The Hankel transform of the sum of consecutive
generalized Catalan numbers

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We discuss the properties of the Hankel transformation of a sequence whose elements are the sums of consecutive generalized Catalan numbers and find their values in the closed form.

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1. Introduction

The Hankel transform of a given sequence \( A = \{a_0, a_1, a_2, \ldots\} \) is the sequence of Hankel determinants \( \{h_0, h_1, h_2, \ldots\} \) (see [7–10]) where \( h_n = |a_{i+j-2}|_{i,j=1}^n \), i.e.

\[
A = \{a_n\}_{n \in \mathbb{N}_0} \rightarrow h = \{h_n\}_{n \in \mathbb{N}_0} : h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \ddots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}. \tag{1}
\]

In this paper, we will consider the sequence of the sums of two adjacent generalized Catalan numbers with parameter \( L \):

\[
a_0 = L + 1, \quad a_n = a_n(L) = c(n; L) + c(n + 1; L) \quad (n \in \mathbb{N}), \tag{2}
\]

where

\[
c(n; L) = T(2n, n; L) - T(2n, n - 1; L) \tag{3}
\]

with

\[
T(n, k; L) = \sum_{j=0}^{n-k} \binom{k}{j} \binom{n-k}{j} L^j. \tag{4}
\]

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The properties of \( \{T(n, k; L)\} \) were studied by P. Barry in [1].

**Example 1.1** Let \( L = 1 \). Vandermonde’s convolution identity implies that

\[
\binom{n}{k} = \sum_j \binom{k}{j} \binom{n-k}{j}.
\]

Hence

\[
T(2n, n; 1) = \binom{2n}{n}, \quad T(2n, n-1; 1) = \binom{2n}{n-1},
\]

wherefrom we get Catalan numbers

\[
c(n) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}
\]

and

\[
a_n = c(n) + c(n+1) = \frac{(2n)!(5n+4)}{n!(n+2)!} \quad (n = 0, 1, 2, \ldots).
\]

In paper [2], Cvetković et al. have proved that the Hankel transform of \( a_n \) equals sequence of Fibonacci numbers with odd indices

\[
h_n = F_{2n+1} = \frac{1}{\sqrt{5}} \binom{2n+1}{n} \left\{ (\sqrt{5}+1)(3+\sqrt{5})^n + (\sqrt{5}-1)(3-\sqrt{5})^n \right\}.
\]

**Example 1.2** For \( L = 2 \) we get like \( a_n(2) \) the next numbers

\[
3, 8, 28, 112, 484, \ldots
\]

and the Hankel transform \( h_n \):

\[
3, 20, 272, 7424, 405504, \ldots
\]

One of us, Barry conjectured that

\[
h_n(2) = 2^{(n^2-n)/2-2} \left\{ (2 + \sqrt{2})^{n+1} + (2 - \sqrt{2})^{n+1} \right\}.
\]

In general, Barry made the conjecture, which we will prove through this paper.

**Theorem 1.3** (The main result) For the generalized Pascal triangle associated to the sequence \( n \mapsto L^n \), the Hankel transform of the sequence

\[
c(n; L) + c(n + 1; L)
\]

is given by

\[
h_n = \frac{L^{(n^2-n)/2}}{2^{n+1}\sqrt{L^2 + 4} - (\sqrt{L^2 + 4} - L)(L + 2 - \sqrt{L^2 + 4})^n. \]
From now till the end, let us denote by

\[ \xi = \sqrt{L^2 + 4}, \quad t_1 = L + 2 + \xi, \quad t_2 = L + 2 - \xi. \]  

(6)

Now, we can write

\[ h_n = \frac{L^{n(n-1)/2}}{2^{n+1} \xi} \cdot ((\xi + L)t_1^n + (\xi - L)t_2^n). \]

Or, introducing

\[ \varphi_n = t_1^n + t_2^n, \quad \psi_n = t_1^n - t_2^n \quad (n \in \mathbb{N}_0), \]

the final statement can be expressed by

\[ h_n = \frac{L^{n(n-1)/2}}{2^{n+1} \xi} \cdot (L\psi_n + \xi \varphi_n). \]

