x)}$ $1,2,5,14,42,\ldots$ (A000108)	$g(x)=\frac{1}{1+3x+\frac{x^2}{1+x}}$ $1,-3,8,-20\ldots$ $f(x)=\frac{1}{1-4x+(1-4x)f(x)}$ $1,3,10,35,126,\ldots$ (A001700)
3	$g(x)=\frac{1}{1+2x+\frac{x^2}{1-x}}$ $1,-2,3,-5,8,-13,21,\ldots$ $f(x)=\frac{1}{1-x+(-1+x)f(x)}$ $1,2,5,15,51,188,\ldots$ (A007317)	$g(x)=\frac{1}{1+3x+x^2}$ $1,-3,8,-21,55,-144,\ldots$ $f(x)=\frac{1}{1-3x-xf(x)}$ $1,3,10,36,\ldots$ (A002212)	$g(x)=\frac{1}{1+4x+\frac{x^2}{1+x}}$ $1,-4,15,-55,200,\ldots$ $f(x)=\frac{1}{1-5x+(1-5x)f(x)}$ $1,4,17,75,339,\ldots$ (A026378)

Chapter 7

Narayana triangles

The Narayana numbers, which are closely related to the ubiquitous Catalan numbers, have an important and growing literature. Their applications are varied. In the next chapter we will look at the mathematics of one application in the area of MIMO (multiple input, multiple output) wireless communication. For our purposes, it is useful to distinguish between three different "Narayana triangles" and their associated "Narayana polynomials". These triangles are documented separately in The On-line Encyclopedia of Integer Sequences [124] along with other variants. We are interested in the Hankel transform [78, 76] of a number of integer sequences that we shall encounter. Other areas where Narayana polynomials and their generalizations find applications include associahedra [23, 57, 105] and secondary RNA structures [43].

7.1 The Narayana Triangles and their generating functions

In this section, we define four separate though related "Narayana triangles", and we describe their (bivariate) generating functions.

According to [166], the number triangle with general term

$$N_0(n,k) = \frac{1}{n+0^n}\binom{n}{k}\binom{n}{k+1} \tag{7.1}$$

has g.f. $\phi_0(x,y)$ which satisfies the equation

$$xy\phi_0^2 + (x+xy-1)\phi_0 + x = 0.$$

Solving for $\phi_0(x,y)$ yields

$$\phi_0(x,y) = \frac{1-x(1+y)-\sqrt{1-2x(1+y)+x^2(1-y)^2}}{2xy}. \tag{7.2}$$

This triangle begins

$$\mathbf{N}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The triangle $\mathbf{N}_1$ with general term

$$N_1(n,k) = 0^{n+k} + \frac{1}{n+0^n}\binom{n}{k}\binom{n}{k+1} \tag{7.3}$$

which begins

$$\mathbf{N}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

clearly has the g.f.

$$\phi_1(x,y) = 1 + \phi_0(x,y) = \frac{1-x(1-y)-\sqrt{1-2x(1+y)+x^2(1-y)^2}}{2xy}. \tag{7.4}$$

The triangle $\mathbf{N}_2$ with general term

$$N_2(n,k) = [k \le n]N_1(n, n-k) = 0^{n+k} + \frac{1}{n+0^{nk}}\binom{n}{k}\binom{n}{k-1} \tag{7.5}$$

begins

$$\mathbf{N}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 1 & 0 & \ldots \\ 0 & 1 & 10 & 20 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This triangle has the g.f.

$$\phi_2(x,y) = 1 + y\phi_0(x,y) = \frac{1 + x(1-y) - \sqrt{1 - 2x(1+y) + x^2(1-y)^2}}{2x}. \tag{7.6}$$

Finally the "Pascal-like" variant $\mathbf{N}_3$ with general term

$$N_3(n,k) = N_0(n+1,k) = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} \tag{7.7}$$

which begins

$$\mathbf{N}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & \ldots \\ 1 & 15 & 50 & 50 & 15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has the g.f.

$$\phi_3(x,y) = \frac{\phi_0(x,y)}{x} = \frac{1 - x(1+y) - \sqrt{1 - 2x(1+y) + x^2(1-y)^2}}{2x^2y}. \tag{7.8}$$

Using the generating functions above, we can relate the Narayana triangles to the process of reverting sequences. We start by calculating the reversion of the expression

$$\frac{x(1-xy)}{1-x(y-1)},$$

considered as a function in x, with parameter y. This amounts to solving the equation

$$\frac{u(1-uy)}{1-u(y-1)} = x$$

for the unknown u. We obtain

$$u = x\phi_1(x,y).$$

Thus we have

$$\phi_1(x,y) = \frac{1}{x}\mathrm{Rev}\frac{x(1-xy)}{1-x(y-1)}. \tag{7.9}$$

In like manner, we obtain

$$\phi_2(x,y) = \frac{1}{x}\mathrm{Rev}\frac{x(1-x)}{1-x(1-y)} \tag{7.10}$$

and

$$\phi_3(x,y) = \frac{1}{x}\mathrm{Rev}\frac{x}{1+(1+y)x+yx^2}. \tag{7.11}$$

7.2 The Narayana Triangles and continued fractions

In this section, we develop continued fraction versions for each of the generating functions ϕ_1, ϕ_2, ϕ_3.

It is easy to see that $\phi_1(x,y)$ obeys the equation [21]

$$xy\phi_1^2 - (xy-x+1)\phi_1 + 1 = 0. \tag{7.12}$$

Thus

$$\phi_1(1-x-xy\phi_1) = 1-xy\phi_1$$

and thus

$$\begin{aligned}\phi_1 &= \frac{1-xy\phi_1}{1-xy\phi_1-x}\\ &= \frac{1}{1-\frac{x}{1-xy\phi_1}}.\end{aligned}$$

We thus obtain the result that $\phi_1(x,y)$ can be expressed as the continued fraction

$$\phi_1(x,y) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cdots}}}}}. \tag{7.13}$$

Similarly for $\phi_2(x,y)$, we have

$$x\phi_2^2 - (1+x-xy)\phi_2 + 1 = 0 \tag{7.14}$$

from which we deduce

$$\phi_2(1-x\phi_2-xy) = 1-x\phi_2$$

and hence

$$\begin{aligned}\phi_2 &= \frac{1-x\phi_2}{1-x\phi_2-xy}\\ &= \frac{1}{1-\frac{xy}{1-x\phi_2}}.\end{aligned}$$

Thus we obtain the result that $\phi_2(x,y)$ can be expressed as the continued fraction

$$\phi_2(x,y) = \cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}. \tag{7.15}$$

In order to find an expression for ϕ_3, we first note that

$$\phi_3 = \frac{\phi_1 - 1}{x} \Rightarrow \phi_1 = 1 + x\phi_3.$$

Substituting into eq. (7.12) and simplifying, we find that

$$\phi_3(1-xy-x^2y\phi_3) = 1 + x\phi_3 \tag{7.16}$$

and hence

$$\begin{aligned}\phi_3 &= \frac{1+x\phi_3}{1-xy(1+x\phi_3)}\\ &= \frac{1}{-xy+\frac{1}{1+x\phi_3}}\\ &= \frac{1}{-xy+\frac{1}{\phi_1}}.\end{aligned}$$

But

$$\frac{1}{\phi_1} = 1 - \cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}.$$

Hence we obtain that

$$\phi_3(x,y) = \cfrac{1}{1-xy-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}. \tag{7.17}$$

We summarize the foregoing results in the next three sections, along with some other relevant information concerning the three Narayana triangles $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$. The Narayana triangles are not Riordan arrays.

7.2.1 The Narayana triangle $\mathbf{N}_1$

We have

$$\mathbf{N}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots\\ 1 & 0 & 0 & 0 & 0 & 0 & \dots\\ 1 & 1 & 0 & 0 & 0 & 0 & \dots\\ 1 & 3 & 1 & 0 & 0 & 0 & \dots\\ 1 & 6 & 6 & 1 & 0 & 0 & \dots\\ 1 & 10 & 20 & 10 & 1 & 0 & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the g.f. [21]

$$\phi_1(x,y) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cdots}}}}}$$

and

$$\phi_1(x,y) = \frac{1}{x}\text{Rev}\frac{x(1-xy)}{1-(y-1)x},$$

or

$$\phi_1(x,y) = \cfrac{1}{1-x-\cfrac{x^2y}{1-x(1+y)-\cfrac{x^2y}{1-x(1+y)-\cfrac{x^2y}{1-\cdots}}}}. \quad (7.18)$$

In closed form, the g.f. can be expressed as

$$\frac{1-(1-y)x-\sqrt{1-2x(1+y)+(1-y)^2x^2}}{2xy}.$$

We have

$$N(n,k) = 0^{n+k} + \frac{1}{n+0^n}\binom{n}{k}\binom{n}{k+1}.$$

This triangle is A131198.

7.2.2 The Narayana triangle $\mathbf{N}_2$

We have

$$\mathbf{N}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 3 & 1 & 0 & 0 & \dots \\ 0 & 1 & 6 & 6 & 1 & 0 & \dots \\ 0 & 1 & 10 & 20 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the g.f.

$$\phi_2(x,y) = \cfrac{1}{1 - \cfrac{xy}{1 - \cfrac{x}{1 - \cfrac{xy}{1 - \cfrac{x}{1 - \cfrac{xy}{1 - \cdots}}}}}}$$

or

$$\phi_2(x,y) = \cfrac{1}{1 - xy - \cfrac{x^2y}{1 - x(1+y) - \cfrac{x^2y}{1 - x(1+y) - \cfrac{x^2y}{1 - \cdots}}}}. \qquad (7.19)$$

In closed form the g.f. is [42]

$$\frac{1 + (1-y)x - \sqrt{1 - 2x(1+y) + (1-y)^2x^2}}{2x}.$$

It has general term

$$0^{n+k} + \frac{1}{n + 0^{nk}}\binom{n}{k}\binom{n}{k-1}$$

which corresponds to

$$[x^{n+1}y^k]\mathrm{Rev}\frac{x(1-x)}{1-(1-y)x}.$$

7.2.3 The Narayana triangle $\mathbf{N}_3$

We have

$$\mathbf{N}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & \ldots \\ 1 & 15 & 50 & 50 & 15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the g.f.

$$\phi_3(x,y) = \cfrac{1}{1 - xy - \cfrac{x}{1 - \cfrac{xy}{1 - \cfrac{x}{1 - \cdots}}}}$$

or

$$\phi_3(x,y) = \cfrac{1}{1 - x(1+y) - \cfrac{x^2y}{1 - x(1+y) - \cfrac{x^2y}{1 - x(1+y) - \cfrac{x^2y}{1 - \cdots}}}}. \tag{7.20}$$

In closed form its g.f. is

$$\frac{1 - x(1+y) - \sqrt{1 - 2x(1+y) + (1-y)^2x^2}}{2x^2y}$$

and its general term is

$$\tilde{N}(n,k) = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} = [x^{n+1}y^k]\text{Rev}\frac{x}{1+(1+y)x+yx^2}.$$

This is (A090181).

7.3 Narayana polynomials

To each of the above triangles, there is a family of "Narayana" polynomials [137, 139], where the triangles take on the role of coefficient arrays. Thus we get the polynomials

$$\begin{aligned}
\mathcal{N}_{1,n}(y) &= \sum_{k=0}^{n} N_1(n,k)y^k \\
\mathcal{N}_{2,n}(y) &= \sum_{k=0}^{n} N_2(n,k)y^k \\
\mathcal{N}_{3,n}(y) &= \sum_{k=0}^{n} N_3(n,k)y^k.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathcal{N}_{1,n}(y) &= [x^{n+1}]\text{Rev}\frac{x(1-xy)}{1-(y-1)x} \\
\mathcal{N}_{2,n}(y) &= [x^{n+1}]\text{Rev}\frac{x(1-x)}{1-(1-y)x} \\
\mathcal{N}_{3,n}(y) &= [x^{n+1}]\text{Rev}\frac{x}{1+(1+y)x+yx^2}.
\end{aligned}$$

Values of these polynomials are often of significant combinatorial interest. Sample values for these polynomials are tabulated below.

y	$\mathcal{N}_{1,0}(y), \mathcal{N}_{1,1}(y), \mathcal{N}_{1,2}(y), \ldots$	A-number
1	$1, 1, 2, 5, 14, 42, \ldots$	(A000108)
2	$1, 1, 3, 11, 45, 197 \ldots$	(A001003)
3	$1, 1, 4, 19, 100, 562, \ldots$	(A007564)
4	$1, 1, 5, 29, 185, 1257, \ldots$	(A059231)

y	$\mathcal{N}_{2,0}(y), \mathcal{N}_{2,1}(y), \mathcal{N}_{2,2}(y), \ldots$	A-number
1	$1, 1, 2, 5, 14, 42, \ldots$	(A000108)
2	$1, 2, 6, 22, 90, 394, \ldots$	(A006318)
3	$1, 3, 12, 57, 300, 1686, \ldots$	(A047891)
4	$1, 4, 20, 116, 740, 5028, \ldots$	(A082298)

y	$\mathcal{N}_{3,0}(y), \mathcal{N}_{3,1}(y), \mathcal{N}_{3,2}(y), \ldots$	A-number
1	$1, 2, 5, 14, 42, 132, \ldots$	(A000108(n+1))
2	$1, 3, 11, 45, 197, 903, \ldots$	(A001003(n+1))
3	$1, 4, 19, 100, 562, 3304, \ldots$	(A007564(n+1))
4	$1, 5, 29, 185, 1257, 8925, \ldots$	(A059231(n+1))

We can derive a moment representation for these polynomials using the generating functions above and the Stieltjes transform. We obtain the following:

$$\begin{aligned}
\mathcal{N}_{1,n}(y) &= \frac{y-1}{y}0^n + \frac{1}{2\pi}\int_{y-2\sqrt{y}+1}^{y+2\sqrt{y}+1} x^n \frac{\sqrt{-x^2+2x(1+y)-(1-y)^2}}{2y}dx, \\
\mathcal{N}_{2,n}(y) &= \frac{1}{2\pi}\int_{y-2\sqrt{y}+1}^{y+2\sqrt{y}+1} x^n \frac{\sqrt{-x^2+2x(1+y)-(1-y)^2}}{x}dx, \\
\mathcal{N}_{3,n}(y) &= \frac{1}{2\pi}\int_{y-2\sqrt{y}+1}^{y+2\sqrt{y}+1} x^n \frac{\sqrt{-x^2+2x(1+y)-(1-y)^2}}{y}dx.
\end{aligned}$$

We can exhibit these families of polynomials as the first columns of three related Riordan arrays. Thus

$\mathcal{N}_{1,n}(y)$ is given by the first column of $\left(\frac{1}{1+x}, \frac{x}{(1+x)(1+yx)}\right)^{-1}$,

$\mathcal{N}_{2,n}(y)$ is given by the first column of $\left(\frac{1}{1+yx}, \frac{x}{(1+x)(1+yx)}\right)^{-1}$

$\mathcal{N}_{3,n}(y)$ is given by the first column of $\left(\frac{1}{(1+x)(1+yx)}, \frac{x}{(1+x)(1+yx)}\right)^{-1}$.

