The Fourier Analysis of Bezier Curves

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Abstract: ***We use the Discrete Fourier Transform to analyze Bezier curves.***

1 Introduction

The Bezier curve is a basic element of many computer graphic toolsets. This is not surprising, as it is easy to define and has well-understood mathematical properties. In this article, we apply the Discrete Fourier Transform to the construction of Bezier curves to gain more insight into their structure. As a Bezier curve is determined by its control polygon, this analysis is intimately linked to the Fourier analysis of the control polygon. An analysis of polygons in the plane has been carried out in [2]. Our work can be seen as a specialization of this.

2 Preliminaries

We let $P_0, P_1, \ldots, P_n$ be n+1 points in the plane. Let t be a parameter, normally an element of [0,1]. Then the Bezier curved defined by the points $P_0, P_1, \ldots, P_n$ is defined by the vector equation

$$P(t) = \sum_{k=0}^{n} C_k^n (1-t)^k t^{n-k} P_i \qquad (1)$$

We recall that the polygon determined by the points $P_0, P_1, \ldots, P_n$ is called the control polygon of the curve. For instance, the line segments P_0P_1 and $P_{n-1}P_n$ are tangents to the start, respectively, the end of the curve.

The above equation results from an iterated process of subdividing line segments in the ratio t:(1-t), starting with the line segments P_kP_{k+1}, k = 0…n. In the case of n = 3, for instance, we have

$$\begin{aligned} P(t) &= (1-t)^3P_0+3t(1-t)^2P_1+3t^2P_2+t^3P_3 \\ &= \begin{pmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3t & 3t^2 & t^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \qquad (2) \end{aligned}$$

We recognize in the square matrix the inverse of the 4×4 binomial matrix. This is the matrix with general form

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

whose (j,k)-th element is equal to C(j-1,k-1).

It is instructive to see the effect of $\mathbf{B}^{-1}$ on the power basis $(1,x,x^2,\ldots)$. In the 4×4 case, for instance, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ x-1 \\ (x-1)^2 \\ (x-1)^3 \end{pmatrix}$$

We now introduce the Discrete Fourier Transform [3]. If we let $f_0, f_1 \ldots f_n$ be n+1 numbers, real or complex, then their Discrete Fourier Transform is the set of n+1 numbers

$$\hat{f}_j = \sum_{k=0}^{n} f_k e^{-2\pi ijk/(n+1)} \qquad (3)$$

where $i = [\sqrt{(-1)}]$. The numbers $[\hat{f}]_j$ are in general complex numbers. We can invert this transformation as follows :

$$f_k = \frac{1}{n+1} \sum_{j=0}^{n} \hat{f}_j e^{2\pi ijk/(n+1)} \qquad (4)$$

We shall refer to the (n+1)×(n+1) matrix with (j,k)-th element $e^{-2\pi ijk/(n+1)}$ as the (discrete) Fourier matrix **F**. For instance, in the case n = 3, we have

$$\begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{pmatrix}$$

3 Analyzing Bezier curves

We now apply the foregoing to Bezier curves in the plane. We observe that a point P in the plane can be regarded as a complex number, so that taking the Fourier transform of a set of points makes sense. We let $\omega_j = e^{-2\pi ij/(n+1)}$ be a solution of the equation

$$z^{n+1}-1 = 0 \qquad (5)$$

Then

$$\begin{aligned} P(t) &= \sum_{k=0}^{n} C_k^n t^k (1-t)^{n-k} P_k \\ &= \sum_{k=0}^{n} C_k^n t^i (1-t)^{n-k} \frac{1}{n+1} \sum_{j=0}^{n} \hat{P}_j \omega_j^k \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{n} C_k^n t^k \omega_j^k (1-t)^{n-k} \hat{P}_j \\ &= \frac{1}{n+1} \sum_{j=0}^{n} \hat{P}_j \sum_{k=0}^{n} C_k^n (t\omega_j)^k (1-t)^{n-k} \\ &= \frac{1}{n+1} \sum_{j=0}^{n} \hat{P}_j ((1-t).1+t\omega_j)^n \qquad (6) \end{aligned}$$

This exhibits P(t) as a linear combination of ``basic" Bezier curves, since we have

$$\begin{aligned} B_j(t) &= ((1-t).1+t\omega_j)^n \\ &= \sum_{k=0}^{n} C_k^n t^k (1-t)^{n-k} \omega_j^k \\ &= \sum_{k=0}^{n} C_k^n t^k (1-t)^{n-k} \omega_k^j \qquad (7) \end{aligned}$$

Hence

$$P(t) = \frac{1}{n+1} \sum_{j=0}^{n} \hat{P}_j B_j(t) \qquad (8)$$

for the Bezier curve with control polygon $(P_0, P_1, \ldots, P_n)$.

