On integer sequences associated to the cyclic and complete graphs

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Abstract

We study integer sequences associated to the cyclic graph C_r and the complete graph K_r. Fourier techniques are used to characterise the sequences that count walks of length n on both these families of graphs. In the case of the cyclic graph, we show that these sequences are associated to an induced colouring of Pascal's triangle. This extends previous results concerning the Jacobsthal numbers.

1 Preliminaries

In [1] we studied the Jacobsthal numbers, and showed that they provide a decomposition (or colouring) of Pascal's triangle. In this article, we shall relate this decomposition to the cyclic graph C_3, and then we will generalize the result to the general cyclic graph on r vertices C_r. In addition, we will study integer sequences related to K_r, the complete graph on n vertices.

Many of the integer sequences that we will encounter are to be found in The On-Line Encyclopedia of Integer Sequences (OEIS) [7], [8].

We begin with a brief recall of the relevant results of [1]. For this, we let the sequence of numbers $J(n)$, be the solution of the recurrence

$$a_n = a_{n-1} + 2a_{n-2}, \ a_0 = 0, \ a_1 = 1,$$

with

n	0	1	2	3	4	5	6	...
$J(n)$	0	1	1	3	5	11	21	...

$$J(n) = \frac{2^n}{3} - \frac{(-1)^n}{3}. \tag{1.1}$$

These are the Jacobsthal numbers [10]. In section 4, we shall relate these numbers to a count of walks on C_3. They have many other applications, as detailed for instance in [3]

and [4]. They appear in the OEIS as A001045. When we change the initial conditions to $a_0 = 1$, $a_1 = 0$, we get a sequence which we will denote by $J_1(n)$, A078008, given by

n	0	1	2	3	4	5	6	$\ldots$
$J_1(n)$	1	0	2	2	6	10	22	$\ldots$

$$J_1(n) = \frac{2^n}{3} + \frac{2(-1)^n}{3}. \tag{1.2}$$

We see that

$$2^n = 2J(n) + J_1(n). \tag{1.3}$$

What is less obvious is that the Jacobsthal numbers are sums of binomial coefficients. In fact, we have

$$J_1(n) = \sum_{(n+k) \bmod 3 = 0} \binom{n}{k}$$

$$J(n) = \sum_{(n+k) \bmod 3 = 1} \binom{n}{k}$$

$$J(n) = \sum_{(n+k) \bmod 3 = 2} \binom{n}{k}.$$

In [1], an inductive argument was used to prove this. We give below a more direct proof, using an identity that will be used later for a more general result. This is

$$\sum_{k \equiv l (\bmod m)} \binom{n}{k} = \frac{1}{m} \sum_{j=0}^{m-1} (1 + \omega^j)^n \omega^{-lj} \tag{1.4}$$

where $\omega = e^{2\pi i/m}$. We then have, for $m = 3$,

$$\begin{aligned}
\sum_{k \equiv -n (\bmod 3)} \binom{n}{k} &= \frac{1}{3} \sum_{j=0}^{2} (1 + \omega^j)^n \omega^{nj} \\
&= \frac{1}{3} (2^n + (1 + \omega)^n \omega^n + (1 + \omega^2)^n \omega^{2n}) \\
&= \frac{1}{3} (2^n + (-\omega^2)^n \omega^n + (-\omega)^n \omega^{2n}) \\
&= \frac{1}{3} (2^n + (-1)^n \omega^{2n} \omega^n + (-1)^n \omega^n \omega^{2n}) \\
&= \frac{1}{3} (2^n + (-1)^n (\omega^{3n} + \omega^{3n})) \\
&= \frac{1}{3} (2^n + 2(-1)^n) = J_1(n).
\end{aligned}$$

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In like manner, we have

$$\sum_{k\equiv(-n+1)(\bmod 3)} \binom{n}{k} = \frac{1}{3}\sum_{j=0}^{2}(1+\omega^j)^n\omega^{(n-1)j}$$

$$= \frac{1}{3}(2^n + (1+\omega)^n\omega^{n-1} + (1+\omega^2)^n\omega^{2(n-1)})$$

$$= \frac{1}{3}(2^n + (-\omega^2)^n\omega^{n-1} + (-\omega)^n\omega^{2(n-1)})$$

$$= \frac{1}{3}(2^n + (-1)^n\omega^{2n}\omega^{n-1} + (-1)^n\omega^n\omega^{2(n-1)})$$

$$= \frac{1}{3}(2^n + (-1)^n(\omega^{3n-1} + \omega^{3n-2}))$$

$$= \frac{1}{3}(2^n + (-1)^n(\omega^{-1} + \omega^{-2}))$$

$$= \frac{1}{3}(2^n + (-1)^n(\omega^2 + \omega^1))$$

$$= \frac{1}{3}(2^n - (-1)^n) = J(n).$$

The third identity is proven in similar fashion.

Since

$$2^n = \sum_{k=0}^{n}\binom{n}{k}$$

we immediately obtain the required decomposition property of Pascal's triangle. We shall express this in terms of lower-triangular matrices, where we identify Pascal's triangle with the Binomial Matrix $\mathbf{B}$ whose (n,k)-th element is equal to $\binom{n}{k}$:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We shall call a lower-triangular matrix a *zero-binomial* matrix if its elements are either 0 or binomial coefficients. We can then state the relevant result of [1] as follows.

Theorem 1.1. *There exists a partition*

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} + \mathbf{B_2}$$

where the $\mathbf{B_i}$ are zero-binomial matrices with row sums equal to $J_1(n)$, $J(n)$, $J(n)$ respectively for $i = 0,\ldots,2$.

We have
$$\mathbf{B}_i = ([n + k \equiv i \bmod 3]\binom{n}{k}),$$

where we use the Iverson notation $[P(j)]$ to denote 1 if the logical expression $P(j)$ is true, and 0 if it is false [5].

We can see this decomposition in the following coloured rendering of $\mathbf{B}$.