(8)

**Lemma 1.4** The values \( \varphi_n \) and \( \psi_n \) satisfy the next relations:

\[ \varphi_j \cdot \varphi_k = \varphi_{j+k} + (4L)^j \varphi_{k-j}, \quad \psi_j \cdot \psi_k = \varphi_{j+k} - (4L)^j \varphi_{k-j} \quad (0 \leq j \leq k), \]

(9)

\[ \varphi_j \cdot \psi_k = \varphi_{j+k} + (4L)^j \psi_{k-j}, \quad \psi_j \cdot \varphi_k = \varphi_{j+k} - (4L)^j \psi_{k-j} \quad (0 \leq j \leq k). \]

(10)

**Corollary 1.5** Assuming that the main theorem is true, the function \( h_n = h_n(L) \) is the next polynomial

\[ h_n(L) = 2^{-n} L^{n(n-1)/2} \]

\[ \cdot \left\{ \sum_{i=0}^{\lceil (n-1)/2 \rceil} \left( \begin{array}{c} n \\ 2i + 1 \end{array} \right) L(L + 2)^{n-2i-1}(L^2 + 4)^i + \sum_{i=0}^{\lfloor n/2 \rfloor} \left( \begin{array}{c} n \\ 2i \end{array} \right) (L + 2)^{n-2i}(L^2 + 4)^i \right\}. \]

**Proof** By previous notation, we can write

\[ (L + \xi)(L + 2 + \xi)^n - (L - \xi)(L + 2 - \xi)^n \]

\[ = (L + \xi) \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (L + 2)^{n-k} \xi^k - (L - \xi) \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) (L + 2)^{n-k} \xi^k \]

\[ = \sum_{k=0}^{n} (1 - (-1)^k) \left( \begin{array}{c} n \\ k \end{array} \right) L(L + 2)^{n-k} \xi^k + \sum_{k=0}^{n} (1 + (-1)^k) \left( \begin{array}{c} n \\ k \end{array} \right) (L + 2)^{n-k} \xi^k \]

\[ = 2 \sum_{i=0}^{\lceil (n-1)/2 \rceil} \left( \begin{array}{c} n \\ 2i + 1 \end{array} \right) L(L + 2)^{n-2i-1} \xi^{2i+1} + 2 \sum_{i=0}^{\lfloor n/2 \rfloor} \left( \begin{array}{c} n \\ 2i \end{array} \right) (L + 2)^{n-2i} \xi^{2i+1} \]

\[ = 2\xi \left\{ \sum_{i=0}^{\lceil (n-1)/2 \rceil} \left( \begin{array}{c} n \\ 2i + 1 \end{array} \right) L(L + 2)^{n-2i-1} \xi^{2i+1} + \sum_{i=0}^{\lfloor n/2 \rfloor} \left( \begin{array}{c} n \\ 2i \end{array} \right) (L + 2)^{n-2i} \xi^{2i+1} \right\}, \]

wherefrom immediately follows the polynomial expression for \( h_n \).
2. The generating function for the sequences of numbers and orthogonal polynomials

The Jacobi polynomials are given by

\[
P_{n}^{(a,b)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+a}{k} \binom{n+b}{n-k} (x-1)^{n-k} (x+1)^k \quad (a, b > -1).
\]

Also, they can be written in the form

\[
P_{n}^{(a,b)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n} \binom{n+a}{k} \binom{n+b}{n-k} \left(\frac{x+1}{x-1}\right)^k.
\]

From the fact

\[
L = \frac{x + 1}{x - 1} \iff x = \frac{L + 1}{L - 1} \quad (x \neq 1, L \neq 1),
\]

we conclude that:

\[
T(2n, n; L) = (L - 1)^n \cdot P_{n}^{(0,0)} \left(\frac{L + 1}{L - 1}\right),
\]

\[
T(2n + 2, n; L) = (L - 1)^n \cdot P_{n}^{(2,0)} \left(\frac{L + 1}{L - 1}\right).
\]