These basic Bezier curves have control polygons determined by the numbers $\{\omega_k^j\}$, and hence the geometry of these curves is determined by the geometry of the corresponding star-polygons [1].

4 A worked example

We let **B** be the 4×4 binomial matrix :

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

Then its inverse is given by

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

The 4×4 Fourier matrix **F** is given by

$$\mathbf{F} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

with inverse

$$\mathbf{F}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_1 & \omega_2 & \omega_3 \\ 1 & \omega_1^2 & \omega_2^2 & \omega_3^2 \\ 1 & \omega_1^3 & \omega_2^3 & \omega_3^3 \end{pmatrix}$$

Then (2) says that

$$\begin{aligned} P(t) &= \begin{pmatrix} 1 & 3t & 3t^2 & t^3 \end{pmatrix} \mathbf{B}^{-1} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3t & 3t^2 & t^3 \end{pmatrix} \mathbf{B}^{-1}\mathbf{F}^{-1}\mathbf{F} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3t & 3t^2 & t^3 \end{pmatrix} \mathbf{B}^{-1}\mathbf{F}^{-1} \begin{pmatrix} \hat{P}_0 \\ \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3t & 3t^2 & t^3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \omega_1-1 & \omega_2-1 & \omega_3-1 \\ 0 & (\omega_1-1)^2 & (\omega_2-1)^2 & (\omega_3-1)^2 \\ 0 & (\omega_1-1)^3 & (\omega_2-1)^3 & (\omega_3-1)^3 \end{pmatrix} \begin{pmatrix} \hat{P}_0 \\ \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \end{pmatrix} \qquad (9) \end{aligned}$$

Hence

$$\begin{aligned} 4P(t) &= \hat{P}_0 + \hat{P}_1 (1+t(\omega_1-1))^3 + \hat{P}_2 (1+t(\omega_2-1))^3 + \hat{P}_3 (1+t(\omega_3-1))^3 \\ &= \hat{P}_0 + \hat{P}_1 ((1-t).1+t\omega_1)^3 + \hat{P}_2 ((1-t).1+t\omega_2)^3 + \hat{P}_3 ((1-t).1+t\omega_3)^3 \qquad (10) \end{aligned}$$

Here, we have, for instance

$$((1-t).1+t\omega_1)^3 = (1-t)^3.1+3(1-t)^2t\omega_1+3(1-t)t^2\omega_1^2+t^3\omega_1^3 \qquad (11)$$

which is the Bezier curve with the four roots of unity $1,\omega_1,\omega_1^2,\omega_1^3$ or 1, -i, -1, i as control points. We then have

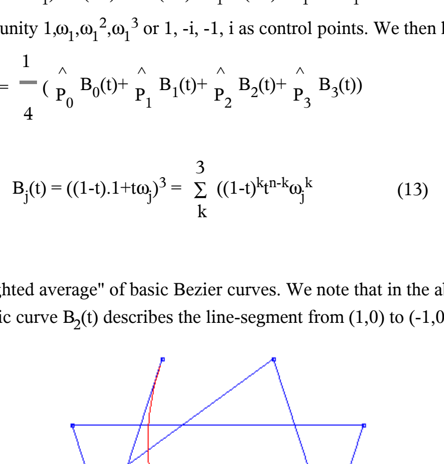
$$P(t) = \frac{1}{4}(\hat{P}_0 B_0(t) + \hat{P}_1 B_1(t) + \hat{P}_2 B_2(t) + \hat{P}_3 B_3(t)) \qquad (12)$$