Chapter 8

Wireless communications

In the first section of this chapter we introduce MIMO channels. Over the past decade random matrices have been studied in the calculation of channel capacity in wireless communications systems. MIMO (multi-input multi-output) channels are channels which offer an increase in capacity compared to single - input single output channels. Due to the ever increasing popularity of wireless communications and the increased desire for efficient use of bandwidth MIMO systems have become an important research area over the past 10 years.

In the second section of this chapter we see links to previous chapters in this document, specifically that relating to the Narayana polynomials. The role of the Catalan numbers and more recently the Narayana polynomials in the elucidation of the behaviour of certain families of random matrices, along with applications to areas such as MIMO wireless communication, is one such application. See for instance [46, 47, 65, 94, 95, 123, 146]. The final section is inspired by MIMO applications in [94] and [146] and calls on results introduced in the last chapter.

The foundation for digital communications was established by Claude Shannon in 1948 [114]. Shannon's pioneering work gave some fundamental results among which is his channel capacity theorem [114],

Definition 8.0.1. *For any information rate R, less than the channel capacity C, it is* *possible to send information at a rate C with error less than some pre-assigned measure ϵ where C is defined to be*

$$C = W \log \big(1 + \frac{P}{N}\big) \tag{8.1}$$

for an additive white Gaussian noise (AWGN) channel, where W is the bandwidth and P/N is the signal to noise ratio (SNR).

8.1 MIMO (multi-input multi-output) channels

In wireless communications, antenna arrays allow a significant increase in the information rate per communication link. This increases information without additional bandwidth and allows for more bits per second to be transmitted. The first applications of MIMO were in line-of-sight microwave links, an application of radio in the 1970's, devised due to a need for more efficient bandwidth utilization. It was in the 1990's that pioneering work on MIMO systems was completed in Bell Labs in New Jersey where it was proved that with multipath transmission, the capacity and spectral efficiency of a MIMO system could be increased indefinitely. Emre Telatar, one of the researchers at Bell Labs, published a paper in 1998 regarding capacity calculations for multi-antenna channels [143]. This paper remains one of the most referenced works in this area today. In 2000, Ralf Muller published a paper entitled random matrix Theory [94], which explains the theory behind random matrices and why it can be used in calculations for MIMO systems. In 2001, Muller also published a paper [94] which uses random matrix results to calculate the Signal to Noise Error for the MIMO system. Verdu and Tulino published a book in 2004 entitled Random Matrix Theory and Wireless Communications [148], in which they give an overview of the classical results in random matrix theory along with looking closely at its application in the world of wireless communications, at applications in both single and multiple antenna receivers for code division multiple access(CDMA) systems. CDMA is a wideband system where the interference is as close as possible to white Gaussian noise [145], an assumption which helps simplify calculations of the channel matrix. More recently, Khan and Henegan have combined results from random matrix theory with Grassman variables to calculate MIMO channel capacities. Relevant more recent work (2008) involves inverting the MIMO channel [94] which involves transmit processing, rather than receive processing [98]. We refer the reader to Ralf Muller's most recent(2011) MIMO publication [96].

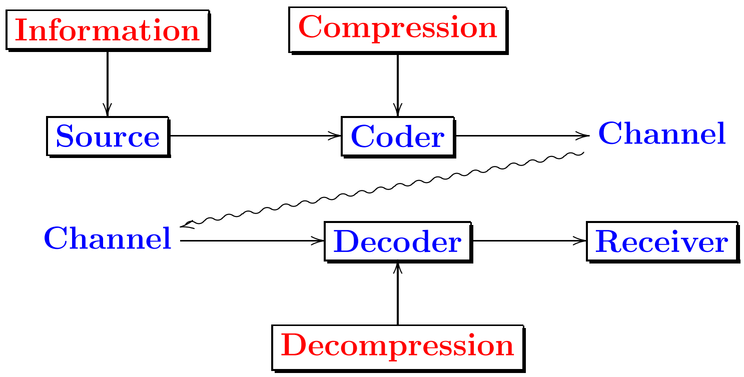


Figure 8.1: Wireless communication system

The basic model of MIMO is

$$y = \mathbf{H}x + n$$

where $y = [y_1, y_2, y_3, \ldots, y_K]$, with y_r denoting the signal received at the r^{th} terminal, $x = [x_1, x_2, \ldots, x_N]$, x_m the zero mean Gaussian signal transmitted and $n = [n_1, n_2, n_3, \ldots, n_K]$ is white noise and $\mathbf{H} \in \mathbf{C}^{N \times K}$ is the channel matrix

$$\mathbf{H} = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,K} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,K} \\ \vdots & \vdots & \vdots & \vdots \\ h_{N,1} & h_{N,2} & \ldots & h_{N,K} \end{pmatrix}.$$

N and K take on different roles depending on the channel access method used and whether it is a single user or multi-user channel. If the channel is a single user narrow band channel, N and K represent the number of antennas at the transmitter and receiver respectively. In the DS - CDMA (Direct sequence - Code division multiple access) channel, which assigns codes to users to allow multiple use of the channel at the same time, K is the number of users and N the spread gain. These are the simpler channels for calculations with $\mathbf{H}$, as in these cases $\mathbf{H}$ has independent and identically distributed entries (i.i.d), which is not the case with other access channels [148]. The capacity of a CDMA MIMO system is

$$\begin{aligned} C_{MIMO} &= \frac{1}{N}\log_2 det(\mathbf{I}_T + \mathbf{HH}^*) \\ &= \frac{1}{N}\sum_{i=1}^{N} log(1 + (\mathbf{SNR})\lambda_i(\mathbf{HH}^*)) \\ &= \int_0^\infty log(1 + (\mathbf{SNR})x)dF^N_{\mathbf{HH}^*}(x)). \end{aligned}$$

The integration is with respect to the distribution function of the eigenvalues of the matrix $\mathbf{HH}^*$. We recall the following result about the cumulative distribution function of the eigenvalues of a random matrix $\mathbf{H}$:

Theorem 1. *Let* $\mathbf{H}$ *be a* $N \times K$ *random matrix,* f *be a function mapping* $\mathbf{H}$ *to a* $N \times N$ *matrix* $f(\mathbf{H})$*, and* $\ell_f(\mathbf{H})$ *be the set containing the eigenvalues of* $f(\mathbf{H})$*. Then, under some weak conditions on the random entries of* $\mathbf{H}$ *and the function* $f()$*, the eigenvalue distribution*

$$F_H(x) \triangleq \frac{1}{N}|\{\lambda \in \ell_f(\mathbf{H}) : \lambda < x\}|$$

converges in probability to a fixed non random distribution as $N, K \to \infty$*, but* $N/K \to \beta < \infty$.

Calculating the cumulative distribution of these matrices is straightforward for the more simple matrix models, however become significantly harder to calculate with more complicated matrices. Work carried out by Marčenko and Pastur yielded a number of results which help simplify calculations of the distribution of the channel matrix.

Definition 8.1.1. *The empirical cumulative distribution function of the eigenvalues (or empirical distribution) of an* $n \times n$ *Hermitian matrix* A*, denoted by* F^n_A*, is defined as [148]*

$$F^n_A = \frac{1}{n}\sum_{i=1}^{n} 1\{\lambda_i(A) \leq x\}$$

where $\lambda_1(A), \ldots, \lambda_n(A)$ *are the eigenvalues of* A *and* $\{.\}$ *is the indicator function.*

Theorem 8.1.1. *Consider an $N \times K$ matrix $\mathbf{H}$ whose entries are zero mean complex random variables with variance $\frac{1}{N}$. As $K, N \to \infty$ with $\frac{N}{K} \to \beta$, the empirical distribution of $\mathbf{HH^T}$ converges almost surely to a non random limiting distribution with density*

$$f_\beta(x) = \left(1 - \frac{1}{\beta}\right)^+ \sigma(x) + \frac{\sqrt{(x-a)^+(b-x)^+}}{2\pi\beta x}$$

where

$$a = (1-\sqrt{\beta})^2 \quad b = (1+\sqrt{\beta})^2.$$

Here, $(1-\frac{1}{\beta})^+$ denotes $\max\{0, 1-\frac{1}{\beta}\}$. Marčenko and Pastur used the Stieltjes transform to simplify their calculations. Before we look at this result we define the Stieltjes transform,

Definition 8.1.2. *For all non real z, the Stieltjes transform of the probability measure $F^H()$ is given by*

$$G_{\mathbf{R}}(z) = \int \frac{1}{\lambda - z} dF^{\mathbf{R}}(\lambda) = E[\frac{1}{\mathbf{R} - z}] = \frac{-1}{z} \sum_{k=0}^{\infty} \frac{E[\mathbf{R}^k]}{z^k}$$

where $F^R(\lambda)$ is the eigenvalue distribution function of the random matrix R,

whose importance consists in the calculation of the statistical moments according to a series expansion

$$\frac{G_{\mathbf{R}}(z^{-1})}{-z} = \sum_{k=0}^{\infty} m_k z^k.$$

Marčenko and Pastur also gave the following important result using the Stieltjes transform, which enables capacity calculations for a more complex channel matrix,

Theorem 8.1.2. *Given a matrix of the form $\mathbf{R} = \mathbf{N} + \mathbf{HPH}^*$ with $\mathbf{R} \in \mathbb{C}^{R\times R}$ composed of Hermitian $\mathbf{N} \in \mathbb{C}^{R\times R}$,$\mathbf{H} \in \mathbb{C}^{R\times T}$ and $\mathbf{P} \in \mathbb{C}^{T\times T}$. As $\mathbf{N} \to \infty$, $\mathbf{R}$ converges to a distribution function of $\mathbf{N}$. Letting $G_{\mathbf{R}}(z)$ denote the Stieltjes transform of $\mathbf{R}$, and*

$G_{\mathbf{N}}(z)$ *the Stieltjes transform of* $\mathbf{N}$*, the equation is given by*

$$G_{\mathbf{R}}(z) = \int \frac{1}{x-z} f_{\mathbf{R}}(x)dx = G_{\mathbf{N}}\big(z - \beta \int \frac{x f_{\mathbf{P}}(x)}{1+xG_{\mathbf{R}}(z)}dx\big).$$

The formation of the matrix in MIMO allows the use of this result so we can proceed to solve as follows. The Stieltjes transform of $\mathbf{P} = \mathbf{I}_T$ and $\mathbf{N} = \mathbf{0}_{R\times R}$ can be calculated as

$$G_{\mathbf{N}}(z) = -\frac{1}{z}, \quad G_{\mathbf{P}}(z) = -\frac{1}{1-z}$$

and

$$\begin{aligned} G_{\mathbf{R}}(z) &= G_{\mathbf{N}}\left(z - \beta \int \frac{x f_{\mathbf{P}} m(x)}{1+xG_{\mathbf{R}}(z)}dx\right) \\ &= \frac{1}{z - \beta \int \frac{x f_{\mathbf{P}}(x)}{1+xG_{\mathbf{R}}(z)}dx} \end{aligned}$$

so we have

$$\begin{aligned} -\frac{1}{G_{\mathbf{R}}(z)} &= z - \beta \int \frac{x f_{\mathbf{P}}(x)}{1+xG_{\mathbf{R}}(z)}dx \\ &= z - \beta/(1+G_R(z)). \end{aligned}$$

As $f_P(x) = \delta(x-1)$ we get the quadratic equation

$$(1+z)G_R(z)^2 + (\beta - z - 2)G_R(z) - 1 = 0.$$

Solving gives,

$$G_R(z) = -\frac{1}{2} + \frac{\beta-1}{2z} + \sqrt{\frac{(1-\beta)^2}{4z^2} + \frac{1}{4} - \frac{\beta+1}{2z}}. \tag{8.2}$$

We note that

$$g_R(z) = \frac{G_{\mathbf{R}}(z^{-1})}{-z} = \frac{1+(1-\beta)z - \sqrt{1-2z(1+\beta)+(1-\beta)^2z^2}}{2z}.$$

8.2 The Narayana triangle N_2 and MIMO

From eq. (8.2) we have

$$G_\beta(z) = -\frac{1}{2} + \frac{\beta - 1}{2z} + \sqrt{\frac{(1-\beta)^2}{4z^2} + \frac{1}{4} - \frac{1+\beta}{2z}}.$$

We recall, in terms of wireless transmission,

$$\beta = \frac{T}{R}$$

where we have T transmit antennas and R receive antennas. In this section it can be treated as a parameter. Again we recall from the last section that the function

$$g_\beta(x) = -\frac{1}{x} G_\beta(\frac{1}{x})$$

which satisfies

$$g_\beta(x) = \frac{1 + (1-\beta)x - \sqrt{1 - 2x(1+\beta) + (1-\beta)^2 x^2}}{2x}$$

and generates the sequence

$$1, \beta, \beta(\beta+1), \beta(\beta^2 + 3\beta + 1), \beta(\beta^3 + 6\beta^2 + 6\beta + 1), \ldots$$

In other words, $g_\beta(x)$ is the g.f. of the sequence

$$a_n^{(\beta)} = \sum_{k=0}^{n} N_2(n,k)\beta^k.$$

We recall from section 7.2.2 that

$$N_2(n,k) = 0^{n+k} + \frac{1}{n + 0^{nk}} \binom{n}{k} \binom{n}{k-1}$$

are the Narayana numbers, which form the array

$$\mathbf{N}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 1 & 0 & \ldots \\ 0 & 1 & 10 & 20 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus

$$g_\beta(x) = \phi_2(x, \beta).$$

We have the following moment representation:

$$\begin{aligned}
a_n^{(\beta)} &= \frac{1}{2\pi}\int_{1+\beta-2\sqrt{\beta}}^{1+\beta+2\sqrt{\beta}} x^n \frac{\sqrt{-x^2+2x(1+\beta)-(1-\beta)^2}}{x}dx \\
&= \frac{1}{2\pi}\int_{(1-\sqrt{\beta})^2}^{(1+\sqrt{\beta})^2} x^n \frac{\sqrt{((1-\sqrt{\beta})^2-x)(x-(1+\sqrt{\beta})^2)}}{x}dx \\
&= \frac{\sqrt{\beta}}{\pi}\int_{(1-\sqrt{\beta})^2}^{(1+\sqrt{\beta})^2} x^n \frac{\sqrt{1-\left(\frac{1+\beta-x}{2\sqrt{\beta}}\right)^2}}{x}dx \\
&= \frac{\sqrt{\beta}}{\pi}\int_{(1-\sqrt{\beta})^2}^{(1+\sqrt{\beta})^2} x^n \frac{w_U\left(\frac{1+\beta-x}{2\sqrt{\beta}}\right)}{x}dx
\end{aligned}$$

where $w_U(x) = \sqrt{1-x^2}$ is the weight function for the Chebyshev polynomials of the second kind.

