WATERFORD INSTITUTE OF TECHNOLOGY

DOCTORAL THESIS

A study in Algebraic properties of Riordan arrays

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in the

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Department of Computing and Mathematics

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Declaration of Authorship

I, Nikolaos PANTELIDIS, declare that this thesis titled, "A study in Algebraic properties of Riordan arrays" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
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"Arithmetic is being able to count up to twenty without taking off your shoes."

Mickey Mouse

WATERFORD INSTITUTE OF TECHNOLOGY

Abstract

School of Scinece and Computing
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Doctor of Philosophy

A study in Algebraic properties of Riordan arrays

by Nikolaos PANTELIDIS

The main objects of our study are the algebraic structure of Riordan arrays, the properties of subgroups of the Riordan group and relationships among directly related or unrelated theories and objects. Firstly, we introduce formal power series, generating functions and their links to lattice paths and continued fractions. We then present the Riordan group, focusing on the important Riordan subgroups and extending the related theory. These results lead us to the study of quasi-involutions, a special type of Riordan arrays, that we link it to a form of orthogonal polynomials. In the following chapter, we present our findings in the theory of almost-Riordan arrays, a newly discovered type of Riordan arrays. In the final chapter, we concern ourselves with the Linear Algebra of Riordan arrays and their analysis through their eigenvalues and eigenvectors.

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List of Abbreviations

...e	**e**ven function
...o	**o**dd function
bgf	**b**ivariate **g**enerating **f**unction
egf	**e**xponential **g**enerating **f**unction
eq	**e**quation
Fig	**F**igure
fps	**f**ormal **p**ower **s**eries
FTRA	**F**undamental **T**heorem (of) **R**iordan **A**rrays
FTa-RA	**F**undamental **T**heorem (of) **a**l-**R**iordan **A**rrays
gf	**g**enerating **f**unction
LHS	**L**eft **H**and **S**ide
ODE	**O**rdinary **D**ifferential **E**quation
OEIS	**O**n-Line **E**ncyclopedia (of) **I**nteger **S**equences
RHS	**R**ight **H**and **S**ide

List of Symbols

$\triangleleft$	normal subgroup
$\lfloor\rfloor$	floor function
$\lceil\rceil$	ceiling function
$\ltimes$	semi-direct product
$\mathbb{N}$	the set of natural numbers
$...^*$	a known set of numbers, excluding 0
$\mathbb{Z}$	the set of Integers
$\mathbb{Q}$	the set of quotient numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{F}$	set of formal power series over a field
$\mathbb{F}_k$	set of formal power series over a field, of order k
$\big(g(z), f(z)\big)$	Ordinary Riordan array
$\big[g(z), f(z)\big]$	Exponential Riordan array
$\big(g(z); f_1(z), f_2(z)\big)$	Double Riordan array
$\big(a(z)\big\vert g(z), f(z)\big)$	Almost-Riordan array
δ	Kronecker's symbol
$\epsilon\mathcal{R}$	Exponential Riordan group

$\alpha\mathcal{R}$	Almost-Riordan group
$\alpha\mathcal{R}(k)$	Almost-Riordan group of level k
$d\mathcal{R}$	Double Riordan group
$d\mathcal{R}(k)$	Double Riordan group of level k

To the loving memory of my grandpa,
Spyros

Chapter 1

Introduction

1.1 A historical time-line of Riordan arrays

Searching for the origins of Riordan arrays, we need to go back at the beginning of Combinatorics, as this new found field is inseparable to this area of Discrete Mathematics. Combinatorics is the part of Mathematics which is based on counting, which is one of the fundamentals of the science of Mathematics. This area was developed because of the necessity for a common theory that will serve the needs for counting, and help in solving problems related to counting, which arise in pure Mathematics, notably in areas such as Algebra, Analysis, Topology, and Geometry.
The earliest recorded use of combinatorial skills is dated back to 16th century BC in Egypt, and centuries later in Ancient Greece. Nevertheless, most historians agree that the most significant work of early combinatorics has been done in the 9th-13th century AD, in India, China, Persia and the Middle East. Some of the findings of that time include the Fibonacci numbers, the Binomial theorem, and an arithmetical triangle that presents the binomial coefficients, which is known nowadays as Pascal's Triangle.
A few years later, Leonardo of Pisa and Jordanus de Nemore from Italy, came in touch with the knowledge of mathematicians from the East, and gradually the research of Combinatorics spread around Europe. This was the dawn of a new era in Combinatorics, as European mathematicians decided to work in this field. In 1666, Gottfried Wilhelm Leibniz was the first who used the term combinatorial. Blaise Pascal and Pierre de Fermat developed some classical combinatorial results which were related to the theory of probability, and of course Leonhard Euler and his famous problem about the seven bridges of Königsberg (known as Kalinigrand, nowadays), in 1735, which can be considered as the foundation of graph theory. Euler was also the one who introduced generating functions in order to solve a problem related to the partition of a number [1, 2, 105].
During the 19th century, there were even more important contributions in the field from George Boole who used combinatorial methods in his work, Arthur Cayley and the development of enumerative graph theory, and James Joseph Sylvester who introduced the term matrix, in 1850. Later, in the 20th century,