The generating function \(G(x, t)\) for the Jacobi polynomials is

\[
G^{(a,b)}(x, t) = \sum_{n=0}^{\infty} P_{n}^{(a,b)}(x) t^n = \frac{2^{a+b}}{\phi \cdot (1-t+\phi)^a \cdot (1+t+\phi)^b},
\]

where

\[
\phi = \phi(x, t) = \sqrt{1 - 2xt + t^2}.
\]

Now,

\[
\sum_{n=0}^{\infty} T(2n, n; L) t^n = \sum_{n=0}^{\infty} P_{n}^{(0,0)} \left(\frac{L + 1}{L - 1}\right) ((L - 1)t)^n = G^{(0,0)} \left(\frac{L + 1}{L - 1}, (L - 1)t\right),
\]

\[
\sum_{n=0}^{\infty} T(2n + 2, n; L) t^n = \sum_{n=0}^{\infty} P_{n}^{(2,0)} \left(\frac{L + 1}{L - 1}\right) ((L - 1)t)^n = G^{(2,0)} \left(\frac{L + 1}{L - 1}, (L - 1)t\right).
\]

Also,

\[
\sum_{n=0}^{\infty} T(2n, n - 1; L) t^n = t \cdot \left\{G^{(2,0)} \left(\frac{L + 1}{L - 1}, (L - 1)t\right) - 1\right\},
\]

\[
\sum_{n=0}^{\infty} T(2n + 2, n + 1; L) t^n = \frac{1}{t} \cdot \left\{G^{(0,0)} \left(\frac{L + 1}{L - 1}, (L - 1)t\right) - 1\right\}.
\]

The generating function \(G(t; L)\) for the sequence \(\{a_n\}_{n \geq 0}\) is given by

\[
G(t; L) = \sum_{n=0}^{\infty} a_n t^n = \frac{t + 1}{t} \cdot G^{(0,0)} \left(\frac{L + 1}{L - 1}, (L - 1)t\right)
\]

\[
- (t + 1) G^{(2,0)} \left(\frac{L + 1}{L - 1}, (L - 1)t\right) - \frac{1}{t}.
\]
THEOREM 2.1  The generating function $G(t; L)$ for the sequence $\{a_n\}_{n \geq 0}$ is
\begin{equation}
G(t; L) = \frac{t + 1}{\rho(t; L)} \left\{ \frac{1}{t} - \frac{4}{(1 - (L - 1)t + \rho(t; L))^2} \right\} - \frac{1}{t},
\end{equation}
where
\begin{equation}
\rho(t; L) = \phi \left( \frac{L + 1}{L - 1}, (L - 1)t \right) = \sqrt{1 - 2(L + 1)t + (L - 1)^2t^2}.
\end{equation}

The function $\rho(t; L)$ has domain
\begin{align*}
D_\rho &= \left( -\infty, \frac{1}{1 - 2\sqrt{L + 1}} \right) \cup \left( \frac{1}{1 - 2L + L^2}, +\infty \right) \quad (L \neq 1) \\
D_\rho &= \left( -\infty, \frac{1}{4} \right) \quad (L = 1).
\end{align*}

Example 2.2  For $L = 1$, we get
\begin{equation}
G(t; 1) = \sum_{n=0}^{\infty} a_n(1) t^n = \frac{1}{t} \left( \frac{(1 - \sqrt{1 - 4t})(1 + t)}{2t} - 1 \right)
\end{equation}
and for $L = 2$, we find
\begin{equation}
G(t; 2) = \sum_{n=0}^{\infty} a_n(2) t^n = -\frac{1}{t} + \frac{t + 1}{\sqrt{t^2 - 6t + 1}} \left\{ \frac{1}{t} - \frac{4}{(1 - t + \sqrt{t^2 - 6t + 1})^2} \right\}.
\end{equation}

3. The weight function corresponding to the functional

It is known (for example, see Krattenthaler [3]) that the Hankel determinant $h_n$ of order $n$ of the sequence $\{a_n\}_{n \geq 0}$ equals
\begin{equation}
h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^{2} \beta_{n-1},
\end{equation}
where $\{\beta_n\}_{n \geq 1}$ is the sequence given by:
\begin{equation}
G(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{a_0}{1 + \alpha_0 x - (\beta_1 x^2/(1 + \alpha_1 x - (\beta_2 x^2/(1 + \alpha_2 x - \cdots))))}.
\end{equation}