8.2.1 Calculation of MIMO capacity

We follow [65] to derive an expression for MIMO capacity in a special case. Thus we assume we have R receive antennas and T transmit antennas, modeled by

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n}$$

where $\mathbf{r}$ is the receive signal vector, $\mathbf{s}$ is the source signal vector, $\mathbf{n}$ is an additive white Gaussian noise (AWGN) vector, which is a realization of a complex normal distribution $N(\mathbf{0}, \sigma^2\mathbf{I}_R)$, and the channel is represented by the complex matrix $\mathbf{H} \in \mathbb{C}^{R\times T}$. We have the eigenvalue decomposition

$$\mathbf{H}^H\mathbf{H} = \frac{1}{T}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H.$$

We assume $T < R$. Then the capacity of the uncorrelated MIMO channels is given by [94]

$$\begin{aligned}
C_{MIMO} &= \frac{1}{R}\log_2 \det(\mathbf{I}_T + \mathbf{H}^H(\sigma^2\mathbf{I}_R)^{-1}\mathbf{H}) \\
&= \frac{1}{R}\log_2 \det(\mathbf{I}_T + \frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H}) \\
&= \frac{1}{R}\log_2 \det(\mathbf{I}_T + \frac{1}{\sigma^2}\frac{1}{T}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H) \\
&= \frac{1}{R}\log_2 \det(\mathbf{I}_T + \frac{1}{\sigma^2 T}\mathbf{\Lambda}) \\
&= \frac{T}{R}\frac{1}{T}\sum_{i=1}^{T}\log_2(1 + \frac{1}{\sigma^2 T}\lambda_i) \\
&= \frac{\beta}{\ln 2}\frac{1}{T}\sum_{i=1}^{T}\ln(1 + \frac{1}{\sigma^2 T}\lambda_i)
\end{aligned}$$

where we have set

$$\beta = \frac{T}{R}.$$

Now

$$\begin{aligned}
\ln(1+x) &= \ln(1+x_0) + \sum_{k=1}^{N}(-1)^{k-1}\frac{(x-x_0)^k}{k(1+x_0)^k}, \qquad |x-x_0| < 1 \\
&= \ln(1+x_0) + \sum_{k=1}^{N}\frac{(-1)^{k-1}}{k(1+x_0)^k}\sum_{j=0}^{k}\binom{k}{j}x^j(-1)^{k-j}x_0^{k-j} \\
&= \ln(1+x_0) + \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j-1}\frac{x_0^{k-j}}{k(1+x_0)^k}x^j \\
&= \sum_{k=0}^{N}p_k x^k,
\end{aligned}$$

where it is appropriate to take $x_0 = \frac{1}{\sigma^2}$. We thus obtain

$$\begin{aligned} C_{MIMO} &= \frac{\beta}{\ln 2}\frac{1}{T}\sum_{i=1}^{T}\sum_{k=0}^{N} p_k \left(\frac{\lambda_i}{\sigma^2 T}\right)^k \\ &= \frac{\beta}{\ln 2}\sum_{k=0}^{N}\frac{p_k}{(\sigma^2 T)^k}\left(\frac{1}{T}\sum_{i=1}^{T}\lambda_i^k\right) \\ &= \frac{\beta}{\ln 2}\sum_{k=0}^{N}\frac{p_k}{(\sigma^2 T)^k} m_k \\ &= \frac{\beta}{\ln 2}\sum_{k=0}^{N}\frac{p_k}{(\sigma^2 T)^k}\sum_{j=0}^{k} N_2(k,j)\beta^j. \end{aligned}$$

Thus

$$C_{MIMO} = \frac{\beta}{\ln 2}\sum_{k=0}^{N}\frac{p_k}{(\sigma^2 T)^k}[x^{k+1}]\mathrm{Rev}_x\left[\frac{x(1-x)}{1-(1-\beta)x}\right]. \tag{8.3}$$

We note from eq. (10.1) that $xg_\beta(x)$ is the series reversion of the function

$$\frac{x(1-x)}{1+(\beta-1)x}.$$

This simple form leads us to investigate the nature of the coefficient array of the orthogonal polynomials $P_n^{(\beta)}(x)$ associated to the weight function

$$w(x) = \frac{1}{2\pi}\frac{\sqrt{-x^2+2x(1+\beta)-(1-\beta)^2}}{x} = \frac{1}{2\pi}\frac{\sqrt{4\beta-(x-1-\beta)^2}}{x}dx$$

for which the elements

$$a_n^{(\beta)} = \sum_{k=0}^{n} N_2(n,k)\beta^k$$

are the moments. Put otherwise, these are the family of orthogonal polynomials associated to the Narayana polynomials $\mathcal{N}_{2,n}$. These polynomials can be expressed in terms of the Hankel determinants associated to the sequence $a_n^{(\beta)}$. We find that the coefficient array of the polynomials $P_n^{(\beta)}(x)$ is given by the Riordan array

$$\left(\frac{1}{1+\beta x}, \frac{x}{1+(1+\beta)x+\beta x^2}\right)$$

whose inverse is given by

$$\mathbf{L} = \left(g_\beta(x), \frac{g_\beta(x)-1}{\beta}\right).$$

The Jacobi-Stieltjes array [37, 103] for $\mathbf{L}$ is found to be

$$\left(\begin{array}{ccccccc} \beta & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta & \beta+1 & 1 & 0 & 0 & 0 & \dots \\ 0 & \beta & \beta+1 & 1 & 0 & 0 & \dots \\ 0 & 0 & \beta & \beta+1 & 1 & 0 & \dots \\ 0 & 0 & 0 & \beta & \beta+1 & 1 & \dots \\ 0 & 0 & 0 & 0 & \beta & \beta+1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

indicating that the Hankel transform of the sequence $a_n^{(\beta)}$ is $\beta^{\binom{n+1}{2}}$, and that

$$g_\beta(x) = \cfrac{1}{1-\beta x - \cfrac{\beta x^2}{1-(\beta+1)x - \cfrac{\beta x^2}{1-(\beta+1)x - \cfrac{\beta x^2}{1-\cdots}}}}.$$

We note that the coefficient array $\mathbf{L}^{-1}$ can be factorized as follows:

$$\mathbf{L}^{-1} = \left(\frac{1}{1+\beta x}, \frac{x}{1+(1+\beta)x+\beta x^2}\right) = \left(1, \frac{x}{1+x}\right)\left(\frac{1-x}{1+(\beta-1)x}, \frac{x(1-x)}{1+(\beta-1)x}\right). \tag{8.4}$$

Hence

$$\begin{aligned} \mathbf{L} &= \left(\frac{1-x}{1+(\beta-1)x}, \frac{x(1-x)}{1+(\beta-1)x}\right)^{-1}\left(1, \frac{x}{1+x}\right)^{-1} \\ &= (g_\beta(x), x g_\beta(x)) \cdot \left(1, \frac{x}{1-x}\right). \end{aligned}$$

The general term of the matrix

$$\left(\frac{1-x}{1+(\beta-1)x}, \frac{x(1-x)}{1+(\beta-1)x}\right)^{-1} = (g_\beta(x), xg_\beta(x))$$

is given by

$$\frac{k+1}{n+1}\sum_{j=0}^{n-k}\binom{n+1}{k+j+1}\binom{n+j}{j}(\beta-1)^{n-k-j} = \sum_{j=0}^{n-k}\frac{k+1}{k+j+1}\binom{n}{k+j}\binom{n+j}{j}(\beta-1)^{n-k-j}.$$

For instance, when $\beta = 1$, which is the case of the matrix $(1-x, x(1-x))^{-1}$, we get the expression

$$\frac{k+1}{n+1}\binom{2n-k}{n-k}$$

for the general term. Now the general term of the matrix $\left(1, \frac{1}{1-x}\right)$ is given by

$$\binom{n-1}{k-1} + 0^n(-1)^k.$$

Hence the general term of $\mathbf{L}$ is given by

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{j+1}{i+j+1}\binom{n}{i+j}\binom{n+i}{i}(\beta-1)^{n-j-i}(\binom{j-1}{k-1} + 0^j(-1)^k). \tag{8.5}$$

It is interesting to note that

$$\mathbf{L}^{-1} = \left(\frac{1+x}{(1+x)(1+\beta x)}, \frac{x}{(1+x)(1+\beta x)}\right). \tag{8.6}$$

We can use the factorization in eq. (8.4) to express the orthogonal polynomials $P_n^{(\beta)}(x)$ in terms of the Chebyshev polynomials of the second kind $U_n(x)$. Thus we recognize that the Riordan array

$$\left(\frac{1}{1+(\beta+1)x+\beta x^2}, \frac{x}{1+(1+\beta)x+\beta x^2}\right)$$

is the coefficient array of the modified Chebyshev polynomials of the second kind $\beta^{\frac{n}{2}}U_n(\frac{x-(\beta+1)}{2\sqrt{\beta}})$. Hence by the factorization in eq. (8.4) we obtain

$$P_n^{(\beta)}(x) = \beta^{\frac{n}{2}}U_n(\frac{x-(\beta+1)}{2\sqrt{\beta}}) + \beta^{\frac{n-1}{2}}U_{n-1}(\frac{x-(\beta+1)}{2\sqrt{\beta}}). \tag{8.7}$$

8.3 The R Transform

Along with the Stieltjes transform, another transform which has been shown to simplify calculations in random matrix theory is the R transform. The R transform, in relation to the Stieltjes transform can be defined as follows

Definition 8.3.1. *Given $P(x)$, some probability distribution, with Stieltjes transform*

$$G(s) = \int \frac{dP(x)}{x - s}$$

then the R transform of $P(X)$ is

$$R(w) = G^{-1}(-w) - \frac{1}{w}$$

Example. *The R transform of the Marčenko and Pastur law is*

$$R(w) = \frac{1}{1 - \alpha w}$$

Looking at an example [95], let $P(x)$ be a probability distribution with

$$G_{X^{-1}}(s) = \int \frac{dP_X(x)}{\frac{1}{x} - s}$$

existing for some complex s with $\mathfrak{Im}(s) > 0$

$$G_{X^{-1}}\left(\frac{1}{s}\right) = -s(1 + sG_X(s))$$

Let $s = G_X^{-1}(-w)$, so we find

$$\begin{aligned} G_{X^{-1}}\left(\frac{1}{G_X^{-1}(-w)}\right) &= -G_X^{-1}(-w)(1 - wG_X^{-1}(-w)) \\ \left(\frac{1}{G_X^{-1}(-w)}\right) &= G_{X^{-1}}^{-1}\big(- G_X^{-1}(-w)(1 - wG_X^{-1}(-w))\big) \end{aligned}$$

Using the above definition we have the following

$$\frac{1}{R_X(w)+\frac{1}{w}} \quad = \quad R_{X^{-1}}\Big(-wR_X(w)\Big(R_X(w)+\frac{1}{w}\Big)\Big)-\frac{1}{wR_X(w)\big(R_X(w)+\frac{1}{w}\big)}$$

and

$$\frac{1}{R_X(w)} \quad = \quad R_{X^{-1}}\Big(-R_X(w)\Big(1+wR_X(w)\Big)\Big)$$

as we have seen previously, for a $K \times N$ random matrix $\mathbf{H}$ with i.i.d. entries of variance $1/N$, $R_{\mathbf{HH}^T}(w) = \frac{1}{1-\alpha w}$ letting $X^{-1} = \mathbf{HH}^T$, so

$$R^{-1}_{\mathbf{HH}^T}(w) = 1 + \alpha R^{-1}_{\mathbf{HH}^T}(w)\Big(1 + wR^{-1}_{\mathbf{HH}^T}(w)\Big)$$

Solving gives

$$R(w) = \frac{1-\alpha-\sqrt{(1-\alpha)^2-4\alpha w}}{2\alpha w}$$

Note that $R(w)$ is the series reversion of the bivariate g.f.,

$$\sum_{n=0}^{\infty} T_n(y)\alpha^n = \frac{\alpha y^2}{\alpha y - y + 1}$$

where the first few components of the sequence $R_n(y)$ are,

$$T_0(y) = y, \quad T_1(y) = y(1-y), \quad T_2(y) = y(1-2y+y^2), \quad T_3(y) = y(1-3y+3y^2-y^3).$$

This is the bivariate g.f. of the Riordan array,

$$L = L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & \ldots \\ 1 & -3 & 3 & -1 & 0 & \ldots \\ 1 & -4 & 6 & -4 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(\frac{1}{1-y}, \frac{y}{y-1}\right).$$

Chapter 9

The Euler-Seidel matrix

The Euler-Seidel matrix [39, 40, 41, 49, 91] of a sequence $(a_n)_{n\geq 0}$, which we will denote by $\mathbf{E} = \mathbf{E}_a$, is defined to be the rectangular array $(a_{n,k})_{n,k\geq 0}$ determined by the recurrence $a_{0,k} = a_k$ $(k \geq 0)$ and

$$a_{n,k} = a_{n-1,k} + a_{n-1,k+1} \quad (n \geq 1, k \geq 0). \tag{9.1}$$

The sequence $(a_{0,k})$, the first row of the matrix, is usually called the *initial sequence*, while the sequence $(a_{n,0})$, first column of the matrix, is called the *final sequence*. They are related by the binomial transform (or Euler transform, after Euler, who first proved this [53]). We recall that the binomial transform of a sequence a_n has general term $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. Thus the first row and column of the matrix are determined from eq. (9.1) as follows:

$$a_{n,0} = \sum_{k=0}^{n} \binom{n}{k} a_{0,k}, \tag{9.2}$$

$$a_{0,n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_{k,0}. \tag{9.3}$$

In general, we have

$$a_{n,k} = \sum_{i=0}^{n} \binom{n}{i} a_{0,i+k} = \sum_{i=0}^{n} \binom{n}{i} a_{i+k}. \tag{9.4}$$

Example. *We take* $a_{0,n} = a_n = C_n = \frac{1}{n+1}\binom{2n}{n}$*, the Catalan numbers. Thus the initial sequence, or first row, is the Catalan numbers, while the final sequence, or first column, will be the binomial transform of the Catalan numbers. We obtain the following matrix:*

$$\begin{pmatrix} 1 & 1 & 2 & 5 & 14 & 42 & \dots \\ 2 & 3 & 7 & 19 & 56 & 174 & \dots \\ 5 & 10 & 26 & 75 & 230 & 735 & \dots \\ 15 & 36 & 101 & 305 & 965 & 3155 & \dots \\ 51 & 137 & 406 & 1270 & 4120 & 13726 & \dots \\ 188 & 543 & 1676 & 5390 & 17846 & 60398 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now remark that the Catalan numbers C_n*, (A000108) have the following moment representation:*

$$C_n = \frac{1}{2\pi}\int_0^4 x^n \frac{\sqrt{x(4-x)}}{x}\,dx. \qquad (9.5)$$

Many of the sequences we will discuss have a moment representation of the form

$$a_n = \int_{\mathbf{R}} x^n d\mu_a$$

for a suitable measure $d\mu_a$.