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 3 \\ 6 & 5 & 5 \\ 10 & 11 & 11 \\ \cdot & \cdot & \cdot \end{matrix}$$

2   Graphs

We now wish to draw a link between the foregoing and graph theory. To this end, we recall a number of relevant definitions [6], [12]. A graph X is a triple consisting of a vertex set $V = V(X)$, an edge set $E = E(X)$, and a map that associates to each edge two vertices (not necessarily distinct) called its endpoints. A *loop* is an edge whose endpoints are equal. To any graph, we may associate the *adjacency matrix*, which is an $n \times n$ matrix, where $|V| = n$ with rows and columns indexed by the elements of the vertex set V and the (x, y)-th element is the number of edges connecting x and y. As defined, graphs are undirected, so this matrix is symmetric. We will restrict ourselves to *simple* graphs, with no loops or multiple edges.

The *degree* of a vertex v, denoted $\deg(v)$, is the number of edges incident with v. A graph is called k-regular if every vertex has degree k. The adjacency matrix of a *k-regular* graph will then have row sums and column sums equal to k.

If $x, y \in V$ then an *x-y walk* in X is a (loop-free) finite alternating sequence

$$x = x_0, e_1, x_1, e_2, x_2, e_3, \ldots, e_{r-1}, x_{r-1}, e_r, x_r = y$$

of vertices and edges from X, starting at the vertex x and ending at the vertex y and involving the r edges $e_i = \{x_{i-1}, x_i\}$, where $1 \leq i \leq r$. The *length* of this walk is r. If $x = y$, the walk is *closed*, otherwise it is *open*. If no edge in the x-y walk is repeated, then the walk is called an *x-y trail*. A closed x-x trail is called a *circuit*. If no vertex of the x-y walk is repeated, then the walk is called an *x-y path*. When $x = y$, a closed path is called a *cycle*. The number of walks from x to y of length n is given by the x, y-th entry of $\mathbf{A}^n$, where $\mathbf{A}$ is the adjacency matrix of the graph X.

The *cyclic graph* C_r on r vertices is the graph with r vertices and r edges such that if we number the vertices $0, 1, \ldots, r - 1$, then vertex i is connected to the two adjacent vertices $i + 1$ and $i - 1 \pmod r$. The *complete graph* K_r on r vertices is the loop-free graph where for all $x, y \in V, x \neq y$, there is an edge $\{x, y\}$.

We note that $C_3 = K_3$.

A final graph concept that will be useful is that of the *chromatic polynomial* of a graph. If $X = (V, E)$ is an undirected graph, a *proper colouring* of X occurs when we colour the vertices of X so that if $\{x, y\}$ is an edge in X, then x and y are coloured with different colours. The minimum number of colours needed to properly colour X is called the *chromatic number* of X and is written $\chi(X)$. For $k \in \mathbf{Z}^+$, we define the polynomial $P(X, k)$ as the number of different ways that we can properly colour the vertices of X with k colours.

For example,

$$P(K_r, k) = k(k-1) \ldots (k - r + 1) \tag{2.1}$$

and

$$P(C_r, k) = (k-1)^r + (-1)^r (k-1). \tag{2.2}$$

3 Notation

In this note, we shall employ the following notation. r will denote the number of vertices in a graph. Note that the adjacenty matrix A of a graph with r vertices will then be an $r \times r$ matrix. We shall reserve the number variable n to index the elements of a sequence, as in a_n, the n-th element of the sequence $\mathbf{a} = \{a_n\}$, or as the n-th power of a number or a matrix (normally this will be related to the n-th term of a sequence). The notation 0^n signifies the integer sequence with generating function 1, which has elements $1, 0, 0, 0, \ldots$.

Note that the Binomial matrix $\mathbf{B}$ and the Fourier matrix $\mathbf{F}_r$ (see below) are indexed from $(0, 0)$, that is, the leftmost element of the first row is the $0, 0$-th element. This allows us to give the simplest form of their general n, k-th element ($\binom{n}{k}$ and $e^{-\frac{2\pi i n k}{r}}$ respectively).

The adjacency matrix of a graph, normally denoted by $\mathbf{A}$, will be indexed as usual from $(1, 1)$. Similarly the eigenvalues of the adjacency matrix will be labelled $\lambda_1, \lambda_2, \ldots, \lambda_r$.

4 Circulant matrices

We now provide a quick overview of the theory of circulant matrices [2], as these will be encountered shortly. An $r \times r$ circulant matrix $\mathbf{C}$ is a matrix whose rows are obtained by shifting the previous row one place to the right, with wraparound, in the following precise sense. If the elements of the first row are $(c_1, \ldots, c_r)$ then

$$c_{jk} = c_{k-j+1}$$

where subscripts are taken modulo n. Circulant matrices are diagonalised by the discrete Fourier transform, whose matrix $\mathbf{F}_r$ is defined as follows : let $\omega(r) = e^{-2\pi i/r}$ where $i = \sqrt{-1}$. Then $\mathbf{F}_r$ has i, j-th element $\omega^{i \cdot j}$, $0 \leq i, j \leq r - 1$.

We can write the above matrix as $\mathbf{C} = \text{circ}(c_1, \ldots, c_r)$. The permutation matrix $\pi = \text{circ}(0, 1, 0, \ldots, 0)$ plays a special role. If we let p be the polynomial $p(x) = c_1 + c_2 x + \ldots + c_r x^{r-1}$, then $\mathbf{C} = p(\pi)$.

We have, for $\mathbf{C}$ a circulant matrix,

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^{-1} \boldsymbol{\Lambda} \mathbf{F}, \\ \boldsymbol{\Lambda} &= \text{diag}(p(1), p(\omega), \ldots, p(\omega^{r-1})). \end{aligned}$$

The i-th eigenvalue of $\mathbf{C}$ is $\lambda_i = p(\omega^{i-1})$, $1 \leq i \leq r$.

5 The graph C_3 and Jacobsthal numbers

We let $\mathbf{A}$ be the adjacency matrix of the cyclic graph C_3. We have

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

We note that this matrix is circulant. We shall be interested both in the powers $\mathbf{A}^n$ of $\mathbf{A}$ and its eigenvalues. There is the following connection between these entities:

$$\mathrm{trace}(\mathbf{A}^n) = \sum_{j=1}^{r} \lambda_j^n$$

where $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of $\mathbf{A}$. Here, $r = 3$. In order to obtain the eigenvalues of $\mathbf{A}$, we use $\mathbf{F}_3$ to diagonalize it. We obtain

$$\mathbf{F}_3^{-1}\mathbf{A}\mathbf{F}_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We immediately have

$$\mathrm{trace}(\mathbf{A}^n) = 2^n + 2(-1)^n$$

Comparing with (1.2), we have

Proposition 5.1. $J_1(n) = \frac{1}{3}trace(\mathbf{A}^n)$

Our next observation relates $J_1(n)$ to 3-colourings of C_r. For this, we recall that $P(C_r, k) = (k-1)^r + (-1)^r(k-1)$. Letting $k = 3$, we get $P(C_r, 3) = 2^r + 2(-1)^r$.