The sequences $\{a_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 1}$ are the coefficients in the recurrence relation
\begin{equation}
Q_{n+1}(x) = (x - \alpha_n) Q_n(x) - \beta_n Q_{n-1}(x),
\end{equation}
where $\{Q_n(x)\}_{n \geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional $\mathcal{U}$ determined by
\begin{equation}
\mathcal{U}[x^n] = a_n \quad (n = 0, 1, 2, \ldots).
\end{equation}

In this section, the functional will be constructed for the sum of consecutive generalized Catalan numbers. We would like to express $\mathcal{U}[f]$ in the form:
\begin{equation}
\mathcal{U}[f(x)] = \int_{\mathbb{R}} f(x) \, d\psi(x),
\end{equation}
where $\psi(x)$ is a distribution, or, even more, to find the weight function $w(x)$ such that $w(x) = \psi'(x)$. 


Denote by $F(z; L)$ the function

$$F(z; L) = \sum_{k=0}^{\infty} a_k z^{-k-1}.$$  

From the generating function (13), we have:

$$F(z; L) = z^{-1} \mathcal{G}(z^{-1}; L)$$  \hspace{1cm} (21)

and after some simplifications we obtain that

$$F(z; L) = -1 + \frac{2(z + 1)}{L - 1 + z + \sqrt{L^2 + (z - 1)^2 - 2L(z + 1)}}$$

$$= -1 + \frac{2(z + 1)}{L - 1 + z(1 + z \rho(\frac{1}{z}, L))}.$$  

**Example 3.1** From (15) and (16), we yield:

$$F(z; 1) = z^{-1} \mathcal{G}(z^{-1}; 1) = \frac{1}{2} \left\{ z - 1 - (z + 1) \sqrt{1 - \frac{4}{z}} \right\},$$

$$F(z; 2) = \frac{-1}{2z} \left\{ 1 + z \left( 2 - z + (z + 1) \sqrt{1 - \frac{6}{z} + \frac{1}{z^2}} \right) \right\}.$$  

Notice that

$$\int F(z; 2) \, dz = z + \frac{1}{4} z(z - 1) \rho\left(\frac{1}{z}, 2\right) + \log(z)$$

$$- \frac{1}{2} \log \left(1 + z \left( \rho\left(\frac{1}{z}, 2\right) - 3 \right) \right) - \frac{7}{2} \log \left(z - 3 + z \rho\left(\frac{1}{z}, 2\right) \right).$$

It will be the impulse for further discussion.

Denote by

$$R(z; L) = z \rho\left(\frac{1}{z}, L\right) = \sqrt{L^2 + (z - 1)^2 - 2L(z + 1)}.$$  

From the theory of distribution functions (see Chihara [4]), especially by the Stieltjes inversion formula

$$\psi(t) - \psi(0) = -\frac{1}{\pi} \lim_{s \to 0^+} \int_0^t \Im F(x + iy; L) \, dx,$$  \hspace{1cm} (22)

we conclude that holds

$$\mathcal{F}(z; L) = \int F(z; L) \, dz = \frac{1}{4} \left[ z^2 - 2Lz - (z - L + 1)R(z; L) - l_1(z) + l_2(z) \right].$$  \hspace{1cm} (23)
where
\[
\begin{align*}
l_1(z) &= 2(3L + 1) \log[z - (L + 1) + R(z; L)], \\
l_2(z) &= 2(L - 1) \log \left[ \frac{-(L - 1)R(z; L) - (L - 1)^2 + z(L + 1)}{z^2(L - 1)^3} \right].
\end{align*}
\]

Rewriting the function \( R(z; L) \) in the form
\[
R(z; L) = \sqrt{(z - L - 1)^2 - 4L}
\]
and replacing \( z = x + iy \), we have
\[
R(x; L) = \lim_{y \to 0^+} R(x + iy; L) = \begin{cases} 
    i\sqrt{4L - (x - L - 1)^2}, & x \in (a, b), \\
    \sqrt{(x - L - 1)^2 - 4L}, & \text{otherwise},
\end{cases}
\]
where
\[
a = (\sqrt{L} - 1)^2, \quad b = (\sqrt{L} + 1)^2. \tag{24}
\]