Example. ***The aerated Catalan numbers****. We have seen (see eq.* (9.5)*) that the Catalan numbers are a moment sequence. Similarly, the aerated Catalan numbers*

$$1, 0, 1, 0, 2, 0, 5, 0, 14, \dots$$

can by represented by

$$C_{\frac{n}{2}}\frac{1+(-1)^n}{2} = \frac{1}{2\pi}\int_{-2}^{2} x^n\sqrt{4-x^2}\,dx.$$

Example. ***The factorial numbers*** $n!$.

We have the well-known integral representation of $n!$, *(A000142)*

$$n! = \int_0^\infty x^n e^{-x}\, dx.$$

Example. ***The aerated double factorials***.

We recall that the double factorials (A001147) are given by

$$(2n-1)!! = \prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{n!\, 2^n}.$$

The aerated double factorials, which begin

$$1, 0, 3, 0, 5, 0, 15, 0, 105, 0, 945, \ldots$$

have integral representation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n\, e^{-\frac{x^2}{2}}\, dx.$$

The aerated double factorial numbers have e.g.f. $e^{\frac{x^2}{2}}$.

9.1 The Euler-Seidel matrix and Hankel matrix for moment sequences

We recall that for a sequence $(a_n)_{n\ge 0}$, its Hankel matrix is the matrix $\mathbf{H} = \mathbf{H}_a$ with general term a_{n+k}. Note that if a_n has o.g.f. $A(x)$ then the bivariate g.f. of $\mathbf{H}_a$ is given by

$$\frac{xA(x) - yA(y)}{x-y}.$$

If a_n has an e.g.f. $G(x)$, then the n-th row (and n-th column) of $\mathbf{H}_a$ has e.g.f. given by

$$\frac{d^n}{dx^n} G(x).$$

Example. *We have seen that the aerated double factorial numbers have e.g.f.* $e^{\frac{x^2}{2}}$. *Thus the n-th row of the Hankel matrix associated to them has e.g.f.*

$$\frac{d^n}{dx^n}e^{\frac{x^2}{2}} = e^{\frac{x^2}{2}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2}(2k-1)!!x^{n-2k}.$$

Note that if

$$a_n = \int x^n d\mu_a$$

then

$$a_{n+k} = \int x^{n+k} d\mu_a = \int x^n x^k d\mu_a.$$

We recall that the binomial matrix is the matrix $\mathbf{B}$ with general term $\binom{n}{k}$. The binomial transform of a sequence a_n is the sequence with general term

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

In this case, the sequence b_n has o.g.f. given by

$$\frac{1}{1-x} A\left(\frac{x}{1-x}\right).$$

The sequence $(b_n)_{n\geq 0}$ can be viewed as

$$\mathbf{B} \cdot (a_n)^t.$$

Note that we have

$$\begin{aligned} b_n &= \sum_{k=0}^{n} \binom{n}{k} a_k \\ &= \sum_{k=0}^{n} \binom{n}{k} \int x^k \, d\mu_a \\ &= \int \sum_{k=0}^{n} \binom{n}{k} x^k \, d\mu_a \\ &= \int (1+x)^n \, d\mu_a. \end{aligned}$$

In similar fashion, we have

$$a_n = \int (x-1)^n \, d\mu_b.$$

Proposition 9.1.1. *We have*

$$\mathbf{E}_a = \mathbf{BH}_a. \tag{9.6}$$

Proof. We have

$$\mathbf{BH}_a = \left(\binom{n}{k} \right) \cdot (a_{n+k}).$$

The result follows from eq. (9.4). ☐

We now let

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k,$$

the binomial transform of a_n. We are interested in the product $\mathbf{B}^{-1}\mathbf{H}_b$.

Example. *Taking $b_n = \sum_{k=0}^{n} \binom{n}{k} C_k$ (A0007317), the binomial transform of the Catalan numbers C_n, we obtain*

$$\mathbf{H}_b = \begin{pmatrix} 1 & 2 & 5 & 15 & 51 & 188 & \dots \\ 2 & 5 & 15 & 51 & 188 & 731 & \dots \\ 5 & 15 & 51 & 188 & 731 & 2950 & \dots \\ 15 & 51 & 188 & 731 & 2950 & 12235 & \dots \\ 51 & 188 & 731 & 2950 & 12235 & 51822 & \dots \\ 188 & 731 & 2950 & 12235 & 51822 & 223191 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Multiplying by $\mathbf{B}^{-1}$*, we obtain*

$$\mathbf{B}^{-1}\mathbf{H}_b = \begin{pmatrix} 1 & 2 & 5 & 15 & 51 & 188 & \dots \\ 1 & 3 & 10 & 36 & 137 & 543 & \dots \\ 2 & 7 & 26 & 101 & 406 & 1676 & \dots \\ 5 & 19 & 75 & 305 & 1270 & 5390 & \dots \\ 14 & 56 & 230 & 965 & 4120 & 17846 & \dots \\ 42 & 174 & 735 & 3155 & 13726 & 60398 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is the transpose of the Euler-Seidel matrix for C_n*.*

This result is general. In order to prove this, we will use the follow lemma.

Lemma 9.1.2.

$$x^n(1+x)^k = \sum_{i=0}^{k}\binom{k}{i}x^{i+n} = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{j+k}\binom{j+k}{i}x^i. \tag{9.7}$$

Proof. Since $(1+x)^k = \sum_{i=0}^{k}\binom{k}{i}x^i$ by the binomial theorem, we immediately have

$$x^n(1+x)^k = x^n\sum_{i=0}^{k}\binom{k}{i}x^i = \sum_{i=0}^{k}\binom{k}{i}x^{i+n}.$$

But also, we have

$$\begin{aligned} \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{j+k}\binom{j+k}{i}x^i &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(1+x)^{j+k} \\ &= (1+x)^k\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(1+x)^j \\ &= (1+x)^k x^n. \end{aligned}$$

□

Proposition 9.1.3. *The Euler-Seidel matrix of the sequence* $(a_n)_{n\geq 0}$ *is equal to the transpose of the matrix given by* $\mathbf{B}^{-1}\mathbf{H}_b$*, where* $\mathbf{H}_b$ *is the Hankel matrix of the binomial transform*

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$$

of the initial sequence a_n*. That is,*

$$\mathbf{E}_a^t = \mathbf{B}^{-1}\mathbf{H}_b. \tag{9.8}$$

Proof. The general element of

$$\mathbf{B}^{-1}\mathbf{H} = \left((-1)^{n-k}\binom{n}{k}\right) \cdot (b_{n+k})$$

is given by

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} b_{j+k}.$$

Now

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} b_{j+k} \quad = \quad \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{j+k}\binom{j+k}{i} a_i.$$

Proposition 9.1.4.

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{j+k}\binom{j+k}{i} a_i = \sum_{i=0}^{k}\binom{k}{i} a_{i+n}. \tag{9.9}$$

Proof. Expanding 9.9, if $k \leq n-2$ we have (If $k > n-2$, the middle term below disappears)

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left(\sum_{i=0}^{k}\binom{j+k}{i} a_i + \sum_{i=k+1}^{n-1}\binom{j+k}{i} a_i + \sum_{i=n}^{n+k}\binom{j+k}{i} a_i\right).$$

Now, we endeavor to prove 9.9, by showing

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{k}\binom{j+k}{i}a_i = 0 \tag{9.10}$$

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=k+1}^{n-1}\binom{j+k}{i}a_i = 0 \tag{9.11}$$

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=n}^{n+k}\binom{j+k}{i}a_i = \sum_{i=0}^{k}\binom{k}{i}a_{i+n} \tag{9.12}$$

By a change of summation eq. (9.10) becomes

$$\begin{aligned}\sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{j+k}{i}a_i &= \sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{j}\binom{n}{n-j}\binom{n-j+k}{i}a_i \\ &= \sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\binom{n-j+k}{i}a_i.\end{aligned}$$

Using Vandermonde's identity and a further change of summation we have

$$\begin{aligned}\sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\binom{n-j+k}{i}a_i &= \sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\sum_{r=0}^{i}\binom{n-j}{r}\binom{k}{i-r}a_i \\ &= \sum_{i=0}^{k}\sum_{r=0}^{i}\binom{k}{i-r}a_i\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\binom{n-j}{r} \\ &= \sum_{i=0}^{k}\sum_{r=0}^{i}\binom{k}{i-r}a_i\sum_{j=0}^{n}(-1)^{j}\binom{n}{n-j}\binom{n-j}{r}.\end{aligned}$$

Now, by the cross product of binomial coefficients we have

$$
\begin{aligned}
&= \sum_{i=0}^{k}\sum_{r=0}^{i}\binom{k}{i-r}a_i\sum_{j=0}^{n}(-1)^j\binom{n}{n-j}\binom{n-j}{r} \\
&= \sum_{i=0}^{k}\sum_{r=0}^{i}\binom{k}{i-r}a_i\sum_{j=0}^{n}(-1)^j\binom{n}{r}\binom{n-r}{n-j-r} \\
&= \sum_{i=0}^{k}\sum_{r=0}^{i}\binom{k}{i-r}\binom{n}{r}a_i\sum_{j=0}^{n}(-1)^j\binom{n-r}{j} \\
&= \sum_{i=0}^{k}\sum_{r=0}^{i}\binom{k}{i-r}\binom{n}{r}a_i\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}.
\end{aligned}
$$

Now as $\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}=0$, eq. (9.10) is satisfied. Now, with a change of summation eq. (9.11) becomes

$$
\begin{aligned}
\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=k+1}^{n-1}\binom{j+k}{i}a_i &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{n-k-2}\binom{j+k}{i+k+1}a_{i+k+1} \\
&= \sum_{i=0}^{n-k-2}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{k+j}{k+i+1}a_{i+k+1}.
\end{aligned}
$$

Expanding using Vandermonde's identity and a further change of summation gives

$$
\begin{aligned}
&= \sum_{i=0}^{n-k-2}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{r=0}^{k+i+1}\binom{k}{r}\binom{j}{k+i+1-r}a_{k+i+1} \\
&= \sum_{i=0}^{n-k-2}\sum_{r=0}^{k+i+1}\binom{k}{r}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{j}{k+i+1-r}a_{k+i+1} \\
&= \sum_{i=0}^{n-k-2}a_{k+i+1}\sum_{r=0}^{k+i+1}\binom{k}{r}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{j}{k+i+1-r}.
\end{aligned}
$$

Now, by the cross product of binomial coefficients we have

$$
\begin{aligned}
&= \sum_{i=0}^{n-k-2}a_{k+i+1}\sum_{r=0}^{k+i+1}\binom{k}{r}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{k+i+1-r}\binom{n-(k+i+1-r)}{j-(k+i+1-r)} \\
&= \sum_{i=0}^{n-k-2}a_{k+i+1}\sum_{r=0}^{k+i+1}\binom{k}{r}\binom{n}{k+i+1-r}\sum_{j=0}^{n}(-1)^{n-j}\binom{n-(k+i+1-r)}{n-j}.
\end{aligned}
$$

Now $r \leq i$ so we have

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n-(k+i+1-r)}{n-j} = \sum_{j=0}^{k+i+1-r}(-1)^{n-j}\binom{n-(k+i+1-r)}{n-j} = 0,$$

so eq. (9.11) is satisfied. Lastly, eq. (9.12) with a summation change becomes

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{i=0}^{k}\binom{j+k}{i+n}a_{i+n} \;=\; \sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{j+k}{i+n}a_{i+n}.$$

Expanding with Vandermonde's identity and a further change of summation gives

$$\begin{aligned}
&= \sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{r=0}^{i+n}\binom{j}{r}\binom{k}{i+n-r}a_{i+n} \\
&= \sum_{i=0}^{k}\sum_{r=0}^{i+n}\binom{k}{i+n-r}a_{i+n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{j}{r}. \\
&= \sum_{i=0}^{k}\sum_{r=0}^{i+n}\binom{k}{i+n-r}a_{i+n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{r}\binom{n-r}{j-r}.
\end{aligned}$$