Proposition 5.2. $J_1(r) = \frac{1}{3}P(C_r, 3)$.

Since $\mathbf{A}$ is circulant, it and its powers $\mathbf{A}^n$ are determined by the elements of their first rows. We shall look at the integer sequences determined by the first row elements of $\mathbf{A}^n$ - that is, we shall look at the sequences $a_{1j}^{(n)}$, for $j = 1, 2, 3$.

Theorem 5.1.

$$a_{11}^{(n)} = J_1(n), \ a_{12}^{(n)} = J(n), \ a_{13}^{(n)} = J(n)$$

Proof. We use the fact that

$$\mathbf{A}^n = \mathbf{F}^{-1} \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix} \mathbf{F} \tag{5.1}$$

Then

$$
\begin{aligned}
\mathbf{A}^n &= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 2^n & (-1)^n & (-1)^n \\ \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 2^n + 2(-1)^n & (2^n + (-1)^n\omega_3 + (-1)^n\omega_3^2) & (2^n + (-1)^n\omega_3^2 + (-1)^n\omega_3^4) \\ \vdots & \vdots & \vdots \end{pmatrix}
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
a_{11}^{(n)} &= (2^n + 2(-1)^n)/3 \\
a_{12}^{(n)} &= (2^n + (-1)^n\omega_3 + (-1)^n\omega_3^2)/3 \\
a_{13}^{(n)} &= (2^n + (-1)^n\omega_3^2 + (-1)^n\omega_3^4)/3.
\end{aligned}
$$

The result now follows from the fact that

$$
1 + \omega_3 + \omega_3^2 = 1 + \omega_3^2 + \omega_3^4 = 0.
$$

$\square$

Corollary 5.1. *The Jacobsthal numbers count the number of walks on C_3. In particular, $J_1(n)$ counts the number of closed walks of length n on the edges of a triangle based at a vertex. $J(n)$ counts the number of walks of length n starting and finishing at different vertices.*

An immediate calculation gives

Corollary 5.2.
$$
2^n = a_{11}^{(n)} + a_{13}^{(n)} + a_{13}^{(n)}.
$$

The identity
$$
2^n = 2J(n) + J_1(n)
$$
now becomes a consequence of the identity
$$
2^n = a_{11}^{(n)} + a_{13}^{(n)} + a_{13}^{(n)}.
$$

This is a consequence of the fact that C_3 is 2-regular. We have arrived at a link between the Jacobsthal partition (or colouring) of Pascal's triangle and the cyclic graph C_3. We recall that this comes about since $2^n = J_1(n) + 2J(n)$, and the fact that $J_1(n)$ and $J(n)$ are expressible as sums of binomial coefficients.

We note that although $C_3 = K_3$, it is the cyclic nature of the graph and the fact that it is 2-regular that links it to this partition. We shall elaborate on this later in the article.

In terms of ordinary generating functions, we have the identity

$$\frac{1}{1-2x} = \frac{1-x}{(1+x)(1-2x)} + \frac{x}{(1+x)(1-2x)} + \frac{x}{(1+x)(1-2x)}$$

and in terms of exponential generating functions, we have

$$\exp(2x) = \frac{2}{3}\exp(-x)(1 + \exp(\frac{3x}{2})\sinh(\frac{3x}{2})) + 2\frac{2}{3}\exp(\frac{x}{2})\sinh(\frac{3x}{2})$$

or more simply,

$$\exp(2x) = \frac{1}{3}(\exp(2x) + 2\exp(-x)) + 2\frac{1}{3}(\exp(2x) - \exp(-x)).$$

An examination of the calculations above and the fact that $\mathbf{F}$ is symmetric allows us to state

Corollary 5.3.

$$\begin{pmatrix} a_{11}^{(n)} \\ a_{12}^{(n)} \\ a_{13}^{(n)} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}\begin{pmatrix} 2^n \\ (-1)^n \\ (-1)^n \end{pmatrix}$$

In fact, this result can be easily generalized to give the following

$$\begin{pmatrix} a_{11}^{(n)} \\ a_{12}^{(n)} \\ \vdots \\ a_{1r}^{(n)} \end{pmatrix} = \frac{1}{r}\mathbf{F}_r\begin{pmatrix} \lambda_1^n \\ \lambda_2^n \\ \vdots \\ \lambda_r^n \end{pmatrix} \qquad (5.2)$$

so that

$$\mathbf{A}^n = \mathrm{circ}(a_{11}^{(n)}, a_{12}^{(n)}, \ldots, a_{1r}^{(n)}) = \mathrm{circ}(\frac{1}{r}\mathbf{F}_r(\lambda_1^n, \ldots, \lambda_r^n)').$$

It is instructive to work out the generating function of these sequences. For instance, we have

$$a_{12}^{(n)} = 1.2^n + \omega.(-1)^n + \omega^2.(-1)^n.$$

This implies that $a_{12}^{(n)}$ has generating function

$$\begin{aligned} g_{12}(x) &= \frac{1}{1-2x} + \frac{\omega}{1+x} + \frac{\omega^2}{1+x} \\ &= \frac{(1+x)^2 + \omega(1+x)(1-2x) + \omega^2(1-2x)(1+x)}{(1-2x)(1+x)(1+x)} \\ &= \frac{3x(1+x)}{3(1-2x)(1+x)^2} \\ &= \frac{x}{1-x-2x^2}. \end{aligned}$$

This is as expected, but it also highlights the importance of

$$(1-2x)(1+x)(1+x) = (1-\lambda_1 x)(1-\lambda_2 x)(1-\lambda_3 x) = 1 - 3x^2 - 2x^3.$$

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Hence each of these sequences not only obeys the Jacobsthal recurrence

$$a_n = a_{n-1} + 2a_{n-2}$$

but also

$$a_n = 3a_{n-2} + 2a_{n-3}.$$

Of course,

$$1 - 3x^2 - 2x^3 = (1 - p(\omega_3^0)x)(1 - p(\omega_3^1)x)(1 - p(\omega_3^2)x)$$

where $p(x) = x + x^2$.