where

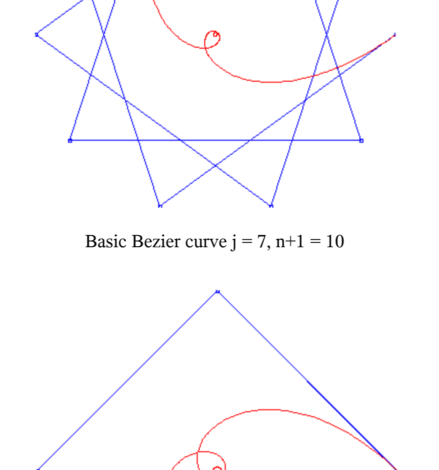
$$B_j(t) = ((1-t).1+t\omega_j)^3 = \sum_{k}^{3} ((1-t)^k t^{n-k} \omega_j^k \qquad (13)$$

with $\omega_j = e^{-2\pi ij/4}$.

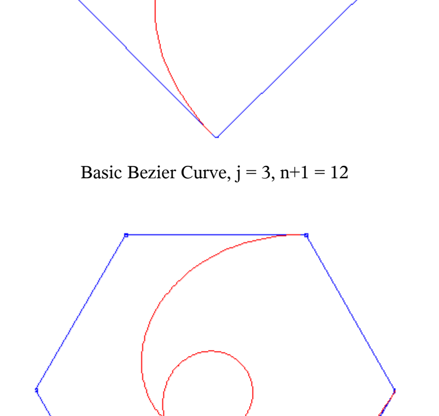
This shows that the Bezier curve P(t) is a ``weighted average" of basic Bezier curves. We note that in the above case, the basic curve $B_0(t)$ degenerates to the single point (1,0) in the plane, while the basic curve $B_2(t)$ describes the line-segment from (1,0) to (-1,0) and back again.



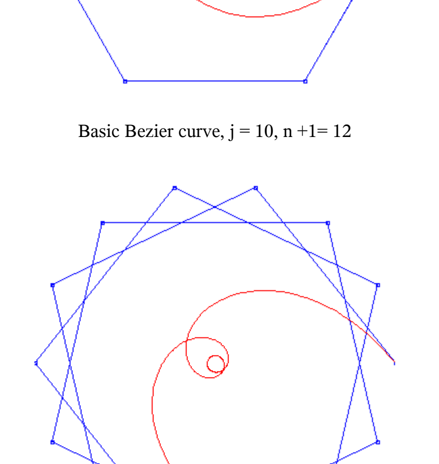
Basic Bezier curve j = 7, n+1 = 10



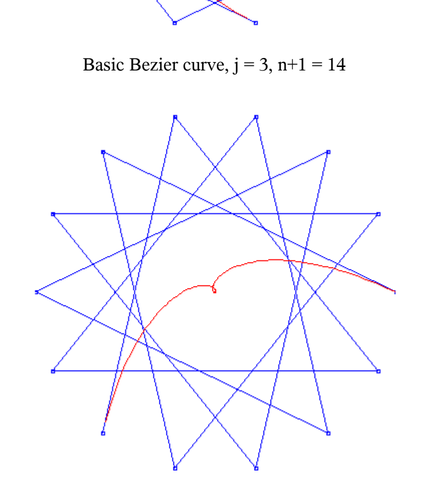
Basic Bezier Curve, j = 3, n+1 = 12



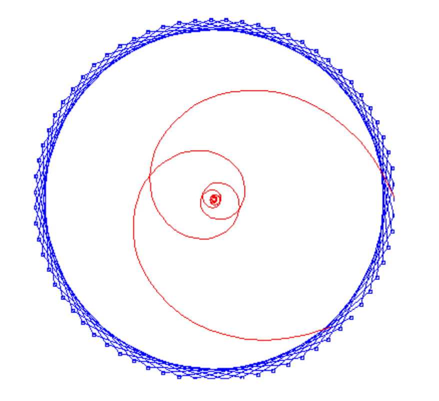
Basic Bezier curve, j = 10, n +1= 12



Basic Bezier curve, j = 3, n+1 = 14



Basic Bezier curve, j = 5, n+1 = 14



Basic Bezier curve, j = 8, n +1= 71

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