Paul Erdős contributed extensively in Combinatorics by solving many open combinatorial problems, and pushed the boundaries further, by inspiring the new generation of mathematicians. Simultaneously, the discovery of the computer provided the mathematical community with a powerful tool for counting, and gave Combinatorics a great opportunity to grow further through this, by projecting the research of all these centuries in programming, and by developing and playing an important role in algorithm design. Fruits of this evolution were the work about the fundamentals of computer science as it was written by Donald E. Knuth [39], Richard P. Stanley [93], Robert Sedgewick and Philippe Flajolet [36, 37], and Gian-Carlo Rota as a series of ten papers on the"Foundations of Combinatorics".

Riordan arrays came as a combination of Combinatorics, Linear Algebra, Group Theory and Functional Analysis. The research that led to their discovery, started at the end of the 1970s, when D.G. Rogers was inspired by Louis Shapiro's work in the discovery of a lower triangular matrix that defined an array generated by its first two columns [77]. In his work, D.G. Rogers tried to determine a family of triangular arrays with arithmetic properties analogous of the Pascal triangle, which he named Renewal arrays.

Riordan arrays have been researched since the early 1990s and are called after John Riordan, an American mathematician who was one of the pioneering researchers in Combinatorics. The first paper entirely based on Riordan arrays was published under the title "The Riordan Group" by L. Shapiro et al. in 1991 [83]. Shapiro's paper was based on "Pascal Triangles, Catalan Numbers and Renewal arrays" by D.G. Rogers [77], and "The Umbral Calculus" by Steven M. Roman and Gian-Carlo Rota [78], and it is considered the foundation stone in the formulation of the theory of Riordan arrays. It contained the initial definition of a Riordan matrix, the way that such matrices can be constructed, and also what is now called the Fundamental Theorem of Riordan arrays. The latter defines a multiplication operation between Riordan matrices, and as a consequence it was shown that the set of these matrices together with this operation, form a group.

Since then, the research community of Riordan arrays has continued to expand by scientists all over the world. A recently updated list of related bibliography [88] by Renzo Sprugnoli contains almost 100 references of published papers from mathematicians from Europe, Asia, Australia, North and South America, and it constitutes a proof of the high scientific interest of this new found field. A research group specialising in Riordan arrays and related topics, named the Applied Algebra and Optimization Research Center (AORC) has been founded at Sunkyunkwan University, South Korea in 2016. Moreover, the first book devoted to Riordan arrays, written by Paul Barry [12], was published in 2017.

The first international Riordan arrays conference was held as an invited mini symposium during the International Congress of Mathematicians in Seoul,

South Korea, in 2014, and since then researchers in the field have been gathering at an annual Riordan arrays symposium. In the following list, we present all the previous Riordan conferences, and the cities that have hosted them:

- Seoul, South Korea (2014)
- Lecco, Italy (2015)
- Bloomington, USA (2016)
- Madrid, Spain (2017)
- Busan, South Korea (2018)
- Sanya, China (2019)

Applications of Riordan arrays have been found in many areas of computing such as algorithm analysis [5], error correcting codes [7] and wireless communications [45]. Additionally, Riordan arrays have been used in different scientific areas beyond the borders of Mathematics as parts of their theory and techniques have been successfully applied in Molecular Biology for RNA secondary structure enumeration [66] and also in Chemistry [24].

1.2 Background and Aim of our study

Although the mathematical community that studies Riordan arrays and related topics has been growing over the last few decades, most of the research has been focused on the combinatorial properties of those matrices. Only a handful of papers have investigated the Riordan group and the algebraic properties of Riordan arrays in general [17, 19, 22, 49, 71, 73, 85].
Our study can be analysed in two main sections, the theoretical approach of the group structure of Riordan arrays, and the combinatorial behaviour of those mathematical constructions. Additionally, we have to mention that the order of those two sections has to be exclusively as above, for the simple reason that our results on the Riordan group determine our further research and guide us on which path to follow.
One of the main goals of our research is to study the structure of the Riordan group, to categorise its elements and to characterise possible subsets and subgroups of interest. In addition, we study the combinatorial properties of each significant Riordan subgroup individually, and try to create possible links between their algebraic structures, both by extending existing research and by formulating novel contributions in the area of the group theory of Riordan arrays.

Computation involving large data structures plays a vital role in the field of Riordan arrays and in Combinatorics in general. We have been using the symbolic mathematical computation program of Mathematica, to extend our computational abilities and provide us with accurate results. An invaluable online resource that helps on our research is the On-Line Encyclopedia of Integer Sequences (OEIS - `oeis.org`), which identifies known sequence of integers and gathers information on results including them.