6 The case of C_4

For the cyclic graph on four vertices C_4 we have the following adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \tag{6.1}$$

We can carry out a similar analysis as for the case $n = 3$. Using $\mathbf{F}$ to diagonalize $\mathbf{A}$, we obtain

$$\mathbf{F}^{-1}\mathbf{A}\mathbf{F} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this, we can immediately deduce the following result.

Theorem 6.1.

$$\frac{1}{4}\text{trace}\mathbf{A}^n = \frac{1}{4}(2^n + (-2)^n + 2.0^n) = 1, 0, 2, 0, 8, 0, 32, \ldots$$

Theorem 6.2.

$$\begin{aligned}
a_{11}^{(n)} &= (2^n + (-2)^n + 2.0^n)/4 = 1, 0, 2, 0, 8, 0, 32, \ldots \\
a_{12}^{(n)} &= (2^n - (-2)^n)/4 = 0, 1, 0, 4, 0, 16, 0 \ldots \\
a_{13}^{(n)} &= (2^n + (-2)^n - 2.0^n)/4 = 0, 0, 2, 0, 8, 0, 32, \ldots \\
a_{14}^{(n)} &= (2^n - (-2)^n)/4 = 0, 1, 0, 4, 0, 16, 0, \ldots
\end{aligned}$$

Proof. We have

$$\mathbf{A}^n = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 & 0 \\ 0 & 0^n & 0 & 0 \\ 0 & 0 & (-2)^n & 0 \\ 0 & 0 & 0 & 0^n \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

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From this we obtain

$$
\begin{aligned}
a_{11}^{(n)} &= (2^n + 0^n + (-2)^n + 0^n)/4 = (2^n + (-2)^n + 2.0^n)/4 \\
a_{12}^{(n)} &= (2^n - i.0^n - (-2)^n + i.0^n)/4 = (2^n - (-2)^n)/4 \\
a_{13}^{(n)} &= (2^n - 1.0^n + (-2)^n - 1.0^n)/4 = (2^n + (-2)^n - 2.0^n)/4 \\
a_{14}^{(n)} &= (2^n + i.0^n - (-2)^n - i.0^n)/4 = (2^n - (-2)^n)/4.
\end{aligned}
$$

$\square$

Corollary 6.1. *The sequences above count the number of walks on the graph C_4. In particular, $a_{11}^{(n)}$ counts the number of closed walks on the edges of a quadrilateral based at a vertex.*

An easy calculation also gives us the important

Corollary 6.2.
$$
2^n = a_{11}^{(n)} + a_{12}^{(n)} + a_{13}^{(n)} + a_{14}^{(n)}.
$$

In terms of ordinary generating functions of the sequences $a_{11}^{(n)}$, $a_{12}^{(n)}$, $a_{13}^{(n)}$, $a_{14}^{(n)}$, we have the following algebraic expression

$$
\frac{1}{1-2x} = \frac{1}{1-2x}\left(\frac{1-2x^2}{1+2x} + \frac{x}{1+2x} + \frac{2x^2}{1+2x} + \frac{x}{1+2x} \right).
$$

Anticipating the general case, we can state

Theorem 6.3. *There exists a partition*

$$
\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} + \mathbf{B_2} + \mathbf{B_3}
$$

where the $\mathbf{B_i}$ are zero-binomial matrices with row sums equal to $a_{11}^{(n)}$, $a_{12}^{(n)}$, $a_{13}^{(n)}$, $a_{14}^{(n)}$, respectively, for $i = 0\dots3$.

In fact, we have

$$
\begin{aligned}
a_{11}^{(n)} &= \sum_{2k-n\equiv 0 \bmod 4} \binom{n}{k} \\
a_{12}^{(n)} &= \sum_{2k-n\equiv 1 \bmod 4} \binom{n}{k} \\
a_{13}^{(n)} &= \sum_{2k-n\equiv 2 \bmod 4} \binom{n}{k} \\
a_{14}^{(n)} &= \sum_{2k-n\equiv 3 \bmod 4} \binom{n}{k}.
\end{aligned}
$$

We shall provide a proof for this later, when we look at the general case. Each of these sequences satisfy the recurrence
$$
a_n = 4a_{n-2}.
$$

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We can see the decomposition induced from C_4 in the following coloured rendering of $\mathbf{B}$.

$$
\mathbf{B} = \left(
\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\
1 & 4 & 6 & 4 & 1 & 0 & 0 & \cdots \\
1 & 5 & 10 & 10 & 5 & 1 & 0 & \cdots \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\right)
\quad
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
2 & 0 & 2 & 0 \\
0 & 4 & 0 & 4 \\
8 & 0 & 8 & 0 \\
0 & 16 & 0 & 16 \\
32 & 0 & 32 & 0 \\
. & . & . & .
\end{array}
$$

7 The case of C_5

This case is worth noting, in the context of integer sequences, as there is a link with the Fibonacci numbers. For the cyclic graph on five vertices C_5 we have the following adjacency matrix

$$
\mathbf{A} = \left(
\begin{array}{ccccc}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{array}
\right)
\tag{7.1}
$$

Diagonalizing with $\mathbf{F}$, we obtain

$$
\mathbf{\Lambda} = \left(
\begin{array}{ccccc}
2 & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{5}}{2} - \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & -\frac{\sqrt{5}}{2} - \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{5}}{2} - \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2} - \frac{1}{2}
\end{array}
\right)
$$

Proposition 7.1. $\frac{1}{5}\mathrm{trace}(\mathbf{A}^n) = (2^n + 2(-1)^n(F(n+1) + F(n-1)))/5$.

Proof. We have $\frac{1}{5}\mathrm{trace}(\mathbf{A}^n) = (2^n + 2(\frac{\sqrt{5}}{2} - \frac{1}{2})^n + 2(-\frac{\sqrt{5}}{2} - \frac{1}{2})^n)/5$. An easy manipulation produces the result. $\qquad\square$

This sequence is [A054877](#).