Now

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n-r}{n-j} = \begin{cases} 0 & \text{if } 0 \leq r < n \\ 1 & \text{if } r = n \end{cases}$$

so

$$\sum_{i=0}^{k}\sum_{r=0}^{i+n}\binom{k}{i+n-r}a_{i+n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{r}\binom{n-r}{j-r} = \sum_{i=0}^{k}\binom{k}{i}a_{i+n}.$$

Eq. (9.12) is satisfied. □

The result follows from eq. (9.4). □

Corollary 9.1.5. *The n^{th} row of $\mathbf{E}_a$, for an e.g.f. $A(x)$ is*

$$e^x \frac{d^n}{dx}\left(A(x)\right).$$

Proof. Using the fact that $\mathbf{E}_a = \mathbf{B}\mathbf{H}_a$ where $\mathbf{H}_a$ is the Hankel matrix of the first column of the Euler Seidel matrix $\mathbf{E}_a$, for an e.g.f. $A(x)$, the columns of the Euler Seidel matrix have the form

$$[e^x, x]\frac{d}{dx}(A(x))$$

and using the fundamental theorem of Riordan arrays the n^{th} column have the g.f.

$$e^x \frac{d^n}{dx}(A(x)).$$

□

Example. *The aerated double factorial numbers which have the e.g.f.* $e^{\frac{x^2}{2}}$ *have first few rows of* $\mathbf{E}_a$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & \dots \\ 1 & 5 & 10 & 10 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 3 & \dots \\ 0 & 1 & 0 & 3 & 0 & \dots \\ 1 & 0 & 3 & 0 & 15 & \dots \\ 0 & 3 & 0 & 15 & 0 & \dots \\ 3 & 0 & 15 & 0 & 105 & \dots \\ 0 & 15 & 0 & 105 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 3 & \dots \\ 1 & 1 & 1 & 3 & 3 & \dots \\ 2 & 2 & 4 & 6 & 18 & \dots \\ 4 & 6 & 10 & 24 & 48 & \dots \\ 10 & 16 & 34 & 72 & 198 & \dots \\ 26 & 50 & 106 & 270 & 678 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

So the n^{th} *column of* $\mathbf{E}_a$ *is*

$$[e^x, x]\frac{d^n}{dx}\left(e^{\frac{x^2}{2}}\right) = e^x \frac{d^n}{dx}\left(e^{\frac{x^2}{2}}\right).$$

The n^{th} *derivative of* $\frac{d^n}{dx}\left(e^{\frac{x^2}{2}}\right)$ *as we have seen in proposition 9.13 can be expressed as*

$$e^{\frac{x^2}{2}} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n-2r}(2r-1)!!x^{n-2r}.$$

Thus the columns of $\mathbf{E}_a$ have the form

$$\begin{aligned} e^x \frac{d^n}{dx}\left(e^{\frac{x^2}{2}}\right) &= e^x e^{\frac{x^2}{2}} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n-2r} (2r-1)!!x^{n-2r} \\ &= e^{\frac{x^2}{2}+x} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n-2r} (2r-1)!!x^{n-2r}. \end{aligned}$$

Since $e^{\frac{x^2}{2}+x}$ is the e.g.f. of the Young Tableaux numbers, A000085 we have

$$\begin{aligned} e^x \frac{d^n}{dx}\left(e^{\frac{x^2}{2}}\right) &= e^{\frac{x^2}{2}+x} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n-2r} (2r-1)!!x^{n-2r} \\ &= \sum_{n=0}^{\infty} Y_n \frac{x^n}{n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n-2r} (2r-1)!!x^{n-2r}. \end{aligned}$$

We generalize this result to the e.g.f. $e^{\alpha x^2}$. As the n^{th} row of a Hankel matrix generated from the e.g.f. $e^{\alpha x^2}$ is the n^{th} derivative of $e^{\alpha x^2}$ we introduce the following proposition.

Proposition 9.1.6.

$$\frac{d^{n+1}}{dx^{n+1}}\left(e^{\alpha x^2}\right) = \frac{d}{dx}\left(e^{\alpha x^2} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2\alpha)^{n-r}\binom{n}{n-2r}(2r)!}{r!2^r} x^{n-2r}\right) = e^{\alpha x^2} \sum_{r=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(2\alpha)^{n+1-r}\binom{n+1}{n+1-2r}(2r)!}{r!2^r} x^{n+1-2r}. \tag{9.13}$$

Proof. Firstly, let us illustrate the expansion of the e.g.f. for the first few values of n

$$\begin{aligned}
\frac{d}{dx}\left(e^{\alpha x^2}\right) &= e^{\alpha x^2}(2\alpha x) \\
&= e^{\alpha x^2}\left(2\alpha\binom{1}{1}x\right) \\
\frac{d^2}{dx}\left(e^{\alpha x^2}\right) &= e^{\alpha x^2}(4\alpha^2x^2+2\alpha) \\
&= e^{\alpha x^2}\left(2^2\alpha^2\binom{2}{2}x^2+2\alpha\binom{2}{0}\right) \\
\frac{d^3}{dx}\left(e^{\alpha x^2}\right) &= e^{\alpha x^2}(8\alpha^3x^3+12\alpha^2x) \\
&= e^{\alpha x^2}\left(2^3\alpha^3\binom{3}{3}x^3+2^2\alpha^2\binom{3}{1}x\right) \\
\frac{d^4}{dx}\left(e^{\alpha x^2}\right) &= e^{\alpha x^2}(16\alpha^4x^4+48\alpha^3x^2+12\alpha^2) \\
&= e^{\alpha x^2}\left(2^4\alpha^4\binom{4}{4}x^4+2^3\alpha^3\binom{4}{2}x^2+2^2\alpha^2 3\binom{4}{0}\right) \\
\frac{d^5}{dx}\left(e^{\alpha x^2}\right) &= e^{\alpha x^2}(e^{\alpha x^2}(32\alpha^5x^5+160\alpha^4x^3+120\alpha^3x) \\
&= e^{\alpha x^2}\left(2^5\alpha^5\binom{5}{5}x^5+2^4\alpha^4\binom{5}{3}x^3+2^3\alpha^3 3\binom{5}{1}x\right).
\end{aligned}$$

Expanding

$$\frac{d}{dx}\left(e^{\alpha x^2}\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(2\alpha)^{n-r}\binom{n}{n-2r}(2r)!}{r!2^r}x^{n-2r}\right)$$

we have

$$e^{\alpha x^2}x^{n-1}(2\alpha)^n n!\left\{\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\frac{\alpha^{-r+1}2^{-2r+1}x^{-2r+2}}{r!(n-2r)!}-\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\frac{\alpha^{-r}2^{-2r+1}x^{-2r}}{(r-1)!(n-2r)!}+\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n\alpha^{-r}2^{-2r}x^{-2r}}{r!(n-2r)!}\right\},$$

and gathering similar terms we have

$$e^{mx^2}x^{n-1}(2\alpha)^n n!\left\{\frac{2\alpha x^2}{n!}+\sum_{r=1}^{\lfloor\frac{n}{2}\rfloor}\frac{\alpha^{-r+1}2^{-2r+1}x^{-2r+2}(n+1)}{r!(n-2r+1)!}+\frac{2^{-2\lfloor\frac{n}{2}\rfloor}\alpha^{\lfloor\frac{n}{2}\rfloor+n}x^{\lfloor\frac{n}{2}\rfloor+n-1}n!}{\lfloor\frac{n}{2}\rfloor!(n-2(\lfloor\frac{n}{2}\rfloor)+1)!}\right\}$$

which summarizes to

$$= e^{\alpha x^2} \sum_{r=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(2\alpha)^{n+1-r} \binom{n+1}{n+1-2r} (2r)!}{r! 2^r} x^{n+1-2r}.$$

□

We are interested in characterising the main diagonal of the Euler-Seidel matrix of a_n, which by the above is the same as the main diagonal of $\mathbf{B}^{-1}\mathbf{H}_b$, where $\mathbf{H}$ is the Hankel matrix of b_n, the binomial transform of a_n.

Note that the diagonal is given by

$$a_{n,n} = \sum_{i=0}^{n} \binom{n}{i} a_{n+i}.$$

Example. *We have seen that the diagonal of the Euler-Seidel matrix for the Catalan numbers C_n begins*

$$1, 3, 26, 305, 4120, 60398, 934064, \ldots$$

By the above, the general term of this sequence is

$$d_n = \sum_{i=0}^{n} \binom{n}{i} C_{n+i}.$$

Now consider the moment representation of the Catalan numbers given by

$$C_n = \int x^n d\mu = \frac{1}{2\pi} \int_0^4 x^n \frac{\sqrt{x(4-x)}}{x} dx.$$

We claim that

$$d_n = \int (x(1+x))^n d\mu = \frac{1}{2\pi} \int_0^4 (x(1+x))^n \frac{\sqrt{x(4-x)}}{x} dx.$$

This follows from the result above, or directly, since

$$\begin{aligned}
\int (x(1+x))^n d\mu &= = \int (x+x^2)^n d\mu \\
&= \int \sum_{i=0}^{n} \binom{n}{i} x^{2i} x^{n-i} d\mu \\
&= \sum_{i=0}^{n} \binom{n}{i} \int x^{n+i} d\mu \\
&= \sum_{i=0}^{n} \binom{n}{i} C_{n+i}.
\end{aligned}$$

Note that by the change of variable $y = x(1+x)$ we obtain in this case the alternative moment representation for d_n given by

$$d_n = \frac{1}{2\pi} \int_0^{20} y^n \frac{\sqrt{2}(1+\sqrt{1+4y})\sqrt{5\sqrt{1+4y}-2y-5}}{4y\sqrt{1+4y}}\, dy.$$

The above method of proof is easily generalised. Thus we have

Proposition 9.1.7. *Let a_n be a sequence which can be represented as the sequence of moments of a measure:*

$$a_n = \int x^n d\mu_a.$$

Then the elements d_n of the main diagonal of the Euler-Seidel matrix have moment representation given by

$$d_n = \int (x(1+x))^n d\mu_a.$$

9.2 Related Hankel matrices and orthogonal polynomials

From the last section, we have

$$\mathbf{E}_a = \mathbf{B}\mathbf{H}_a$$

and

$$\mathbf{E}_a^t = \mathbf{B}^{-1}\mathbf{H}_b$$

where b_n is the binomial transform of a_n. The second equation shows us that

$$\begin{aligned}\mathbf{E}_a &= (\mathbf{B}^{-1}\mathbf{H}_b)^t \\ &= \mathbf{H}_b^t(\mathbf{B}^{-1})^t \\ &= \mathbf{H}_b(\mathbf{B}^t)^{-1},\end{aligned}$$

since $\mathbf{H}_b$ is symmetric. Thus we obtain

$$\mathbf{B}\mathbf{H}_a = \mathbf{H}_b(\mathbf{B}^t)^{-1},$$

which implies that

$$\mathbf{H}_b = \mathbf{B}\mathbf{H}_a\mathbf{B}^t. \tag{9.14}$$

Since $\det(\mathbf{B}) = 1$, we deduce once again that the Hankel transform of b_n is equal to that of a_n. We can also use this result to relate the **LDU** decomposition of $\mathbf{H}_b$ [103, 164] to that of $\mathbf{H}_a$. Thus we have

$$\begin{aligned}\mathbf{H}_b &= \mathbf{B}\mathbf{H}_a\mathbf{B}^t \\ &= \mathbf{B}\cdot\mathbf{L}_a\mathbf{D}_a\mathbf{L}_a^t\cdot\mathbf{B}^t \\ &= (\mathbf{B}\mathbf{L}_a)\mathbf{D}_a(\mathbf{B}\mathbf{L}_a)^t.\end{aligned}$$

One consequence of this is that the coefficient triangle of the polynomials orthogonal with respect to $d\mu_b$ is given by

$$\mathbf{L}_a^{-1}\mathbf{B}^{-1},$$

where $\mathbf{L}_a^{-1}$ is the coefficient array of the polynomials orthogonal with respect to $d\mu_a$.

Example. *We take the example of the Catalan numbers $a_n = C_n$ and their binomial transform $b_n = \sum_{k=0}^n C_k$. It is well known that the Hankel transform of C_n is the all 1's sequence, which implies that $\mathbf{D}_a$ is the identity matrix. Thus in this case,*

$$\mathbf{H}_a = \mathbf{L_a}\mathbf{L_a}^t$$

where

$$\mathbf{L}_a = \mathbf{L}_{C_n} = (c(x), xc(x)^2) \qquad (\underline{A039599})$$

with

$$\mathbf{L}_a^{-1} = \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right) \qquad (\underline{A129818})$$

with general term $(-1)^{n-k}\binom{n+k}{2k}$*, where we have used the notation of Riordan arrays. The polynomials*

$$P_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} x^k$$

are thus a family of polynomials orthogonal on $[0,4]$ *with respect to the density function* $\frac{1}{2\pi}\frac{\sqrt{x(4-x)}}{x}$ *[157]. It is known that the bivariate g.f. of the inverse of the* n*-th principal minor of* $\mathbf{H}_a$ *is given by the Christoffel-Darboux quotient*

$$\frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{x-y}.$$

We deduce that the orthogonal polynomials defined by b_n *have coefficient matrix*

$$\begin{aligned}
\mathbf{L}_b^{-1} &= \mathbf{L}_a^{-1}\mathbf{B}^{-1} \\
&= \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right) \cdot \left(\frac{1}{1+x}, \frac{x}{1+x}\right) \\
&= \left(\frac{1+x}{1+3x+x^2}, \frac{x}{1+3x+x^2}\right).
\end{aligned}$$

It turns out that these polynomials $Q_n(x)$ *are given simply by*

$$Q_n(x) = P_n(x-1).$$

Thus $\mathbf{H}_b^{-1}$ *has* n*-th principal minor generated by*

$$\frac{Q_{n+1}(x)Q_n(y) - Q_{n+1}(y)Q_n(x)}{x-y}.$$

In similar manner, we can deduce that the Euler-Seidel matrix $\mathbf{E}_a = \mathbf{E}_{C_n}$ *is such that the* n*-th principal minor of* $\mathbf{E}_a^{-1}$ *is generated by*

$$\frac{P_{n+1}(x)P_n(y-1) - P_{n+1}(y-1)P_n(x)}{x-y} = \frac{P_{n+1}(x)Q_n(y) - Q_{n+1}(y)P_n(x)}{x-y}.$$

Example. *For the aerated double factorials, we have*

$$\mathbf{H}_a = \mathbf{L}_a \mathbf{D}_a \mathbf{L}_a^t$$

where

$$\mathbf{L}_a = [e^{\frac{x^2}{2}}, x], \quad \mathbf{D}_a = diag(n!).$$

The associated orthogonal polynomials (which are scaled Hermite polynomials) have coefficient matrix

$$\mathbf{L}_a^{-1} = [e^{-\frac{x^2}{2}}, x],$$

and we have

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k-1)!!(-1)^k x^{n-2k},$$

or equivalently,

$$P_n(x) = \sum_{k=0}^{n} Bessel^* \left(\frac{n+k}{2}, k \right) (-1)^{\frac{n-k}{2}} \frac{1+(-1)^{n-k}}{2} x^k,$$

where

$$Bessel^*(n,k) = \frac{(2n-k)!}{k!(n-k)!2^{n-k}} = \binom{n+k}{2k} (2k-1)!!,$$

(see [9]). With

$$Q_n(x) = P_n(x-1)$$

we again have that the Euler-Seidel matrix $\mathbf{E}_a$ is such the n-th principal minor of $\mathbf{E}_a^{-1}$ is generated by

$$\frac{P_{n+1}(x)P_n(y-1) - P_{n+1}(y-1)P_n(x)}{x-y} = \frac{P_{n+1}(x)Q_n(y) - Q_{n+1}(y)P_n(x)}{x-y}.$$

Chapter 10

Conclusions and future directions

Riordan arrays and orthogonal polynomials have been areas of particular interest throughout this study. In Chapter 3 we focused on algebraic structures in Hankel matrices and Hankel-plus-Toeplitz matrices relating to classical orthogonal polynomials, in particular the Chebyshev polynomials. Future work involves extending our research to establish links between such algebraic structures and other classical and semi-classical orthogonal polynomials.