1.3 Preliminaries

In order to determine a Riordan array, we introduce some special topics and methods from Analysis, that we are going to use from now on.

1.3.1 Sequences

A **sequence** is a mapping from the set $\mathbb{N}$ of natural numbers into some other known set of numbers $\mathbb{S}$, such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ [90]. A sequence is usually denoted as $(a_n)_{n\in\mathbb{N}}$ or $\{a_n | n \in \mathbb{N}\}$, and for the mapping a, we have that

$$a : \mathbb{N} \to \mathbb{S}$$

where

$$k \mapsto a_k,$$

for $k \in \mathbb{N}$.
For Riordan arrays, we are going to use a double sequence of integers, where we define a mapping of the set $\mathbb{N} \times \mathbb{N}$ to the set of integers, as

$$a : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$$

where

$$(n, k) \mapsto a_{n,k},$$

and $(n, k) \in \mathbb{N} \times \mathbb{N}$ represents the rows and columns of a matrix, respectively and $a_{n,k} \in \mathbb{Z}$ represents the entry of the matrix in this position. A double sequence is usually denoted as $(a_{n,k})_{n,k\in\mathbb{N}}$ or $\{a_{n,k} | n, k \in \mathbb{N}\}$.

1.3.2 Formal Power Series

Let $\mathbb{C}$ be the field of complex numbers and let z be any indeterminate over $\mathbb{C}$. A **formal power series** (fps) in $\mathbb{C}$ has the form

$$f_0 z^0 + f_1 z^1 + f_2 z^2 + f_3 z^3 + \cdots = \sum_{n=0}^{\infty} f_n z^n, \tag{1.1}$$

where $f_0, f_1, f_2, \ldots$ are all complex numbers [90]. The set of fps over a field $\mathbb{F}$ is denoted by $\mathbb{F}[[z]]$.
Let $\mathbb{F} = \mathbb{K}[[z]]$ be the ring of fps, with coefficients in $\mathbb{K}$, where $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$, with operations

$$\begin{aligned} a(z) + b(z) &= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n \\ &= \sum_{n=0}^{\infty} (a_n + b_n) z^n \end{aligned}$$

$$\begin{aligned} a(z) \cdot b(z) &= \sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_k b_{n-k}) z^n \end{aligned}$$

for the fps

$$a(z) = \sum_{n=0}^{\infty} a_n z^n \text{ , and } b(z) = \sum_{n=0}^{\infty} b_n z^n.$$

In Combinatorics, the coefficients $f_0, f_1, f_2, \ldots$ of the fps (1.1) are mostly used to count objects, and therefore usually we have $f_k \in \mathbb{N}$, where $k \in \mathbb{N}$, or in some cases $f_k \in \mathbb{Q}^+$, where $k \in \mathbb{N}$ [90].
Now, suppose that $f(z)$ is an fps such that $f(z) \in \mathbb{F}$, then the **order of an fps**

$$\operatorname{ord}\big(f(z)\big) = k, \text{ where } k \in \mathbb{N},$$

is defined as the lowest index of k for which $f_k \neq 0$. The **set of fps of order** k is denoted by $\mathbb{F}_k$ [90]. For the multiplicative inverse of an fps, we have the following theorem.

Theorem 1.3.1. *[90] Any fps, $f(z)$, is invertible if and only if $f(z) \in \mathbb{F}_0$.*

A fundamental operation for the theory of Riordan arrays, is the composition of fps [12]. Let

$$g(z) = \sum_{k=0}^{\infty} g_k z^k = g_0 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots,$$

and

$$f(z) = \sum_{k=1}^{\infty} f_k z^k = f_1 z + f_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots$$

be two fps, where $f(z) \in \mathbb{F}_1$ and $g(z) \in \mathbb{F}_0$. Then the composition of $g(z)$ and $f(z)$ is defined as follows

$$(g \circ f)(z) = g\big(f(z)\big) = \sum_{k=0}^{\infty} g_k \big(f(z)\big)^k = \sum_{k=0}^{\infty} \epsilon_k z^k,$$

where

$$\epsilon_k = \sum_{k \in \mathbb{N}_0, j} g_k f_{j_1} f_{j_2} \cdots f_{j_n},$$

and the sum is over all (k, j) with $k \in \mathbb{N}_0$ and $j \in \mathbb{N}^k$ with

$$|j| = j_1 + j_2 + j_3 + \cdots + j_k = n.$$

We also define $\bar{f}$ as the **compositional inverse** of f, that is

$$(f \circ \bar{f})(z) = (\bar{f} \circ f)(z) = z.$$

To avoid any possible confusion, we simply refer to the multiplicative inverse as the inverse.
By the definition of the composition of fps, we need to have $\operatorname{ord}\big(f(z)\big) \neq 0$, otherwise for two fps in $\mathbb{F}_0$,