In the case when \( x \notin ((\sqrt{L} - 1)^2, (\sqrt{L} + 1)^2) \), value \( R(x; L) \) is real. Therefore we can calculate imaginary part of \( \mathcal{F}(x; L) = \lim_{y \to 0^+} \mathcal{F}(x + iy; L) \):
\[
\Im \mathcal{F}(x; L) = \Im [l_2(x) - l_1(x)] = 0.
\]

Otherwise, if \( x \in ((\sqrt{L} - 1)^2, (\sqrt{L} + 1)^2) \) we have that:
\[
\begin{align*}
l_1(x) &= 2(3L + 1) \log \left[ x - (L + 1) \pm i\sqrt{4L - (x - L - 1)^2} \right], \\
\Im l_1(x) &= \begin{cases} 
    2(3L + 1) \arctan \frac{\sqrt{4L - (x - L - 1)^2}}{x - (L + 1)}, & x \geq L + 1, \\
    2(3L + 1) \left( \pi + \arctan \frac{\sqrt{4L - (x - L - 1)^2}}{x - (L + 1)} \right), & x < L + 1,
\end{cases}
\end{align*}
\[
l_2(x) &= 2(L - 1) \log \left[ \frac{-(L - 1)^2 + 2x(L + 1) - i(L - 1)\sqrt{4L - (x - L - 1)^2}}{x^2(L - 1)^3} \right], \\
\Im l_1(x) &= \begin{cases} 
    2(L - 1) \left( 2\pi + \arctan \frac{x(L + 1) - (L - 1)^2}{\sqrt{4L - (x - L - 1)^2}} \right), & x \geq \frac{(L - 1)^2}{L + 1}, \\
    2(L - 1) \left( \pi + \arctan \frac{x(L + 1) - (L - 1)^2}{\sqrt{4L - (x - L - 1)^2}} \right), & x < \frac{(L - 1)^2}{L + 1}.
\end{cases}
\]

After substituting all considered cases in (23), we finally obtain the value
\[
\Im \mathcal{F}(x; L) = \Im \mathcal{F}(x + iy; L) = \Im [l_2(x) - l_1(x) - (x - L + 1)\sqrt{4L - (x - L - 1)^2}].
\]

From the relation (22), we conclude that
\[
\omega(x; L) = \psi'(x; L) = -\frac{1}{\pi} \frac{d}{dx} \Im \mathcal{F}(x; L) \tag{25}
\]
and finally, we obtain
\[\omega(x; L) = \frac{1}{2\pi} \left( 1 + \frac{1}{x} \right) \sqrt{4L - (x - L - 1)^2}\]
\[= \frac{\sqrt{L}}{\pi} \left( 1 + \frac{1}{x} \right) \sqrt{1 - \left( \frac{x - L - 1}{2\sqrt{L}} \right)^2}.\]  
(26)

The previous formula holds for \(x \in (a, b)\), and otherwise is \(\omega(x; L) = 0\).

### 4. Determining the three-term recurrence relation

The crucial moment in our proof of the conjecture is to determine the sequence of polynomials \(\{Q_n(x)\}\) orthogonal with respect to the weight \(w(x; L)\) given by (26) on the interval \((a, b)\) and to find the sequences \(\{\alpha_n\}\) \(\{\beta_n\}\) in the three-term recurrence relation.

**Example 4.1**  
For \(L = 4\), we can find the first members:

\[Q_0(x) = 1, \quad \|Q_0\|^2 = 5,\]
\[Q_1(x) = x - \frac{24}{5}, \quad \|Q_1\|^2 = \frac{104}{5},\]
\[Q_2(x) = x^2 - \frac{127}{13}x + \frac{256}{13}, \quad \|Q_2\|^2 = \frac{1088}{13},\]
\[Q_3(x) = x^3 - \frac{541}{17}x^2 + \frac{1096}{17}x - \frac{1344}{17}, \quad \|Q_3\|^2 = \frac{5696}{17},\]

wherefrom
\[\alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169}.\]

Hence
\[h_1 = a_0 = 5, \quad h_2 = a_0^2\beta_1 = 104, \quad h_3 = a_0^3\beta_1^2\beta_2 = 5^3 \left( \frac{104}{25} \right)^2 \frac{680}{169} = 8704.\]

At the beginning, we will notice that in the definition of the weight function appears the square root member.