Continuing our emphasis on links between Riordan arrays and orthogonal polynomials, we have studied elements of an important property of OPS's, the three-term recurrence relation. Specifically, we have studied the coefficients of the recurrence relations, related paths, and continued fraction expansions, all in the context of associated Riordan arrays. In future work we can extend this focus on the coefficients to study how the existence of recursive relations between these coefficients can be linked to the corresponding Riordan arrays. Another area of possible exploration is the link between the "ladder operator" [25] approach and the Riordan array approach.

Riordan arrays and lattice paths have also been of interest in this thesis. We have been especially interested in lattice paths corresponding to Riordan arrays that are related to orthogonal polynomials and how the structure of these paths relate to general Riordan arrays. In [14] we have shown that Riordan arrays of the form

$$\left(\frac{1-\lambda x-\mu x^2}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$$

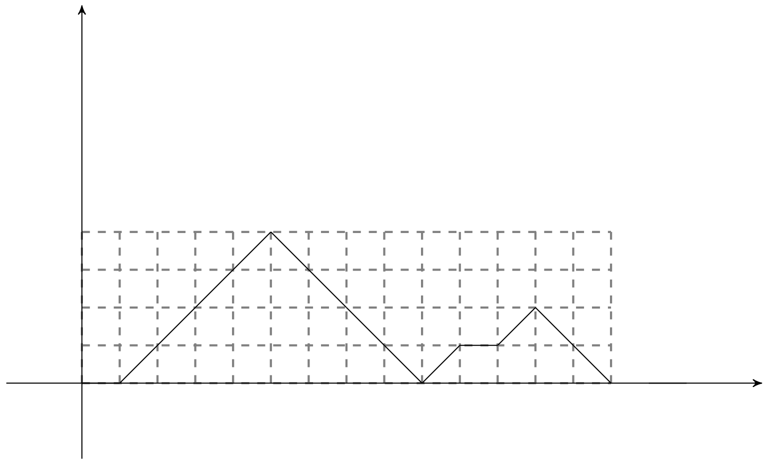


Figure 10.1: A Dyck path of weight $s^3(\mu+s)rs(\mu+s)$

are the coefficient arrays of orthogonal polynomials. The related Stieltjes matrix has the form

$$\begin{pmatrix} \lambda+r & 1 & 0 & 0\ldots \\ \mu+s & r & 1 & 0\ldots \\ 0 & s & r & 1\ldots \\ 0 & 0 & s & r\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

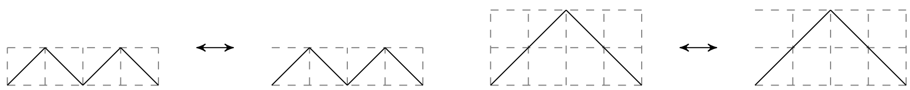
The corresponding Motzkin paths have weighting $\lambda+r$ for level steps on the x axis, $\mu+s$ for S-E steps returning to the x axis, and all other S-E steps of weight s and level steps of weight r. Fig. (10.1) shows an example of one such path.

We have shown that the Hankel matrix generated from the sequence formed from the first column of the inverse of the Riordan array above, can de decomposed in such a way to yield a Riordan array with related non-tridiagonal Stieljtes matrix of the form

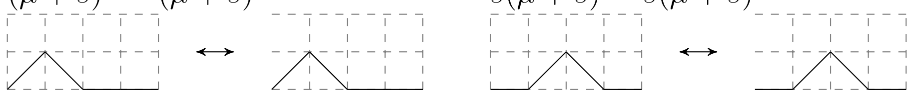
$$\begin{pmatrix} \lambda+r & 1 & 0 & 0\ldots \\ \mu+s & r-1 & 1 & 0\ldots \\ \mu+s & s & r-1 & 1\ldots \\ \mu+s & s & s & r-1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have shown that this matrix relates to Łukasiewicz paths of weighting $\lambda + r$ for level steps on the x axis, $\mu + s$ for S-E Łukasiewicz steps returning to the x axis, and all other S-E Łukasiewicz steps of weight s and level steps of weight $r - 1$.

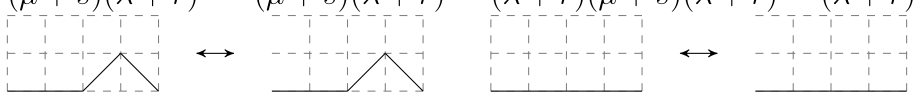
We exhibited a bijection between these two types of Motkzin and Łukasiewicz paths. Below we show the associated weights for both Łukasiewicz and Motzkin paths for a path of length 4.



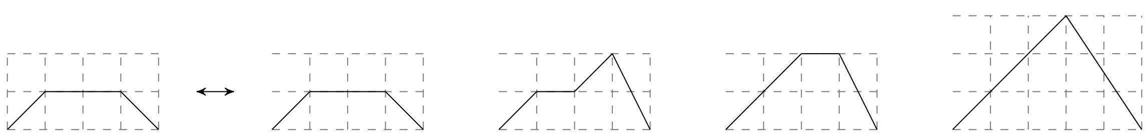
$(\mu + s)^2 = (\mu + s)^2$ $\quad$ $s(\mu + s) = s(\mu + s)$



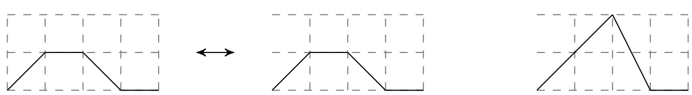
$(\mu + s)(\lambda + r)^2 = (\mu + s)(\lambda + r)^2$ $\quad$ $(\lambda + r)(\mu + s)(\lambda + r) = (\lambda + r)(\mu + s)(\lambda + r)$



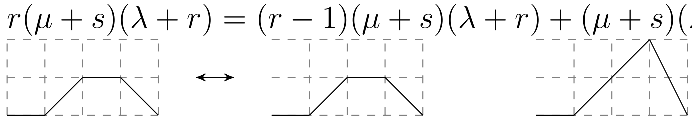
$(\lambda + r)^2(\mu + s) = (\lambda + r)^2(\mu + s)$ $\quad$ $(\lambda + r)^4 = (\lambda + r)^4$



$r^2(\mu + s) \quad = \quad (r - 1)^2(\mu + s) + (r - 1)(\mu + s) + (r - 1)(\mu + s) + (\mu + s)$



$r(\mu + s)(\lambda + r) = (r - 1)(\mu + s)(\lambda + r) + (\mu + s)(\lambda + r)$



$(\lambda + r)(\mu + s)(r) = (\lambda + r)(r - 1)(\mu + s) + (\lambda + r)(\mu + s)$

Riordan arrays relating to the paths above are ordinary Riordan arrays. Some of the above work extended to exponential Riordan arrays. Future work could involve establishing bijections between Motkin and Łukasiewicz paths relating to exponential Riordan arrays.

Recent work carried out by Dan Drake [45] shows bijections between weighted Dyck paths and Schröder paths. Drake also studied bijections between matchings and certain paths. In a similar manner, we can explore the possibilities of extending our combinatorial interpretations to matchings and to study bijections arising between these matchings and Łukasiewicz paths. We may also seek to extend our work on Riordan arrays to other combinatorial interpretations, noting that Viennot and Flajolet studied many forms of combinatorial interpretations relating to orthogonal polynomials. Recent work in this field has been carried out by Louis Shapiro who has studied tree structures relating to Riordan arrays [120].

As with the bijections proved in this thesis, we can continue to explore similar bijections in the area of restricted paths. Such paths have been shown to model polymer absorbtion [108].

Using continued fractions expansions, expressions have been derived for q-orthogonal polynomials related to restricted Dyck paths. We can explore the possibility, using the links we have established between Motzkin paths relating to orthogonal polynomials and corresponding Łukasiewicz paths, of extending this to q-orthogonal polynomials.

Finally, we distinguished between three different "Narayana triangles" and their associated "Narayana polynomials"and applied these polynomials in the area of MIMO (multiple input, multiple output) wireless communication. We expressed the channel capacity for a MIMO channel using one of the Narayana polynomials. This gave us the following result:

$$C_{MIMO} = \frac{\beta}{\ln 2} \sum_{k=0}^{N} \frac{p_k}{(\sigma^2 T)^k} [x^{k+1}] \mathrm{Rev}_x \left[\frac{x(1-x)}{1-(1-\beta)x} \right]. \tag{10.1}$$

In calculating the channel capacity, the Stieltjes transform was used. Extending on this work, we looked at the use of other transforms, such as the R [95] transform to establish similar results. We hope to develop on this work.

Appendix A

Appendix

A.1 Published articles

A.1.1 Journal of Integer Sequences, Vol. 12 (2009), Article 09.5.3

Notes on a Family of Riordan Arrays and Associated Integer Hankel Transforms

Abstract: We examine a set of special Riordan arrays, their inverses and associated Hankel transforms.

P. Barry and A. Hennessy, Notes on a Family of Riordan Arrays and Associated Integer Hankel transforms, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **12**, (2009), Article 09.05.3.

A.1.2 Journal of Integer Sequences, Vol. 13 (2010), Article 10.9.4

Meixner – Type Results for Riordan Arrays and Associated Integer Sequence

Abstract: We determine which (ordinary) Riordan arrays are the coefficient arrays of a family of orthogonal polynomials. In so doing, we are led to introduce a family of polynomials, which includes the Boubaker polynomials, and a scaled version of the Chebyshev polynomials, using the techniques of Riordan arrays. We classify these polynomials in terms of the Chebyshev polynomials of the first and second kinds. We also examine the Hankel transforms of sequences associated with the inverse of the polynomial coefficient arrays, including the associated moment sequences.

P. Barry and A. Hennessy, Meixner-type results for Riordan arrays and associated integer sequences, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **13**, (2010), Article 10.9.4.

A.1.3 Journal of Integer Sequences, Vol. 13 (2010), Article 10.8.2

The Euler – Seidel Matrix, Hankel Matrices and Moment Sequences

Abstract: We study the Euler-Seidel matrix of certain integer sequences, using the binomial transform and Hankel matrices. For moment sequences, we give an integral representation of the Euler-Seidel matrix. Links are drawn to Riordan arrays, orthogonal polynomials, and Christoffel-Darboux expressions.

P. Barry and A. Hennessy, The Euler-Seidel Matrix, Hankel Matrices and Moment Sequences, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **13**, (2010), Article 10.8.2.

A.1.4 Journal of Integer Sequences, Vol. 14 (2011), Article 11.3.8

A Note on Narayana Triangles and Related Polynomials, Riordan Arrays, and MIMO Capacity Calculations

Abstract: We study the Narayana triangles and related families of polynomials. We link this study to Riordan arrays and Hankel transforms arising from a special case of capacity calculation related to MIMO communication systems. A link is established between a channel capacity calculation and a series reversion.

P. Barry, A Hennessy, A note on Narayana triangles and related polynomials, Riordan arrays, and MIMO capacity calculations, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **14** (2011), Article 11.3.8.

A.1.5 Journal of Integer Sequences, Vol. 14 (2011), Article 11.8.2

Generalized Stirling Numbers, **Exponential Riordan Arrays**, **and Orthogonal Polynomials**.

Abstract: We define a generalization of the Stirling numbers of the second kind, which depends on two parameters. The matrices of integers that result are exponential Riordan arrays. We explore links to orthogonal polynomials by studying the production matrices of these Riordan arrays. Generalized Bell numbers are also defined, again depending on two parameters, and we determine the Hankel transform of these numbers.

A. Hennessy, P. Barry, Generalized Stirling Numbers, Exponential Riordan Arrays, and Orthogonal Polynomials, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **14** (2011), Article 11.8.2.

A.2 Submitted articles

A.2.1 Cornell University Library, arXiv:1101.2605

Riordan arrays and the LDU decomposition of symmetric Toeplitz plus Hankel matrices

(Submitted on 13 Jan 2011)

Abstract: We examine a result of Basor and Ehrhardt concerning Hankel and Toeplitz plus Hankel matrices, within the context of the Riordan group of lower-triangular matrices. This allows us to determine the LDU decomposition of certain symmetric Toeplitz plus Hankel matrices. We also determine the generating functions and Hankel transforms of associated sequences.

P. Barry and A. Hennessy, Riordan arrays and the LDU decomposition of symmetric Toeplitz-plus-Hankel matrices, published electronically at:
`http://arxiv.org/abs/1101.2605`, 2011.

Bibliography

[1] T. W. Anderson, R. A. Fisher and Multivariate Analysis, *Journal of Statistical Science*, Vol. **11**, No. **1**, (1996), pp. 20–34.