$$\begin{aligned} g(z) &= g_0 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots, \text{ and} \\ f(z) &= f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \cdots, \end{aligned}$$

the composition $(g \circ f)(z)$ is defined as follows

$$\begin{aligned} (g \circ f)(z) &= g\big(f(z)\big) \\ &= g_0 + g_1 f(z) + g_2 f^2(z) + g_3 f^3(z) + \cdots \\ &= g_0 + g_1(f_0 + \cdots) + g_2(f_0 + \cdots)^2 + g_3(f_0 + \cdots)^3 + \cdots \\ &= (g_0 + g_1 f_0 + g_1 f_0^2 + g_2 f_0^3 + \cdots) + \text{modulo}(z), \end{aligned} \tag{1.2}$$

and the constant term of eq 1.2 is written as

$$g_0 + g_1 f_0 + g_1 f_0^2 + g_2 f_0^3 + \cdots = \sum_{k=0}^{\infty} g_k f_0^k.$$

Now, if $f_0 \neq 0$, and k tends to infinity, the power series contain a coefficient which is not in the field $\mathbb{K}$. Hence, we need an fps $f(z)$ that does not have a constant term.

1.3.3 Ordinary generating functions

As Herbert S. Wilf [102] wrote, "generating functions are a bridge between discrete mathematics and continuous analysis". An **(ordinary) generating function** (gf) is an important means of unifying the treatment of combinatorial problems [75]. However, the related theory which has been developing, is not only restricted in Combinatorics. There are a number of analysis problems which have been tackled using gfs, while as Combinatorics plays an important role in computer science, gfs are useful in designing counting algorithms [36, 81]. By using gfs, we are able to describe an infinite sequence of numbers in a neat and less complicated way, as we will see later on. This ability of gfs makes them a powerful tool for solving a variety of counting problems.
Now, suppose that we have a sequence of numbers $a_0, a_1, a_2, a_3, \ldots$. A useful representation of its gf is a closed form expression. Closed form expressions are independent of the general number n of arbitrary elements of the sequence a_n, $n \in \mathbb{N}$. So, we get the following definition:

Definition 1.3.1. *[90] The (ordinary) generating function (gf) of a sequence a_n is the fps*

$$G(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \tag{1.3}$$

where $a_n \in \mathbb{R}$.

For that purpose, usually OEIS provides us directly the gf of the sequence, if this sequence has been studied before.

Example 1.3.2. *[102] Suppose that we have the sequence $0, 1, 3, 7, 15, 31, \ldots$ [OEIS, A000225], which corresponds to the power series*

$$G(z) = z + 3z^2 + 7z^3 + 15z^4 + 31z^5 + \cdots = \sum_{n=0}^{\infty} a_n z^n.$$

Let us try to find $G(z)$. We notice that, the terms of the sequence satisfy the recurrence relation

$$a_{n+1} = 2a_n + 1, (n > 0; a_0 = 0). \tag{1.4}$$

Multiplying the LHS of (1.4) by z^n and summing over the values of n, for $n \geq 0$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n+1}z^n &= a_1 + a_2z + a_3z^2 + \cdots \\ &= \frac{(a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots) - a_0}{z} \\ &= \frac{G(z)}{z}. \end{aligned} \tag{1.5}$$

Doing the same for the RHS of (1.4), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(2a_n + 1)z^n &= 2G(z) + \sum_{n=0}^{\infty} z^n \\ &= 2G(z) + \frac{1}{1-z}, \end{aligned} \tag{1.6}$$

as the $\sum_{n=0}^{\infty} z^n$ *has coefficients correspond to the sequence* $1, 1, 1, 1, 1, \ldots$ *[OEIS, A000012], and has closed form expression of the geometrical series* $\frac{1}{1-z}$*, which is valid for* $|z| < 1$*. Hence, we get the equation*

$$\frac{G(z)}{z} = 2G(z) + \frac{1}{1-z},$$

and solving it for $G(z)$*, we have*

$$G(z) = \frac{z}{(1-z)(1-2z)},$$

which is the closed form gf of the given sequence.

1.3.4 Exponential generating functions

The exponential function $exp(z) = e^z$, can also be expressed as an exponential fps as

$$\begin{aligned} e^z &= exp(z) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \end{aligned}$$

More precisely, e^z can be written as

$$1 + 1 \cdot \frac{z}{1!} + 1 \cdot \frac{z^2}{2!} + 1 \cdot \frac{z^3}{3!} + \cdots$$

which, excluding the factorials on the denominator of each term, gives us the sequence $1, 1, 1, 1, \ldots$. Hence, e^z is the **exponential generating function** (egf) of the sequence of ones [12].
By generalising this concept, we have the exponential function that corresponds to the sequence d_n, for $n \in \mathbb{N}$, which is the egf of this sequence. This leads us to the following definition.