That is why, let us consider the monic orthogonal polynomials \(\{S_n(x)\}\) with respect to the \(p^{(1/2, 1/2)}(x) = \sqrt{1 - x^2}\) on the interval \((-1, 1)\). These polynomials are monic Chebyshev polynomials of the second kind:
\[S_n(x) = \frac{\sin((n + 1) \arccos x)}{2^n \cdot \sqrt{1 - x^2}}.\]

They satisfy the three-term recurrence relation (Chihara [4]):
\[S_{n+1}(x) = (x - \alpha_n^*) S_n(x) - \beta_n^* S_{n-1}(x) \quad (n = 0, 1, \ldots),\]  
(27)

with initial values
\[S_{-1}(x) = 0, \quad S_0(x) = 1,\]
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\[ \alpha_n^* = 0 \quad (n \geq 0) \quad \text{and} \quad \beta_0^* = \frac{\pi}{2}, \quad \beta_n^* = \frac{1}{4} \quad (n \geq 1). \]

If we use the weight function \( \hat{w}(x) = (x - c) \ p^{(1/2, 1/2)}(x) \), then the corresponding coefficients \( \hat{\alpha}_n \) and \( \hat{\beta}_n \) can be evaluated as follows (see, for example, Gautschi [5]):

\[ \lambda_n = S_n(c), \]
\[ \hat{\alpha}_n = c - \frac{\lambda_{n+1}}{\lambda_n} - \frac{\beta_{n+1}^* \lambda_n}{\lambda_{n+1}}, \quad (28) \]
\[ \hat{\beta}_n = \beta_n^* \frac{\lambda_{n-1} \lambda_{n+1}}{\lambda_n^2} \quad (n \in \mathbb{N}_0). \]

From the relation (27), we conclude that the sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) satisfies the following recurrence relation:

\[ 4\lambda_{n+1} - 4c\lambda_n + \lambda_{n-1} = 0 \quad (\lambda_{-1} = 0; \ \lambda_0 = 1). \quad (29) \]

The characteristic equation

\[ 4z^2 - 4cz + 1 = 0 \]

has the solutions

\[ z_{1,2} = \frac{1}{2}(c \pm \sqrt{c^2 - 1}) \]

and the integral solution of (29) is

\[ \lambda_n = E_1 z_1^n + E_2 z_2^n \quad (n \in \mathbb{N}). \]

We evaluate values \( E_1 \) and \( E_2 \) from the initial conditions \( (\lambda_{-1} = 0; \ \lambda_0 = 1) \).

In order to solve our problem, we will choose \( c = -(L + 2/\sqrt{L}). \) Hence

\[ z_k = \frac{-t_k}{4\sqrt{L}} \quad (k = 1, 2), \quad \text{where} \quad t_{1,2} = L + 2 \pm \sqrt{L^2 + 4}. \]

Finally, we obtain:

\[ \lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{n/2} \sqrt{L^2 + 4}} (t_1^{n+1} - t_2^{n+1}) \quad (\lambda = -1, 0, 1, \ldots), \]

i.e.,

\[ \lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{n/2} \xi} \psi_{n+1} \quad (\lambda = -1, 0, 1, \ldots). \]

After replacing in (28), we obtain:

\[ \hat{\alpha}_n = -L + 2 \cdot \frac{1}{2\sqrt{L}} + \frac{1}{4\sqrt{L}} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + \frac{\psi_{n+1}}{\psi_{n+2}}, \quad \hat{\beta}_n = \frac{\psi_n \psi_{n+2}}{4\psi_{n+1}^2}, \quad (30) \]