Theorem 7.1.

$$a_{11}^{(n)} = (2^n + 2(\frac{\sqrt{5}}{2} - \frac{1}{2})^n + 2(-\frac{\sqrt{5}}{2} - \frac{1}{2})^n)/5$$

$$a_{12}^{(n)} = (2^n + (\frac{\sqrt{5}}{2} - \frac{1}{2})^{n+1} + (-\frac{\sqrt{5}}{2} - \frac{1}{2})^{n+1})/5$$

$$a_{13}^{(n)} = (2^n + (\frac{\sqrt{5}}{2} - \frac{1}{2})^n(\frac{\sqrt{5}}{2} + \frac{1}{2}) + (-\frac{\sqrt{5}}{2} - \frac{1}{2})^n(\frac{\sqrt{5}}{2} - \frac{1}{2}))/5$$

$$a_{14}^{(n)} = (2^n + (\frac{\sqrt{5}}{2} - \frac{1}{2})^n(\frac{\sqrt{5}}{2} + \frac{1}{2}) + (-\frac{\sqrt{5}}{2} - \frac{1}{2})^n(\frac{\sqrt{5}}{2} - \frac{1}{2}))/5$$

$$a_{15}^{(n)} = (2^n + (\frac{\sqrt{5}}{2} - \frac{1}{2})^{n+1} + (-\frac{\sqrt{5}}{2} - \frac{1}{2})^{n+1})/5.$$

Proof. The result follows from the fact that the first row of $\mathbf{A}^n$ is given by $\frac{1}{5}\mathbf{F}(\lambda_1^n, \lambda_2^n, \lambda_3^n, \lambda_4^n, \lambda_5^n)'$. $\square$

Corollary 7.1. *The sequences in Theorem 7.1 count walks on C_5. In particular, the sequence $a_{11}^{(n)}$ counts closed walks of length n along the edges of a pentagon, based at a vertex.*

We note that $a_{12}^{(n)} = a_{15}^{(n)}$. This is [A052964](). Similarly $a_{13}^{(n)} = a_{14}^{(n)}$. This is (the absolute value of) [A084179]().

An easy calculation gives us the important result

Corollary 7.2.

$$2^n = a_{11}^{(n)} + a_{12}^{(n)} + a_{13}^{(n)} + a_{14}^{(n)} + a_{15}^{(n)}.$$

In terms of the ordinary generating functions for these sequences, we obtain the following algebraic identity

$$\frac{1}{1-2x} = \frac{1}{1-2x}\left(\frac{1-x-x^2}{1+x-x^2} + \frac{2x(1-x)}{1+x-x^2} + \frac{2x^2}{1+x-x^2}.\right)$$

Anticipating the general case, we can state the

Theorem 7.2. *There exists a partition*

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} + \mathbf{B_2} + \mathbf{B_3} + \mathbf{B_4}$$

where the $\mathbf{B_i}$ are zero-binomial matrices with row sums equal to $a_{11}^{(n)}$, $a_{12}^{(n)}$, $a_{13}^{(n)}$, $a_{14}^{(n)}$, $a_{15}^{(n)}$, respectively, for $i = 0 \ldots 4$.

In fact, we have

$$a_{11}^{(n)} = \sum_{2k-n\equiv 0 \text{ mod } 5} \binom{n}{k}$$

$$a_{12}^{(n)} = \sum_{2k-n\equiv 1 \text{ mod } 5} \binom{n}{k}$$

$$a_{13}^{(n)} = \sum_{2k-n\equiv 2 \text{ mod } 5} \binom{n}{k}$$

$$a_{14}^{(n)} = \sum_{2l-n\equiv 3 \text{ mod } 5} \binom{n}{k}$$

$$a_{15}^{(n)} = \sum_{2k-n\equiv 4 \text{ mod } 5} \binom{n}{k}.$$

We note that

$$(1 - p(\omega_5^0)x)(1 - p(\omega_5^1)x)(1 - p(\omega_5^2)x)(1 - p(\omega_5^3)x)(1 - p(\omega_5^4)x) = 1 - 5x^3 + 5x^4 - 2x^5$$

for $p(x) = x + x^4$ which implies that each of the sequences satisfies the recurrence

$$a_n = 5a_{n-3} - 5a_{n-4} + 2a_{n-5}.$$

8   The General Case of C_r

We begin by remarking that since C_r is a 2-regular graph, its first eigenvalue is 2. We have seen explicit examples of this in the specific cases studied above. We now let $\mathbf{A}$ denote the adjacency matrix of C_r. We have $\mathbf{A} = p(\pi)$ where $p(x) = x + x^{r-1}$, so $\mathbf{A} = \text{circ}(0, 1, 0, \ldots, 0, 1)$.

Theorem 8.1.

$$2^n = \sum_{j=1}^{r} a_{1j}^{(n)}$$

where $a_{1j}^{(n)}$ is the j-th element of the first row of $\mathbf{A}^n$.

Proof. We have

$$(a_{1j}^{(n)})_{1\leq j\leq r} = \frac{1}{r}\mathbf{F}(\lambda_1^n, \lambda_2^n, \ldots, \lambda_r^n)'$$

$$= \frac{1}{r}(\sum_{k=1}^{r} \lambda_k^n \omega^{(j-1)(k-1)}).$$

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Hence we have

$$\sum_{j=1}^{r} a_{1j}^{(n)} = \frac{1}{n} \sum_{j=1}^{r} \sum_{k=1}^{r} \lambda_k^n \omega^{(j-1)(k-1)}$$

$$= \frac{1}{r} \sum_{k=1}^{r} \lambda_k^n \sum_{j=1}^{r} \omega^{(j-1)(k-1)}$$

$$= \frac{1}{r} r \lambda_1^n = \frac{1}{r} r 2^n = 2^n.$$

$\square$

We can now state the main result of this section.

Theorem 8.2. *Let $\mathbf{A}$ be the adjacency matrix of the cyclic graph on r vertices C_r. Let $a_{1j}^{(n)}$ be the first row elements of the matrix $\mathbf{A}^n$. There exists a partition*

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} + \ldots + \mathbf{B_{r-1}}$$

where the $\mathbf{B_i}$ are zero-binomial matrices with row sums equal to the sequences $a_{1,i+1}^{(n)}$, respectively, for $i = 0 \ldots r - 1$.