Definition 1.3.2. *[45] The exponential generating function (egf) $g(z)$, of a sequence d_n is the fps*

$$g(z) = \sum_{n=0}^{\infty} d_n \cdot \frac{z^n}{n!} = d_0 \cdot \frac{z^0}{0!} + d_1 \cdot \frac{z}{1!} + d_2 \cdot \frac{z^2}{2!} + d_3 \cdot \frac{z^3}{3!} + \cdots \tag{1.7}$$

1.3.5 Bivariate generating functions

We can also define generating functions for two variables instead of one. These types of generating functions are called **bivariate** (bgf), and are used instead of the simple ones, on some occasions. The procedure of expansion of a bgf is quite similar to the case of a simple gf, where the only difference is that we use one of the variables as a fixed number, and we expand the gf with respect to the other. As a result, we get a sequence which depends on the "fixed" variable. Expressing this sequence as a column vector, and extracting the coefficients of the "fixed" variable, we get an $n \times n$ (triangular) matrix.

Example 1.3.3. *Suppose that we were given the bivariate gf*

$$f(x, y) = \frac{1}{1 - 3x - xy}$$

Let us keep the variable y fixed, then $f(x, y)$ is written as $\frac{1}{1-(y+3)x}$, which is a geometrical progression with coefficients expanding as

$$1, y + 3, (y + 3)^2, (y + 3)^3, (y + 3)^4, \ldots$$

Expressing this sequence as a column vector, we get

$$\begin{bmatrix} 1 \\ y+3 \\ (y+3)^2 \\ (y+3)^3 \\ (y+3)^4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ y+3 \\ y^2+6y+9 \\ y^3+9y^2+27y+27 \\ y^4+12y^3+54y^2+108y+81 \\ \vdots \end{bmatrix}$$

Extracting the coefficient matrix from it, we get the following $n \times n$ matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 0 & 0 & 0 & \cdots \\ 1 & 6 & 9 & 0 & 0 & \cdots \\ 1 & 9 & 27 & 27 & 0 & \cdots \\ 1 & 12 & 54 & 108 & 81 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

As we saw earlier in Example 1.3.2, the gf of the first column is $\frac{1}{1-x}$. For the second column, we notice that all the entries are the sequence of natural numbers, multiplied by 3, so its gf is

$$\frac{3x}{(1-x)(1-x)}.$$

Similarly, the gf of the third column will be

$$\frac{9x^2}{(1-x)^3},$$

and goes on.

1.3.6 Coefficient extraction

We define an operator [12] to extract the coefficient of each term z^k, $\forall k \in \mathbb{N}$, of a power series

$$f(z) = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \cdots + f_k z^k + \cdots$$

Therefore, we have the **coefficient extraction operator** $[z^n]$, $\forall n \in \mathbb{N}$

$$[z^n] : R[[z]] \to R,$$

$$[z^n] f(z) = [z^n] \sum_{k=0}^{\infty} f_k z^k = f_n.$$

Example 1.3.4. *The coefficient of the term z^2 of the fps $\frac{z}{(1-z)^2}$, is*

$$\begin{aligned} [z^2]\left(\frac{z}{(1-z)^2}\right) &= [z^2](z + 2z^2 + 3z^3 + 4z^4 + \cdots) \\ &= 2 \end{aligned}$$

Now, for the fps $f(z), g(z) \in \mathbb{F}$ and the operator $[z^n], \forall n \in \mathbb{N}$ the following statements hold [64]

- $[z^n]\big(\mu f(z) + \lambda g(z)\big) = \mu[z^n]f(z) + \lambda[z^n]g(z)$, where $\mu, \lambda \in \mathbb{K}$ (Linearity)
- $[z^n]zf(z) = [z^{n-1}]f(z)$ (Shifting)
- $[z^n]f'(z) = (n+1)[z^{n+1}]f(z)$ (Differentiation)
- $[z^n]g(z)f(z) = \sum_{k=0}^{n} \big([z^k]g(z)\big)[z^{n-k}]f(z)$ (Convolution or Vandermonde's identity)
- $[z^n]g\big(f(z)\big) = \sum_{k=0}^{\infty} \big([z^k]g(z)\big)[z^n]f(z)^k$ (Composition)
- $[z^n]\bar{f}(z)^k = \frac{k}{n}[z^{n-k}]\left(\frac{z}{f(z)}\right)^n$ (Inversion)

1.4 Orthogonal polynomials

An **orthogonal polynomial sequence** $\big(p_n(x)\big)_{n\geq 0}$ [9] is an infinite sequence of polynomials $p_n(x)$ where $n \geq 0$, with real coefficients (often integer coefficients) that are mutually orthogonal on an interval $[x_0, x_1]$ (where $x_0 = -\infty$ is allowed, as well as $x_1 = +\infty$), with respect to a weight function $w : [x_0, x_1] \to \mathbb{R}$:

$$\int_{x_0}^{x_1} p_n(x)p_m(x)w(x)dx = \delta_{nm}\sqrt{h_n h_m}$$

where

$$h_n = \int_{x_0}^{x_1} p_n^2(x)w(x)dx.$$

and

$$\delta_{nm} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

We assume that w is strictly positive on the interval (x_0, x_1). The following theorem shows a "three-term recurrence" which is satisfied by any orthogonal polynomial sequence.