If a new weight function \( \tilde{w}(x) \) is introduced by

\[ \tilde{w}(x) = \hat{w}(ax + b), \]
then we have
\[ \tilde{\alpha}_n = \frac{\hat{\alpha}_n - b}{a}, \quad \tilde{\beta}_n = \frac{\hat{\beta}_n}{a^2} \quad (n \geq 0). \]

Now, by using \( x \mapsto \frac{x - L - 1}{2\sqrt{L}} \), i.e., \( a = 1/2\sqrt{L} \) and \( b = -(L + 1/2\sqrt{L}) \), we have the weight function
\[ \tilde{w}(x) = \hat{w}\left(\frac{x - L - 1}{2\sqrt{L}}\right) = \frac{1}{2} \left( \frac{x - L - 1}{2\sqrt{L}} + \frac{L + 2}{2\sqrt{L}} \right) \sqrt{1 - \left( \frac{x - L - 1}{2\sqrt{L}} \right)^2}. \]

Thus
\[ \tilde{\alpha}_n = -1 + \frac{1}{2} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + 2L \cdot \frac{\psi_{n+1}}{\psi_{n+2}} \quad (n \in \mathbb{N}_0) \quad (32) \]
and
\[ \tilde{\beta}_0 = (L + 2) \frac{\pi}{2}, \quad \tilde{\beta}_n = L \frac{\psi_n \psi_{n+2}}{\psi_{n+1}} \quad (n \in \mathbb{N}). \quad (33) \]

**Example 4.2** For \( L = 4 \), we get
\[
\begin{align*}
P_0(x) &= 1, & \|P_0\|^2 &= 3\pi, \\
P_1(x) &= x - \frac{17}{3}, & \|P_1\|^2 &= \frac{32\pi}{3}, \\
P_2(x) &= x^2 - \frac{43}{4} x + \frac{101}{4}, & \|P_2\|^2 &= 42\pi, \\
P_3(x) &= x^3 - \frac{331}{21} x^2 + \frac{1579}{21} x - \frac{2189}{21}, & \|P_3\|^2 &= \frac{3520\pi}{21},
\end{align*}
\]
wherefrom
\[
\begin{align*}
\tilde{\alpha}_0 &= \frac{17}{3}, & \tilde{\beta}_0 &= 3\pi, & \tilde{\alpha}_1 &= \frac{61}{12}, & \tilde{\beta}_1 &= \frac{32}{9}, & \tilde{\alpha}_2 &= \frac{421}{84}, & \tilde{\beta}_2 &= \frac{63}{16}.
\end{align*}
\]

Introducing the weight
\[ \tilde{w}(x) = \frac{2L}{\pi} \hat{w}(x) \]
will not change the monic polynomials and their recurrence relations, only it will multiply the norms by the factor \( 2L/\pi \), i.e.
\[
\begin{align*}
\tilde{P}_k(x) &\equiv P_k(x), & \|\tilde{P}_k\|_{\tilde{w}}^2 &= \int_a^b \tilde{P}_k(x) \tilde{w}(x) \, dx = \frac{2L}{\pi} \|P_k\|^2 \quad (k \in \mathbb{N}_0), \\
\tilde{\beta}_0 &= L(L + 2), & \tilde{\beta}_k &= \tilde{\beta}_k \quad (k \in \mathbb{N}), & \tilde{\alpha}_k &= \tilde{\alpha}_k \quad (k \in \mathbb{N}_0).
\end{align*}
\]
Here is
\[ \tilde{\beta}_0 \tilde{\beta}_1 \cdots \tilde{\beta}_{n-1} = \frac{L^n}{2} \cdot \frac{\psi_{n+1}}{\psi_n}. \quad (34) \]

In [6], Gautschi has treated the next problem: If we know all about the MOPS orthogonal with respect to \( \tilde{w}(x) \) what can we say about the sequence \( \{Q_n(x)\} \) orthogonal with respect to
a weight
\[ w_d(x) = \frac{\tilde{w}(x)}{x - d} \quad (d \notin \text{support} (\tilde{w})) \]