Proof. We have already shown that

$$2^n = \sum_{i=0}^{n} \binom{n}{i} = \sum_{j=1}^{r} a_{1j}^{(n)}.$$

We shall now exhibit a partition of this sum which will complete the proof. For this, we recall that $\mathbf{A} = p(\pi)$, where $p(x) = x + x^{r-1}$. Then

$$(a_{1j}^{(n)})_{1 \le j \le r} = \frac{1}{r} \mathbf{F}_r \begin{pmatrix} p(\omega_r^0)^n \\ p(\omega_r^1)^n \\ p(\omega_r^2)^n \\ \vdots \\ p(\omega_r^{r-1})^n \end{pmatrix}$$

$$= \frac{1}{r} \mathbf{F}_r \begin{pmatrix} (\omega^0 + \omega^{0.(r-1)})^n \\ (\omega^1 + \omega^{1.(r-1)})^n \\ (\omega^2 + \omega^{2.(r-1)})^n \\ \vdots \\ (\omega^{r-1} + \omega^{(r-1).(r-1)})^n \end{pmatrix}$$

$$= \frac{1}{r} \mathbf{F}_r \begin{pmatrix} (\omega^0 + \omega^{-0})^n \\ (\omega^1 + \omega^{-1})^n \\ (\omega^2 + \omega^{-2})^n \\ \vdots \\ (\omega^{r-1} + \omega^{-(r-1)})^n \end{pmatrix}$$

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$$a_{1j}^{(n)} = \frac{1}{r} \sum_{l=0}^{r-1} (\omega_r^l + \omega_r^{-l})^n \omega_r^{(j-1)l}$$

$$= \frac{1}{r} \sum_{l=0}^{r-1} \sum_{k=0}^{n} \binom{n}{k} \omega_r^{kl} \omega_r^{-l(n-k)} \omega_r^{(j-1)l}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\frac{1}{r} \sum_{l=0}^{r-1} \omega_r^{kl} \omega_r^{-l(n-k)} \omega_r^{(j-1)l})$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\frac{1}{r} \sum_{l=0}^{r-1} \omega_r^{2kl+l(j-1-n)})$$

$$= \sum_{r|2k+(j-1-n)} \binom{n}{k}$$

$$= \sum_{2k \equiv (n+1-j) \bmod r} \binom{n}{k}.$$

We thus have

$$B_i = [2k \equiv n + 1 - i \bmod r] \binom{n}{k}.$$

$\square$

Corollary 8.1.

$$\mathbf{A}^n = \mathrm{circ}\left(\frac{1}{r}\mathbf{F}_r \left(2^n \cos(\frac{2\pi j}{r})^n\right)_{0 \le j \le r-1}\right).$$

Proof. This comes about since $(\omega^j + \omega^{-j}) = e^{2\pi i j/k} + e^{-2\pi i j/k} = 2\cos(2\pi j/r)$. $\square$

This verifies the well-known fact that the eigenvalues of C_r are given by $2\cos(2\pi j/r)$, for $0 \le j \le r - 1$ [12]. It is clear now that if $\sigma_i = i$-th symmetric function in $2\cos(2\pi j/r)$, $0 \le j \le r - 1$, then the sequences $a_{1j}^{(n)}$, $1 \le j \le r$, satisfy the recurrence

$$a_n = \sigma_2 a_{n-2} - \sigma_3 a_{n-3} + \cdots + (-1)^{r-1}\sigma_{r-1}a_{r-1}.$$

We thus have

Corollary 8.2. *The sequences*

$$a_{1j}^{(n)} = \sum_{2k = \equiv n+1-j \bmod r} \binom{n}{k},$$

which satisfy the recurrence

$$a_n = \sigma_2 a_{n-2} - \sigma_3 a_{n-3} + \cdots + (-1)^{r-1}\sigma_{r-1}a_{r-1}$$

count the number of walks of length n from vertex 1 to vertex j of the cyclic graph C_r.

9 A worked example

We take the case $r = 8$. We wish to characterize the 8 sequences $\sum_{8|2k+(j-1-n)} \binom{n}{k}$ for $j = 1 \ldots 8$. We give details of these sequences in the following table.

sequence	a_n	binomial expression
$1, 0, 2, 0, 6, 0, 20 \ldots$	$(1 + (-1)^n)(0^n + 2.2^{n/2} + 2^n)/8$	$\sum_{n-2k \equiv 0 \bmod 8} \binom{n}{k}$
$0, 1, 0, 3, 0, 10, 0 \ldots$	$(1 - (-1)^n)(2^n + \sqrt{2}(\sqrt{2})^n)/8$	$\sum_{2k-n \equiv 1 \bmod 8} \binom{n}{k}$
$0, 0, 1, 0, 4, 0, 16, \ldots$	$(1 + (-1)^n)2^n/8 - 0^n/4$	$\sum_{2k-n \equiv 2 \bmod 8} \binom{n}{k}$
$0, 0, 0, 1, 0, 6, 0 \ldots$	$(1 - (-1)^n)(2^n - \sqrt{2}(\sqrt{2})^n)/8$	$\sum_{2k-n \equiv 3 \bmod 8} \binom{n}{k}$
$0, 0, 0, 0, 2, 0, 12, \ldots$	$(1 + (-1)^n)(2^n - 2(\sqrt{2})^n)/8 + 0^n/4$	$\sum_{2k-n \equiv 4 \bmod 8} \binom{n}{k}$
$0, 0, 0, 1, 0, 6, 0 \ldots$	$(1 - (-1)^n)(2^n - \sqrt{2}(\sqrt{2})^n)/8$	$\sum_{2k-n \equiv 5 \bmod 8} \binom{n}{k}$
$0, 0, 1, 0, 4, 0, 16, \ldots$	$(1 + (-1)^n)2^n/8 - 0^n/4$	$\sum_{2k-n \equiv 6 \bmod 8} \binom{n}{k}$
$0, 1, 0, 3, 0, 10, 0 \ldots$	$(1 - (-1)^n)(2^n + \sqrt{2}(\sqrt{2})^n)/8$	$\sum_{2k-n \equiv 7 \bmod 8} \binom{n}{k}$