Theorem 1.4.1. *(Favard's Theorem) [52] Let $p_n(x)_{n\geq 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, 2, \ldots$. Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$, with $\beta_n \neq 0$ for all $n \geq 1$, such that the three-term recurrence*

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x),$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - b_0$.

The coefficients a_n, b_n and c_n are dependent on n but not x. We note that if

$$p_j(x) = x^j + k'_j x^{j-1} + ..., \quad \text{for } j = 0, 1, ...$$

then

$$a_n = \frac{k_{n+1}}{k_n}, \quad b_n = a_n \left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n} \right), \quad c_n = a_n \left(\frac{k_{n-1} h_n}{k_n h_{n-1}} \right).$$

Since the degree of $p_n(x)$ is n, the coefficient array of the polynomials is a lower triangular (infinite) matrix. In the case of monic orthogonal polynomials the diagonal elements of this array will all be 1. In this case, we can write the three-term recurrence as

$$p_{n+1}(x) = (x - \beta_n) p_n(x) - \alpha_n p_{n-1}(x), \tag{1.8}$$

with initial conditions

$$p_0(x) = 1, \quad p_1(x) = x - \beta_0.$$

The **moments** associated to the orthogonal polynomial sequence are the numbers

$$\mu_n = \int_{x_0}^{x_1} x^n w(x) dx.$$

Using the terms of the moment sequence $(\mu_n)_{n \in \mathbb{N}}$ in ascending order, we create a square matrix with constant entries along antidiagonals, as follows

$$M_n = \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n} \end{bmatrix}. \tag{1.9}$$

The following theorem is a criterion for the existence of a sequence of an orthogonal polynomial.

Theorem 1.4.2. *[23] Let $|M_n|$ be the determinant of the square matrix of moments M_n, $n \in \mathbb{N}$. A necessary and sufficient condition for the existence of an orthogonal polynomial sequence is*

$$|M_n| = \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_5 \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n} \end{vmatrix} \neq 0, \text{ for } n \in \mathbb{N} \tag{1.10}$$

1.5 Hankel transforms

The matrix 1.9, and its determinant 1.10 of the previous section, are related to the theory of Hankel transforms that we discuss in this section, individually. Now, let $(a_n)_{n\in\mathbb{N}}$ be an integer sequence. The infinite matrix

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\ a_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\ a_3 & a_4 & a_5 & a_6 & a_7 & \cdots \\ a_4 & a_5 & a_6 & a_7 & a_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.11}$$

is called the Hankel matrix, with elements $h_n = a_{i+j-1}$. The Hankel matrix H_n of order n is the upper-left $n \times n$ submatrix of H, and h_n, the Hankel determinant of order n, is the determinant of the corresponding Hankel matrix of order n, $h_n = det(H_n)$ [53].

Example 1.5.1. *Let the sequence* $2, 3, 8, 21, 55, 149, 404, 1097, 2981, ...$ *[OEIS-A004790], the Hankel matrix of order* 5 *is*

$$H_5 = \begin{bmatrix} 2 & 3 & 8 & 21 & 55 \\ 3 & 8 & 21 & 55 & 149 \\ 8 & 21 & 55 & 149 & 404 \\ 21 & 55 & 149 & 404 & 1097 \\ 55 & 149 & 404 & 1097 & 2981 \end{bmatrix}$$

with 5^{th} *order Hankel determinant* $h_5 = 385$. *Calculating the Hankel determinants of the submatrices* H_1, H_2, H_3, H_4, *we get the numbers* $2, 7, -1, -172$, *respectively.*

For the Hankel matrix 1.11, without lost of generality, by taking $a_0 = 1$, and the fact that H is positive definite, its definition will be as the following.

Definition 1.5.1. *[72] The* ***Hankel matrix*** $H = (h_{nk})_{n,k\geq 0}$ *generated by the sequence* $1, a_1, a_2, a_3, ...$ *is given by*

$$h_{00} = 1, h_{nk} = a_{n+k} \text{ for } n \geq 0, k \geq 0. \tag{1.12}$$

The sequence $(h_n)_{n\geq 1} = \{h_1, h_2, h_3, h_4, ...\}$ of Hankel determinants is called the **Hankel transform** of the matrix H_n . The Hankel transform of a sequence a_n, and its binomial transform are equal, while there is a number of integer sequences that have been found to have the same Hankel transform. [53].