Gautshi has proved that, by the auxiliary sequence
\[ r_{-1} = -\int_{\mathbb{R}} w_d(x) \, dx, \quad r_n = d - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots), \]
it can be determined
\[ \alpha_{d,0} = \tilde{\alpha}_0 + r_0, \quad \alpha_{d,k} = \tilde{\alpha}_k + r_k - r_{k-1}, \quad \beta_{d,0} = -r_{-1}, \quad \beta_{d,k} = \frac{\tilde{\beta}_{k-1} r_{k-1}}{r_{k-2}} \quad (k \in \mathbb{N}). \]

In our case it is enough to take \( d = 0 \) to get the final weight
\[ w(x) = \frac{\tilde{w}(x)}{x}. \]

Hence
\[ r_{-1} = -(L + 1), \quad r_n = -\left( \tilde{\alpha}_n + \frac{\tilde{\beta}_n}{r_{n-1}} \right) \quad (n = 0, 1, \ldots). \tag{35} \]

**Lemma 4.3** The parameters \( r_n \) have the explicit form
\[ r_n = -\frac{\psi_{n+1}}{\psi_{n+2}} \cdot \frac{L \psi_{n+2} + \xi \varphi_{n+2}}{L \psi_{n+1} + \xi \varphi_{n+1}} \quad (n \in \mathbb{N}_0). \tag{36} \]

*Proof* We will use the mathematical induction. For \( n = 0 \), we really get the expected value
\[ r_0 = -\frac{L^2 + 2L + 2}{(L + 1)(L + 2)}. \]

Suppose that it is true for \( k = n \). Now, by the properties for \( \varphi_n \) and \( \psi_n \), we have
\[ \tilde{\alpha}_{n+1} \cdot r_n + \tilde{\beta}_{n+1} = -\frac{\psi_{n+1}}{\psi_{n+3}} \cdot \frac{L \psi_{n+3} + \xi \varphi_{n+3}}{L \psi_{n+1} + \xi \varphi_{n+1}}. \]

Dividing with \( r_n \), we conclude that the formula is valid for \( r_{n+1} \). \hfill \square

**Example 4.4** For \( L = 4 \), we get
\[ r_{-1} = -5, \quad r_0 = -\frac{13}{15}, \quad r_1 = -\frac{51}{52}, \quad r_2 = -\frac{356}{357}, \]
wherefrom
\[ \alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169}, \]
just the same as in Example 4.1.
Proof of the main result  

Krattenthaler’s formula (17) can also be written in the form

\[ h_1 = a_0, \quad h_n = \beta_0 \beta_1 \beta_2 \cdots \beta_{n-2} \beta_{n-1} \cdot h_{n-1}. \]  

(37)

From the theory of orthogonal polynomials, it is known that

\[ \|Q_{n-1}\|^2 = \beta_0 \beta_1 \beta_2 \cdots \beta_{n-2} \beta_{n-1} \quad (n = 2, 3, \ldots), \]  

(38)

wherefrom

\[ h_1 = a_0, \quad h_n = \|Q_{n-1}\|^2 \cdot h_{n-1} \quad (n = 2, 3, \ldots). \]  

(39)

\[ \Box \]

Here,

\[ \|Q_{n-1}\|^2 = \beta_0 \frac{r_{n-2}}{r_{n-1}} \prod_{k=0}^{n-2} \hat{\beta}_k = \frac{L_n}{2} \cdot \frac{L \psi_n + \xi \varphi_n}{L \psi_{n-1} + \xi \varphi_{n-1}}. \]  

(40)

We will apply the mathematical induction again. The formula for \( h_n \) is true for \( n = 1 \). Suppose that it is valid for \( k = n - 1 \). Then

\[ h_n = \frac{L_n}{2} \cdot \frac{L \psi_n + \xi \varphi_n}{L \psi_{n-1} + \xi \varphi_{n-1}} \cdot \frac{L^{(n-1)(n-2)/2}}{2^n \xi} \cdot (L \psi_{n-1} + \xi \varphi_{n-1}), \]

wherefrom it follows that the final statement

\[ h_n = \frac{L^{n(n-1)/2}}{2^n \xi} \cdot (L \psi_n + \xi \varphi_n) \quad (n \in \mathbb{N}) \]

is true.

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References