In terms of ordinary generating functions, we have

$$1, 0, 2, 0, 6, 0, 20 \ldots \quad : \quad \frac{1 - 4x + 2x^2}{(1 - 2x^2)(1 - 4x^2)}$$

$$0, 1, 0, 3, 0, 10, 0 \ldots \quad : \quad \frac{x(1 - 3x^2)}{(1 - 2x^2)(1 - 4x^2)}$$

$$0, 0, 1, 0, 4, 0, 16, \ldots \quad : \quad \frac{x^2(1 - 2x^2)}{(1 - 2x^2)(1 - 4x^2)}$$

$$0, 0, 0, 1, 0, 6, 0 \ldots \quad : \quad \frac{x^3}{(1 - 2x^2)(1 - 4x^2)}$$

$$0, 0, 0, 0, 2, 0, 12, \ldots \quad : \quad \frac{2x^4}{(1 - 2x^2)(1 - 4x^2)}$$

$$0, 0, 0, 1, 0, 6, 0 \ldots \quad : \quad \frac{x^3}{(1 - 2x^2)(1 - 4x^2)}$$

$$0, 0, 1, 0, 4, 0, 16, \ldots \quad : \quad \frac{x^2(1 - 2x^2)}{(1 - 2x^2)(1 - 4x^2)}$$

$$0, 1, 0, 3, 0, 10, 0 \ldots \quad : \quad \frac{x(1 - 3x^2)}{(1 - 2x^2)(1 - 4x^2)}$$

All these sequences satisfy the recurrence

$$a_n = 6a_{n-2} - 8a_{n-4}$$

with suitable initial conditions. In particular, the sequence $1, 0, 2, 0, 6, \ldots$ has general term

$$a_{11}^{(n)} = \frac{0^n}{4} + (1 + (-1)^n)(\frac{2^n}{8} + \frac{(\sqrt{2})^n}{4})$$

This counts the number of closed walks at a vertex of an octagon.

The sequences are essentially [A112798](), [A007582](), [A000302](), [A006516](), [A020522](), with interpolated zeros.

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10 The case $n \to \infty$

We recall that the modified Bessel function of the first kind [11], [13] is defined by the integral

$$I_k(z) = \int_0^\pi e^{z\cos(t)} \cos(kt)dt.$$

$I_n(z)$ has the following generating function

$$e^{z(t+1/t)/2} = \sum_{k=-\infty}^{\infty} I_k(z)t^k.$$

Letting $z = 2x$ and $t = 1$, we get

$$e^{2x} = \sum_{k=-\infty}^{\infty} I_k(2x) = I_0(2x) + 2\sum_{k=1}^{\infty} I_k(2x).$$

The functions $I_k(2x)$ are the exponential generating functions for the columns of Pascal's matrix (including 'interpolated' zeros). For instance, $I_0(2x)$ generates the sequence of central binomial coefficients $1, 0, 2, 0, 6, 0, 20, 0, 70, \ldots$ with formula $\binom{n}{n/2}(1 + (-1)^n)/2$. This gives us the limit case of the decompositions of Pascal's triangle - in essence, each of the infinite matrices that sum to $\mathbf{B}_\infty$ corresponds to a matrix with only non-zero entries in a single column.

The matrix

$$\mathbf{A}_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

corresponds to the limit cyclic graph C_∞. We can characterize the (infinite) set of sequences that correspond to the row elements of the powers $\mathbf{A}_\infty^n$ as those sequences with exponential generating functions given by the family $I_k(2x)$. We also obtain that trace($\mathbf{A}_\infty^n$) is the set of central binomial numbers (with interpolated zeros) generated by $I_0(2x)$.

11 Sequences associated to K_r

By way of example for what follows, we look at the adjacency matrix $\mathbf{A}$ for K_4. We note that K_4 is 3-regular. $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Again, this matrix is circulant, with defining polynomial $p(x) = x + x^2 + x^3 = x(1 + x + x^2)$. Using $\mathbf{F}$ to diagonalize it, we obtain

$$\mathbf{\Lambda} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This leads us to the sequence $\frac{1}{4}\text{trace}(\mathbf{A}^n) = (3^n + 3(-1)^n)/4 = 1, 0, 3, 6, 21, 60, \ldots$, which is A054878. Comparing this result with the expression $P(C_n, 4) = 3^n + 3(-1)^n$ we see that $\text{trace}(\mathbf{A}^n) = P(C_n, 4)$.

Proposition 11.1.

$$\begin{aligned}
a_{11}^{(n)} &= (3^n + 3(-1)^n)/4 = 1, 0, 3, 6, 21, 60, \ldots \\
a_{12}^{(n)} &= (3^n - (-1)^n)/4 = 0, 1, 2, 7, 20, 61, \ldots \\
a_{13}^{(n)} &= (3^n - (-1)^n)/4 = 0, 1, 2, 7, 20, 61, \ldots \\
a_{14}^{(n)} &= (3^n - (-1)^n)/4 = 0, 1, 2, 7, 20, 61, \ldots
\end{aligned}$$

Proof. Using

$$(a_{1j}^{(n)})_{1 \leq j \leq n} = \frac{1}{n}\mathbf{F}(\lambda_1^n, \lambda_2^n, \ldots, \lambda_n^n)'$$

we obtain, for instance,

$$\begin{aligned}
a_{12}^{(n)} &= (3^n + \omega(-1)^n + \omega^2(-1)^n + \omega^3(-1)^n)/4 \\
&= (3^n + (-1)^n(\omega + \omega^2 + \omega^3))/4 = (3^n - (-1)^n)/4
\end{aligned}$$

$\square$

We note that $a_{11}^{(n)}$ is A054878, while $a_{12}^{(n)} = a_{13}^{(n)} = a_{14}^{(n)}$ are all equal to A015518.

Corollary 11.1.

$$3^n = \mathbf{a}_{11}^{(n)} + \mathbf{a}_{12}^{(n)} + \mathbf{a}_{13}^{(n)} + \mathbf{a}_{14}^{(n)}.$$

Corollary 11.2. *The sequences $a_{1j}^{(n)}$ satisfy the linear recurrence*

$$a_n = 2a_{n-1} + 3a_{n-2}$$

with initial conditions

$$\begin{aligned}
a_0 = 1, \; a_1 = 0, \; j = 0 \\
a_0 = 0, \; a_1 = 1, \; j = 2 \ldots 4.
\end{aligned}$$

This result is typical of the general case, which we now address. Thus we let $\mathbf{A}$ by the adjacency matrix of the complete graph K_r on r vertices.