[2] T. W. Anderson, An Introduction to Multivariate Statistical Analysis, 3rd Edition, ISBN: 978-0-471-36091-9, 2003.

[3] Z. D. Bai, Methodologies in spectral analysis of large dimensional random matrices, a review (with discussion) *Statist. Sinica, Taipei. ISSN 1017-0405*, Vol. **9**, (1999), pp. 611–677.

[4] J. Baik and J. W. Silverstein, Eigenvalues of large sample covariance matrices of spiked population models, *Journal of Multivariate Analysis. ISSN 0047-259*, Vol. **97**, No. **6**, (2006), pp. 1382–1408.

[5] J. Baik, P. A. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations, *Journal of the Amer. Math. Soc., Providence, RI. ISSN 0894-0347*, Vol. **12**, Issue **4**, (1999), pp. 1119–1178.

[6] P. Barry, Continued fractions and transformations of integer sequences, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, Vol. **12**, (2009),Article 9.7.6.

[7] P. Barry, On a One-Parameter Family of Riordan Arrays and the Weight Distribution of MDS Codes, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, Vol. **10**, (2007), Article 07.9.8.

[8] P. Barry, On Integer-Sequence-Based Constructions of Generalized Pascal Triangles, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, Vol. **9**, (2006), Article 06.2.4.

[9] P. Barry, On a family of generalized Pascal's triangles defined by exponential Riordan arrays, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, Vol. **10**, (2000), Article 07.3.5.

[10] P. Barry, The Restricted Toda Chain, Exponential Riordan Arrays, and Hankel Transforms, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, Vol. **13**, (2010),Article 10.8.4.

[11] P. Barry, P. Rajkovic and M. Petkovic, The Hankel transforms based on the central coefficients of certain Pascal-like triangles, *Integral Transforms and Special Functions. An International Journal. Taylor & Francis, Abingdon. ISSN 1065-2469*, Vol. **18**, Issue **4** (2007), pp. 285–296.

[12] P. Barry, P. Rajkovic and M. Petkovic, An application of Sobolev orthogonal polynomials to the computation of a special Hankel Determinant, in W. Gautschi, G. Rassias, M. Themistocles (Eds), *Approximation and Computation*, Springer, 2010.

[13] P. Barry, A Hennessy, A note on Narayana triangles and related polynomials, Riordan arrays, and MIMO capacity calculations, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **14** (2011), Article 11.3.8.

[14] P. Barry and A. Hennessy, Meixner-type results for Riordan arrays and associated integer sequences, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **13**, (2010), Article 10.9.4.

[15] P. Barry and A. Hennessy, The Euler-Seidel Matrix, Hankel Matrices and Moment Sequences, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **13**, (2010), Article 10.9.4.

[16] P. Barry and A. Hennessy, Notes on a Family of Riordan Arrays and Associated Integer Hankel transforms, *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **13**, (2010), Article 09.05.3.

[17] P. Barry and A. Hennessy, Riordan arrays and the LDU decomposition of symmetric Toeplitz-plus-Hankel matrices, published electronically at `http://arxiv.org/abs/1101.2605`, 2011.

[18] E. Basor and T. Ehrhardt, Some identities for determinants of structured matrices. Special issue on structured and infinite systems of linear equations, *Linear Algebra and its Applications. ISSN 0024-3795*, Vol. **343/344**, (2002), pp. 5–19.

[19] C. Berg, Moment problems and orthogonal polynomials, *Series of lectures, University of Copenhagen*, available electronically at `http://www.math.ku.dk/kurser/2005-06/blok2/moment/intro.pdf`, 2005.

[20] J. P. Bouchard and M. Potters, Theory of Financial Risk and Derivative Pricing, *University Press, Cambridge*, 2003.

[21] P. Brändén, A. Claesson & E. Steingrímsson, Catalan continued fractions and increasing subsequences in permutations, available electronically at `http://www.cs.chalmers.se/ claesson/papers/kb13.ps`, 2009.

[22] P. Brändén, The generating function of two-stack sortable permutations by descents is real-rooted, published electronically at `http://arxiv.org/abs/math/0303149`, 2003.

[23] F. Chapoton, Enumerative properties of Generalized Associahedra, *Sém. Lothar. Combin.* Vol. **51**, 2004.

[24] W.Y.C. Chen, S.H.F. Yan and L.L.M. Yang, Identities from weighted 2-Motzkin paths, published electronically at `http://arxiv.org/pdf/math.CO/0410200.pdf`, 2006.

[25] Y. Chen and M. R. Mckay, Perturbed Hankel determinants: Applications to the information theory of MIMO wireless communications, , published electronically at `http://arxiv.org/abs/1007.0496`, 2010.

[26] G. Cheon, H. Kim, L. W. Shapiro, The hitting time subgroup, L ukasiewicz paths and Faber polynomials, European Journal of Combinatorics, Vol. **32**, Issue 1, 2011.

[27] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[28] C. Coker, Enumerating a class of lattice paths, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012-365X*, **271**, (2003) pp. 13–28.

[29] J. B. Conroy, The Riemann Hypothesis, *Notices of the American Institute of Mathematics*, 2003.

[30] T. Conlon, H. J. Ruskin and M. Crane, Random matrix theory and fund of funds portfolio optimisation, *Dublin City University, Physica A,* Vol. **382**, Issue **2** (2007), pp. 565-576.

[31] A. G. Constantine, Some noncentral distribution problems in multivariate analysis, *Annals of Mathematical Statististics*, Vol. **34**, No. **4**, (1963), pp. 1270–1285.

[32] C. Corsani, D. Merlini, R. Sprugnoli, Left-inversion of combinatorial sums, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012-365X*, Vol. **180** (1998), pp. 107–122.

[33] A. Cvetković, P. Rajković and M. Ivković, Catalan Numbers, the Hankel Transform and Fibonacci Numbers, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, Vol. **5**, (2002), Article 02.1.3.

[34] R. Dawkins, River out of Eden, *HarperCollins, Part of the Science Masters series*, ASIN 0465069908, 1995.

[35] E. Deutch and L. Shapiro, Exponential Riordan Arrays, *Lecture Notes, Nankai University*, available electronically at `http://www.combinatorics.net/ppt2004/Louis%20W.%20Shapiro/shapiro.htm`, 2011.

[36] E. Deutsch, L. Ferrari, and S. Rinaldi, Production Matrices, *Advances in Applied Mathematics. Elsevier, Inc., Oxford. ISSN 0196-8858*, Vol. **34**, (2005), pp. 101–122.

[37] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, *Annals of Combinatorics. Birkhauser, Basel. ISSN 0218-0006*, Vol. **13**, No.**1**, (2007), pp. 65–85.

[38] P. Diaconis, A. Gamburd, A, Random matrices, magic squares and matching polynomials. *Electronic J. of Combinatorics*, Vol. **11**, (2004).

[39] A. Dil and V. Kurt, Applications of Euler-Seidel matrices, *Advanced Studies in Contemporary Mathematics (Kyungshang). Memoirs of the Jangjeon Math. Soc., Hapcheon. ISSN 1229-3067*, **18**, (2009), 219–233.

[40] A. Dil and V. Kurt, Investigating exponential and geometric polynomials with Euler-Seidel algorithm, published electronically at `http://arxiv.org/abs/0908.2585`, 2011.

[41] A. Dil, V. Kurt, and M. Cenkci, Algorithms for Bernoulli and allied polynomials, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638* Vol. **10**, (2007), Article 07.5.4.

[42] T. Doslić, Morgan Trees and Dyck Paths, *Croatica Chemica Acta*, **75**, (4), (2002), pp. 881–889.

[43] T. Doslić, D. Svrtan & D. Veljan, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012- 365X*, **285**, (2004), pp. 67–82.

[44] R. B. Dozier and J. W. Silverstein, Analysis of the limiting spectral distribution of large dimensional information-plus-noise type matrices, *North Carolina State University Journal of Multivariate Analysis*, **98**, (2007), pp. 1099 – 1122.

[45] D. Drake, Bijections from Weighted Dyck Paths to Schröder Paths , *J. Integer Seq. ON. ISSN 1530-7638*, Vol. **13** (2010), Article 10.9.2.

[46] I. Dumitriu, Eigenvalue Statistics for Beta-Ensembles, Thesis, MIT, 2003.

[47] I. Dumitriu, E. Rassart, Path counting and Random Matrix Theory, *Electronic Journal of Combinatorics*, **10**, 1, (2003), pp. 35–47.

[48] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, *Journal of Mathematical Physics. Amer. Inst. Phys., Melville, NY. ISSN 0022-2488.*, **43**, (2002), pp. 5830 – 5847.

[49] D. Dumont, Matrices d'Euler-Seidel, *Sém. Lothar. Combin.*, B05c (1981), Available electronically at `http://www.emis.de/journals/SLC/opapers/s05dumont.html`.

[50] F. Dyson, The threefold way. Algebraic structures of symmetry groups and ensembles in Quantum Mechanics, *Journal of Mathematical Physics. Amer. Inst. Phys., Melville, NY. ISSN 0022-2488.*, **3**, (1963), pp. 1199-1215.

[51] A. Edelman and N. R. Rao, Random matrix theory, *Cambridge University Press*, 2005.

[52] S. P. Eu and T. S. Fu, A simple proof of the Aztec diamond, *Electronic Journal of Combinatorics. Electron. J. Combin., San Marcos, CA. ISSN 1077-8926*, **12**, (2005) R18.

[53] L. Euler, De transformatione serierum, *Opera Omnia* ser. 1, Vol. **10**, Teubner, 1913.

[54] R. A. Fisher, The sampling distribution of some statistics obtained from nonlinear equations, *Annals of Eugenics*, **9**, (1939), pp. 238-249.

[55] P. Flajolet and R. Sedgewick, Analytic Combinatorics, Cambridge University Press, Cambridge, UK, 2009.

[56] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012- 365X* , Vol. **32**, (1980), pp. 125–161.

[57] S. Fomin, N. Reading, Root Systems and Generalized Associahedra, published electronically at `http://arxiv.org/abs/math/0505518`, 2005.

[58] G. J. Foschini, Layered Space-Time Architecture for Wireless Communication in a Fading Environment When Using Multi-Element Antennas, *Bell Laboratories Technical Journal*, (1996), pp. 41-59.

[59] G. J. Foschini and M. J. Gans, On limits of wireless communications in a fading environment when using multiple antennas, *Wireless Personal Communications*, Vol. **6**, No. **3**, (1998): pp. 311–335.

[60] G. J. Foschini, G. D. Golden, R. A. Valenzuela and P. W. Wolniansky, Simplified Processing for High Spectral Efficiency Wireless Communication Employing Multi-Element Arrays, *IEEE Journal on Selected Areas in Communications*, Vol. **17**, No. **11**, (1999), pp. 1841–1852.

[61] P. di Francesco, Matrix model combinatorics: Applications to colouring and folding, in Random matrix models and their applications, *Cambridge University Press* (2001), pp. 111–170.

[62] W. Gautshi, Orthogonal polynomials applications and computation, *Acta Numerica 1996, Cambridge University Press*, (1996), pp. 45–119. MR 99i:65019.

[63] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press, Oxford, 2003.

[64] S. Getu, L.W. Shapiro and W.J. Woan, The Riordan group, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012- 365X* , **34**, (1991), pp. 229–239.

[65] I. Groh, S. Plass & S. Stand, Capacity approximation for uncorrelated MIMO channels using random matrix methods In *Proc. 2006 Second International Symposium on Communications, Control and Signal Processing, Marrakech, Morocco*, (2006), pp. 13–16.

[66] A Guionnet, Large Random Matrices: Lectures on Macroscopic Asymptotics, cole d't de Probabilits de Saint-Flour XXXVI, 2006.

[67] M. C. Harding, Explaining the Single Factor Bias of Arbitrage Pricing, *Ecconomics letters*, 2007.

[68] M. C. Harding, Structural Estimation of High-Dimensional Factor Models uncovering the Effect of Global Factors on the US Economy, *Job Market Paper*, 2007.

[69] G. Heinig , Chebyshev-Hankel matrices and the splitting approach for centrosymmetric Toeplitz-plus-Hankel matrices, *Linear Algebra and its Applications. ISSN 0024-3795.*, Vol. **327**, Issues **(1-3)**, (2001), pp. 181–196.

[70] S. Heubach & T. Mansour, Staircase tilings and lattice paths, *Congr. Numer.* 182 (2006), pp. 97–109.

[71] B. Holter, J.E. Håkegård & T.A. Myrvoll, On the use of MIMO in aeronautical communications, *ENRI Int. Workshop on ATM*, 2010.

[72] P. L. Hsu, On the distribution of roots of certain determinantal equations, *Annals of Eugenics*, Vol. **9** (1939), pp. 250–258.

[73] K. Johansson, Non-intersecting paths, random tilings and random matrices, *Probability Theory Related Fields*, Vol. **123**, (2002), pp. 225–280.

[74] O. Khorunzhiy, Rooted trees and moments of large sparse random matrices, *Discrete Mathematics and Theoretical Computer Science*, Vol. **AC**, (2003), pp. 145–154.

[75] V. P. Kostov, A. Martínez-Finkelshtein & B. Z. Shapiro, Narayana numbers and Schur-Szegö composition, *Journal of Approximation Theory*, Vol. **161**, Issue **2**, (2008), pp. 464–476.

[76] C. Krattenthaler, Advanced Determinant Calculus, *Sèminaire Lotharingien Combin.* (B42q), **42**, 1999.

[77] L. Laloux, P. Cizeau and M. Potters and J.P. Bouchaud, Random matrix theory and financial correlations, *International Journal of Theoretical and Applied Finance*, Vol. **3** No. **3**, (2000), pp. 391–397.

[78] J. W. Layman, The Hankel Transform and Some of its properties, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, **4**, (2001), Article 01.1.5.

[79] F. Lehner, *Culmulants, Lattice paths and orthogonal polynomials,Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012- 365X*, Vol. **270**, (2003), pp. 177–191.

[80] F. Lehner, *Free cumulants and enumeration of connected partitions*, *Europ. J. Combin. 22*, Vol. **270**,(2002), pp. 1025–1031.

[81] N. A. Loehr, *Bijective Combinatorics*, CRC Press, Boca Raton, 2011.