1.6 Lattice paths

Some other mathematical objects with significant combinatorial properties are the lattice paths. Integer sequences are able to express the number of paths on a plane, according to given restrictions on the allowable steps. Let $\mathbb{Z}^2$ be the 2-dimensional integer lattice, and S be a subset of $\mathbb{Z}^2$. A **lattice path** L in $\mathbb{Z}^2$ of length k in S is a sequence of points $v_0, v_1, v_2, ..., v_k \in \mathbb{Z}^2$, such that each consecutive difference $v_i - v_{i-1}$ lies in S [93]. A pair of consecutive points is called a **step** of the path. A **valuation** is a function on the set of possible steps $\mathbb{Z}^2 \times \mathbb{Z}^2$. A valuation of a path is the product of the valuations of its steps, and it will be independent of the x-coordinates of the points. Therefore, we can represent a path π by the sequence of its y-coordinates $(\pi(0), \pi(1), \ldots, \pi(n))$ [54].
We give the definitions of different types of paths on a plane.

Definition 1.6.1. *[99] A **Dyck path** is a walk from point $(0,0)$ to point $(2n,0)$ with the diagonal steps $(1,1)$ (a "rise") and $(1,-1)$ (a "fall"), that lies strictly above (but may touch) the x-axis.*

The number of Dyck paths of order n is given by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and the gf

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}$$

i.e.1, 2, 5, 14, 42, 132, ... [OEIS, A000108].

Example 1.6.1. *The five Dyck paths for $n = 3$ are*

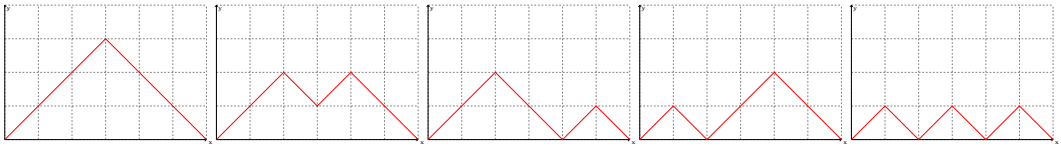


FIGURE 1.1: Dyck paths

Definition 1.6.2. *[54] A **Motzkin path** of length n is a lattice path starting at $(0,0)$ and ending at $(n,0)$ that satisfies the following conditions*

1. *The elementary steps can be $(1,1)$, $(1,0)$ and $(1,-1)$, or north-east(N-E), east(E) and south-east(S-E), respectively.*
2. *Steps never descend below the x-axis.*

The number of Motzkin paths of order n is given by the gf of Motzkin numbers

$$\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$$

i.e. 1, 1, 2, 4, 9, 21, 51,... [OEIS, A001006].

Example 1.6.2. *The four Motzkin paths for $n = 3$ are*

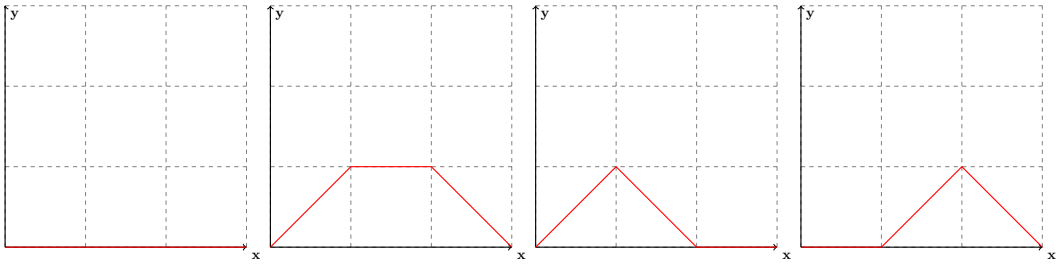


FIGURE 1.2: Motzkin paths

We notice that Dyck paths are Motzkin paths without east steps.

Definition 1.6.3. *[100] A **Schröder path** is the lattice path that starts at the point $(0,0)$ and ends at the point $(2n,0)$. It contains no points below the x-axis, and it is composed only of steps $(1,1)$, $(1,-1)$, and $(2,0)$.*

The number of Schröder paths S_n is given by Large Schröder numbers and the gf

$$G(z) = \frac{1-z-\sqrt{1-6z+z^2}}{2z}$$

which gives the sequence 1, 2, 6, 22, 90, 394,... [OEIS, A006318].

Example 1.6.3. *The six Schröder paths of order $n = 2$ are*

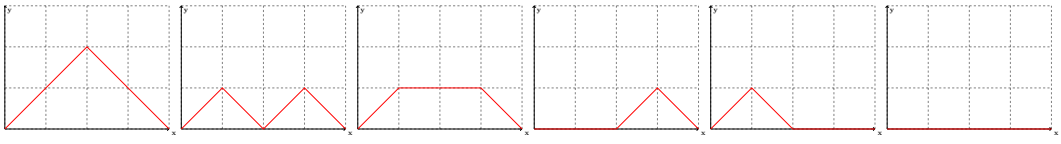


FIGURE 1.3: Schröder paths

We note that Dyck paths are Schröder paths without the step $(1,1)$.