Lemma 11.1. *The eigenvalues of $\mathbf{A}$ are $r - 1, -1, \ldots, -1$.*

Proof. We have $\mathbf{A} = p(\pi)$, where $p(x) = x + x^2 + \ldots + x^{r-1}$. The eigenvalues of $\mathbf{A}$ are $p(1), p(\omega), p(\omega^2), \ldots, p(\omega^{r-1})$, where $\omega^r = 1$. Then $p(1) = 1 + \ldots + 1 = r - 1$. Now

$$p(x) = x + \ldots + x^{r-1} = 1 + x + \ldots + x^{r-1} - 1 = \frac{1 - x^r}{1 - x} - 1.$$

Then

$$p(\omega^j) = \frac{1 - \omega^{rj}}{1 - \omega^j} - 1 = -1$$

since $\omega^{rj} = 1$ for $j \geq 1$. $\qquad\square$

Theorem 11.1. *Let $\mathbf{A}$ be the adjacency matrix of the complete graph K_r on r vertices. Then the r sequences $\mathbf{a}_{1j}^{(n)}$ defined by the first row of $\mathbf{A}^n$ satisfy the recurrence*

$$a_n = (r - 2)a_{n-1} + (r - 1)a_{n-2}$$

with initial conditions

$$a_0 = 1, \ a_1 = 0, \ j = 1$$
$$a_0 = 0, \ a_1 = 1, \ j = 2 \ldots r.$$

In addition, we have

$$(r - 1)^n = \sum_{j=1}^{r} a_{1j}^{(n)}.$$

Proof. We have

$$\begin{pmatrix} a_{11}^{(n)} \\ a_{12}^{(n)} \\ \vdots \\ a_{1r}^{(n)} \end{pmatrix} = \frac{1}{r} \mathbf{F}_r \begin{pmatrix} \lambda_1^n \\ \lambda_2^n \\ \vdots \\ \lambda_r^n \end{pmatrix}$$

$$= \frac{1}{r} \mathbf{F}_r \begin{pmatrix} p(1)^n \\ p(\omega_r^2)^n \\ \vdots \\ p(\omega_r^{r-1})^n \end{pmatrix}$$

$$= \frac{1}{r} \mathbf{F}_r \begin{pmatrix} (r - 1)^n \\ (-1)^n \\ \vdots \\ (-1)^n \end{pmatrix}$$

Hence

$$\begin{aligned} a_{11}^{(n)} &= \frac{1}{r}((r - 1)^n + (-1)^n + \ldots + (-1)^n) \\ &= \frac{1}{r}((r - 1)^n + (r - 1)(-1)^n). \end{aligned}$$

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It is now easy to show that $a_{11}^{(n)}$ satisfies the recurrence

$$a_n = (r-2)a_{n-1} + (r-1)a_{n-2}$$

with $a_0 = 1$ and $a_1 = 0$.

For $j > 1$, we have

$$
\begin{aligned}
a_{1j}^{(n)} &= \frac{1}{r}((r-1)^n + (-1)^n \omega^j + \cdots + (-1)^n \omega^{j(k-1)}) \\
&= \frac{1}{r}((r-1)^n + (-1)^n(\omega^j + \ldots + \omega^{j(k-1)})) \\
&= \frac{1}{r}((r-1)^n + (-1)^n(\frac{1-\omega^{jk}}{1-\omega^j} - 1)) \\
&= \frac{1}{r}((r-1)^n - (-1)^n).
\end{aligned}
$$

This is the solution of the recurrence

$$a_n = (r-2)a_{n-1} + (r-1)a_{n-2}$$

with $a_0 = 0$ and $a_1 = 1$ as required. To prove the final assertion, we note that

$$
\begin{aligned}
\sum_{j=1}^{r} a_{1j}(n) &= a_{11}(n) + (r-1)a_{12}^{(n)} \\
&= \frac{(r-1)^n}{r} + \frac{(-1)^n(r-1)}{r} + (r-1)\left(\frac{(r-1)^n}{r} - \frac{(-1)^n}{r}\right) \\
&= \frac{(r-1)^n}{r}(1+r-1) + \frac{(-1)^n(r-1)}{r} - \frac{(r-1)(-1)^n}{r} \\
&= \frac{(r-1)^n}{r}r = (r-1)^n.
\end{aligned}
$$

$\square$

Thus the recurrences have solutions

$$a_n = \frac{(r-1)^n}{r} + \frac{(-1)^n(r-1)}{r}$$

when

$$a_0 = 1, \ a_1 = 0,$$

and

$$a_n' = \frac{(r-1)^n}{r} - \frac{(-1)^n}{r}$$

for

$$a_0' = 0, \ a_1' = 1.$$

We recognize in the first expression above the formula for the chromatic polynomial $P(C_n, r)$, divided by the factor r. Hence we have

Corollary 11.3. $\frac{1}{r}\text{trace}(\mathbf{A}^n) = a_{11}^{(n)} = \frac{1}{r}P(C_n, r)$.

We list below the first few of these sequences, which count walks of length n on the complete graph K_r. Note that we give the sequences in pairs, as for each value of r, there are only two distinct sequences. The first sequence of each pair counts the number of closed walks from a vertex on K_r. In addition, it counts r-colourings on C_n (when multiplied by r).

$$
\begin{array}{rcl}
r & = & 3 \\
(2^n + 2(-1)^n)/3 & : & 1, 0, 2, 2, 6, 10, 22, \ldots \\
(2^n - (-1)^n)/3 & : & 0, 1, 1, 3, 5, 11, 21, \ldots \\
r & = & 4 \\
(3^n + 3(-1)^n)/4 & : & 1, 0, 3, 6, 21, 60, 183, \ldots \\
(3^n - (-1)^n)/4 & : & 0, 1, 2, 7, 20, 61, 182, \ldots \\
r & = & 5 \\
(4^n + 4(-1)^n)/5 & : & 1, 0, 4, 12, 52, 204, 820, \ldots \\
(4^n - (-1)^n)/5 & : & 0, 1, 3, 13, 51, 205, 819, \ldots \\
r & = & 6 \\
(5^n + 5(-1)^n)/6 & : & 1, 0, 5, 20, 105, 520, 2605, \ldots \\
(5^n - (-1)^n)/6 & : & 0, 1, 4, 21, 104, 521, 2604, \ldots
\end{array}
$$

We have encountered the first four sequences already. The last four sequences are A109499, A015521, A109500 and A015531.

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