[82] B. Mckay, The expected eigenvalue distribution of a large regular graph, *Linear Algebra and its Applications. ISSN 0024-3795*, **40**, (1981), pp. 203 – 216.

[83] V. A. Marčenko and L.A. Pastur, Distribution of eigenvalues for some sets of random matrices, *Math. USSR, Sb.*, Vol. **1**, No. **4**, (1967), pp. 457–483.

[84] M. L. Mehta, Random Matrices, *Academic Press, Boston, third edition*, 1991.

[85] D. Merlini and M.Verri, Generating trees and proper Riordan arrays, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012-365X* , **218**, (2000), pp. 167–183.

[86] D. Merlini , The tennis ball problem, *Journal of Combinatorial Theory. Series A. Elsevier, Inc., Amsterdam. ISSN 0097-3165.*, Series **A**, Vol. **99**, (2002), pp. 307–344.

[87] D. Merlini and R.Sprugnoli, A Riordan array proof of a curious identity, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, 2 A8, 2002.

[88] D. Merlini, R. Sprugnoli and M. Verri, Combinatorial sums and implicit Riordan arrays, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012-365X.*, Vol. **309**, Issue **2**, (2009), pp. 475–486.

[89] D. Merlini, R. Sprugnoli and M. C. Verri, Lagrange Inversion: When and How, *Acta Applicandae Mathematicae*, **94**, (2006), pp. 233–249.

[90] D. Merlini, R. Sprugnoli and M. C. Verri, The Method of Coefficients, *Amer. Math. Monthly*, **114**, (2007), pp. 40–57.

[91] I. Mezö and A. Dil, Euler-Seidel method for certain combinatorial numbers and a new characterization of Fibonacci sequence, *Central European Journal of Mathematics. Versita, Warsaw. ISSN 1895-107*, Vol. **7**, No. **2**, (2009), pp. 310–321.

[92] S. J. Miller, Notes on Random matrix theory, 2004.

[93] S. J. Miller and R. Takloo - Bighash, Introduction to Random matrix theory in Invitation to Modern number theory, *Princeton University Press*, 2007.

[94] R. Müller, A Random Matrix Model of Communications via Antenna Arrays, *IEEE Transactions on Information Theory*, Vol. **48**, No. **9**, (2002) , pp. 2495–2506.

[95] R. Müller, D. Guo & A. Moustakas, Vector precoding for wireless MIMO systems and its replica analysis, *IEEE Journal on Selected Areas in Communications*, Vol. **26**, No. **3**, (2008), pp. 530–540.

[96] B.M. Zaidel, R. Müller, A.L. Moustakas, R. de Miguel, Vector Precoding for Gaussian MIMO Broadcast Channels: Impact of Replica Symmetry Breaking, published electronically at `http://arxiv.org/abs/1001.3790`, 2011.

[97] R. J. Muirhead, Aspects of Multivariate Statistical Theory, *John Wiley and Sons, New York*, 1982.

[98] J. A. Nossek, M. Joham, and W. Utschick Transmit Processing in MIMO Wireless Systems, *IEEE 6th CAS Symp. on Emerging Technologies: Mobile and Wireless Communication*, 2005.

[99] A. Okounkov, Random matrices and random permutations, *International Mathematical Res. Notices*, No. **20**, (2000), pp. 1043–1095.

[100] A. M. Odlyzko, On the distribution of spacings between zeros of the zeta function. *Dynamical, Spectral and Arithmetic Zeta-Functions, Contemporary Mathematics Series*, Vol. **290**, (2001),pp. 139–144.

[101] L. Pastur and V. Vasilchuk, On the Law of Addition of Random Matrices,*Centre de Physique Theorique de CNRS*, 2000.

[102] D. Paul and J. W. Silverstein No eigenvalues outside the support of the limiting empirical spectral distribution of a separable covariance matrix. Journal of Multivariate Analysis 100, available electronically at `http://www4.ncsu.edu/ jack/pub.html`, (2009), pp. 37-57.

[103] P. Peart, W-J. Woan, Generating Functions via Hankel and Stieltjes Matrices, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, Vol. **3**, (2000), Article 00.2.1.

[104] P. C. Phillips Proffessor T. W. Anderson interview, *Ecconometric theory*, 2, 249-288, available electronically at `http://korora.econ.yale.edu/phillips/pubs/art/i002.pdf`, 1986.

[105] A. Postnikov, V. Reiner & L. Williams, Faces of Generalized Permutohedra, *Documenta Math.*, **13**, (2008) pp. 207–273.

[106] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, Numerical recipes in C. The art of scientific computing, *University press, Cambridge,* 2^{nd} *Ed.*, 1992.

[107] A. Owczarek, T. Prellberg, Enumeration of area-weighed Dyck paths with restricted height, published electronically at `http://arxiv.org/abs/1004.1699`, 2010.

[108] R. Brak, G.K Iliev, T. Prellberg, An infinite family of adsorption models and restricted Łukas iewicz paths, published electronically at `http://arxiv.org/abs/1105.0306`, 2010.

[109] C. Radoux, Calcul effectif de certains déterminants de Hankel, Bulletin de la Societe Mathematique de Belgique, 1979.

[110] R. Rao, The Mathematics of Infinite Random Matrices the Stieltjes transform based approach, *Lectures from MIT*, 2004.

[111] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsberichte der Berliner Akademie*, 1859.

[112] T. Rivlin, Chebyshev polynomials: from approximation theory to algebra and number theory, 2. ed., Wiley, New York, 1990.

[113] B. Rosenow, V. Plerou, P. Gopikrishnan, Amaral and H. Eugene Stanley, application of random matrix theory to study cross - correlations of stock prices, *International Journal of Theoretical and Applied Finance*, Vol. **3**, No. **3**, (2000), pp. 399–403.

[114] C. Shannon A Mathematical Theory of Communication, *Bell System Technical Journal*, 1948.

[115] L. W. Shapiro, Bijections and the Riordan group, *Theoretical Computer Science. Elsevier B. V., Amsterdam. ISSN 0304-3975*, Vol. **307**, (2003), pp. 403–413.

[116] R. Sprugnoli, Riordan Array Proofs of Identities in Gould's Book, `http://www.dsi.unifi.it/resp/GouldBK.pdf`, 2006.

[117] R. Sprugnoli, A bibliography of Riordan arrays, `http://www.dsi.unifi.it/ resp/BibRioMio.pdf`, 2011.

[118] L. W. Shapiro, S. Getu, W-J. Woan and L.C. Woodson, The Riordan Group, *Discrete Applied Mathematics. The Journal of Combinatorial Algorithms, Informatics and Computational Sciences. Elsevier Sci. B. V., Amsterdam. ISSN 0166-218X*, Vol. **34**, (1991), pp. 229–239.

[119] L. W. Shapiro, S. Getu, W.-J. Woan and L. Woodson, The Riordan Group, *Discrete Applied Mathematics. The Journal of Combinatorial Algorithms, Informatics and Computational Sciences. Elsevier Sci. B. V., Amsterdam. ISSN 0166-218X*, Vol. **34**, (1991), pp. 229–239.

[120] L. Shapiro, Bijections and the Riordan group, *Theoretical computer science*, Vol. **307**, (2003), pp. 403 – 413

[121] L. Shapiro, A survey of the Riordan Group, Talk at a meeting of the American Mathematical Society, 1994.

[122] L. Shapiro, The uplift principle and the Riordan group, Talk at the KAIST Discete maths seminar, 2011.

[123] J. W. Silverstein, Z. D. Bai, On the Empirical Distribution of Eigenvalues of a Class of Large Dimensional Random Matrices, *J. of Multivariate Analysis*, Vol. **54**, (1995), pp. 175–192.

[124] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at `http://www.oeis.org`, 2011.

[125] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, *Notices of the AMS*, **50**, (2003), pp. 912–915.

[126] P. Sniady, Permutations without long decreasing subsequences and random matrices, *University of Wroclaw*, 2007.

[127] R. Speicher, Free convolution and the random sum of matrices, Research Institute for Mathematical Sciences, Vol. **29**, (1993), pp. 731–744.

[128] R. Speicher, Multiplicative functions on the lattice of non-crossing partitions and free convolution, *Annals of Mathematics*, Vol. **298**, (1994), pp. 611–620.

[129] R. Speicher, Combinatorial theory of the free product with amalgamation and operator-valued free probability theory,*Memoirs of the American Mathematical society*, **627**, 1998.

[130] N. J. A. Sloane, My favorite integer sequences, a paper for the SETA'98 conference on sequences, 1998.

[131] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012-365X*, Vol. **132**, (1994), pp. 267–290.

[132] R. Sprugnoli , R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Mathematics. Elsevier Sci. B. V., Amsterdam. ISSN 0012- 365X*, Vol. **132**(1994), pp. 267–290.

[133] R. Sprugnoli, A unified approach to combinatorial sum inversion, *Technical Report RT 14/94, Dipartimento Sistemi e Informatica*, 1994.

[134] R. Sprugnoli, An Introduction to Mathematical Methods in Combinatorics, `http://www.dsi.unifi.it/7Eresp/Handbook.pdf`, 2006.

[135] R. Sprugnoli, Riordan Array Proofs of Identities in Gould's Book. Available electronically at `http://www.dsi.unifi.it/resp/GouldBK.pdf`, 2007.

[136] R. Stanley, Catalan Addendum, available electronically at `http://www-math.mit.edu/ rstan/ec/catadd.pdf`, 2011.

[137] R. A. Sulanke, Counting lattice paths by Narayana polynomials, *Electronic J. Combin.* **7**, 2000.

[138] R. A. Sulanke, The Narayana distribution, *J. Statist. Plann. and Infer.* **101** (2002) pp. 311–346.

[139] R. A. Sulanke, Generalizing Narayana and Schröder numbers to higher dimensions, *Electron. J. Combin.* **11** (1) (2004), #R54.

[140] B. Sutton, The Mathematics of Infinite Random Matrices Tridiagonal Matrices, Orthogonal Polynomials and the Classical Random Matrix Ensembles, *Handout, MIT lectures*, 2004.

[141] Y. Sun, The Star of David rule, published electronically at `http://arxiv.org/abs/0805.1277`, 2008.

[142] G. Szegö, *Orthogonal Polynomials*, 4th ed. Providence, RI, Amer. Math. Soc., (1975).

[143] E. Teletar, The capacity of Gaussian Mult-antenna channels, *European Transactions on Telecommunications*, 10(6), 1999, pp. 585–596.

[144] V. Totik, Orthogonal polynomials, published electronically at `http://arxiv.org/abs/math/0512424v1`, 2005.

[145] D. Tse and P. Viswanath, Fundamentals of wireless communications, *Cambridge university press*, 2005.

[146] A. Tulino, S. Verdú, *Random Matrix Theory and Wireless Communications*, now Publishers Inc., Hanover, MA., 2004

[147] S. Verdu and S. Shamai, Spectral Efficiency of CDMA with Random Spreading, IEEE Trans. *Information Theory*, Vol. **45**,, No. **2**, (1999), pp. 622–640.

[148] S. Verdu and Tulino, Random Matrix theory and wireless communications, *Now Publisher Inc,ISSN:1567-2190*, 2004.

[149] S. Verdu, Random Matrix Theory and its applications to Statistics and Wireless Communications Eigenvalues and Singular Values of Random Matrices: A Tutorial Introduction, Princeton University, National University of Singapore, *Institute for Mathematical Sciences*, 2006.

[150] S. Verdu, Workshop on Random Matrix Theory and Wireless Communications, *The Interplay between Random Matrices and Information Theory* Chautauqua Park, Boulder, Colorado, Princeton University, 2008.

[151] X. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux, Lecture notes, UQAM, Montreal, Quebec, 1983.

[152] D. Voiculescu, Addition of certain non-commuting random variables, *Journal of Functional Analysis*, **66**, pp. 323–346, 1986.

[153] D. Voiculescu, Limit laws for random matrices and free products, *Invent.math.*,**104**, (1991), pp. 201–220.

[154] D. Voiculescu, Free probability theory: random matrices and von Neumannalgebras, *Proceedings of the ICM 1994, Birkhauser*, (1995), pp. 227–241.

[155] D. Voiculescu, Free Probability Theory, *Fields Institute Communications*, Vol. **12**, 1997.

[156] H. S. Wall, Analytic Theory of Continued Fractions, *Chelsea publishing company*, 1948.

[157] E. Weisstein, Morgan-Voyce Polynomials, MathWorld, available electronically at `http://mathworld.wolfram.com/Morgan-VoycePolynomials.html`, 2009.

[158] H. Widom, On the Limiting Distribution for the Longest Alternating Sequence in a Random Permutation, *Department of Mathematics University of California*, 2005.

[159] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Annals of Mathematics*, Vol. **62**, (1952), pp. 546 –564.

[160] A. S. Wightman, Eugene Paul Wigner 1902-1995, *Notices of the American Mathematical Society*, 1995.

[161] H. S. Wilf, Generatingfunctionology, *department of Mathematics, University of Pennsylvania, Pennsylvania*,1990.

[162] H. S. Wilf, The Longest increasing sub-sequences in pattern-restricted permutations, *Electronic Journal of Combinatorics. Electron. J. Combin., San Marcos, CA. ISSN 1077-8926* v9 i2. R12, 2003.

[163] J. Wishart, On the multivariate distribution in samples from a multivariate population, Vol. **62**, (1952), pp. 546 – 564.

[164] W.-J. Woan, Hankel matrices and lattice paths, *J. Integer Seq., Waterloo, ON. ISSN 1530-7638*, Vol. **4** (2001), Article 01.1.2.

[165] H. Widom, Homepage of Harold Widom available electronically at `http://math.ucsc.edu/ widom/widom.html`,2011.

[166] D. Zeilberger, Six études in generating functions, *International Journal of Computer Mathematics*, **29**, (2-4) (1989), pp. 201–215.

[167] American Statistical society website available electronically at `http://www.ams.org/`, 2011.

[168] Biography on the Institut de Physique Théorique website, available electronically at available electronically at `http://ipht.cea.fr/en/Phocea-SPhT/`, 2011.

[169] School of Mathematics and Statistics, University of St. Andrews, Scotland, Biographies, available electronically at `http://www-history.mcs.st-andrews.ac.uk.`,2011