Additionally, a Schröder path with east step $(1,0)$ instead of $(2,0)$, is called a **royal path** [100], as it can be pictured by the path of a king from $(0,0)$ to (n,n) on a chessboard.

1.7 Continued fractions

The theory of continued fractions that has been developed is related to many different areas of Combinatorics. In the current section, we focus on the link of continuous fractions with orthogonal polynomials, and how they are connected with Hankel transforms [35, 96]. Since the area of continued fractions is quite broad, we find it convenient to review it in the context of our topic. Therefore, we present two types of continued fractions as expansions of a formal power series.

Let the fps $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The **Jacobi continued fraction expansion** (J-fraction) of $f(z)$ is of the form

$$\cfrac{1}{1-b_0 z-\cfrac{a_1 z^2}{1-b_1 z-\cfrac{a_2 z^2}{1-b_2 z-\cfrac{a_3 z^2}{\ddots}}}},$$

and the **Stieltjes continued fraction expansion** (S-fraction) of $f(z)$ is of the form

$$\cfrac{a_0}{1-\cfrac{a_1 z^2}{1-\cfrac{a_2 z^2}{1-\cfrac{a_3 z^2}{\ddots}}}},$$

where $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$ are sequences of real numbers. Continued fractions are used to express the gfs of paths, according to the following theorem.

Theorem 1.7.1. *[35] Let*

$$\mu_n = \sum_{\pi \in \mathcal{M}_n} v(\pi)$$

where the sum is over $\mathcal{M}_n$, the set of Motzkin paths $\big(\pi(0), \pi(1), \ldots, \pi(n)\big)$ of length n. Here $\pi(j)$ is the level after the j^{th} step, and the valuation of a path is the product of the valuations of its steps $v = \prod_{i=1}^{n} v_i$, where

$$v_i = v\big(\pi(i-1), \pi(i)\big) = \begin{cases} 1, & \text{if the } i^{th} \text{ step rises,} \\ \beta_{\pi(i-1)}, & \text{if the } i^{th} \text{ step is horizontal,} \\ \alpha_{\pi(i-1)}, & \text{if the } i^{th} \text{ step falls.} \end{cases}$$

Then the gf of the sequence μ_n is given by

$$M(z) = \sum_{n=0}^{\infty} \mu_n z^n,$$

which is expanded to the continued fraction

$$M(z) = \cfrac{1}{1 - \beta_0 z - \cfrac{\alpha_1 z^2}{1 - \beta_1 z - \cfrac{\alpha_2 z^2}{1 - \beta_2 z - \cfrac{\alpha_3 z^3}{\ddots}}}}. \tag{1.13}$$

Now, since Dyck paths are Motzkin paths without the horizonal step, β_i's of the J-fraction 1.13 will be all 0. So, from Theorem 2.5.2 in [45], we similarly have that the gf of Dyck paths is expanded to the S-fraction

$$\cfrac{1}{1 - \cfrac{\alpha_1 z^2}{1 - \cfrac{\alpha_2 z^2}{1 - \cfrac{\alpha_3 z^2}{\ddots}}}}$$

Finally, the following theorem links the continued fractions to orthogonal polynomials, and Hankel transforms.

Theorem 1.7.2. *[52] Let $(a_n)_{n\geq 0}$ be a sequence of numbers with gf $g(z) = \sum_{k=0}^{\infty} a_k z^k$ that is expressed in its J-form, as*

$$\sum_{k=0}^{\infty} a_k z^k = \cfrac{a_0}{1 - b_0 z - \cfrac{c_1 z^2}{1 - b_1 z - \cfrac{c_2 z^2}{1 - b_2 z - \cfrac{c_3 z^3}{\ddots}}}}.$$

Then the Hankel determinant h_n of order n of the sequence $(a_n)_{n\geq 0}$ is given by

$$h_n = a_0^{n+1} c_1^n c_2^{n-1} \cdots c_{n-1}^2 c_n = a_0^{n+1} \prod_{k=1}^{n} c_k^{n+1-k}, \tag{1.14}$$

where the sequences $\{c_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 0}$ *are the coefficients in the recurrence relation*

$$p_n(z) = (z - b_n)p_{n-1}(z) - c_n p_{n-2}(z), n = 1, 2, 3, 4, ... \tag{1.15}$$

of the family of orthogonal polynomials p_n *for which* a_n *forms the moment sequence.*

We observe that eq 1.14 is independent from b_n, hence eq 1.15 can also be satisfied for the case of an S-form continuous fraction. Additionally, we note that (1.15) from the above Theorem is (1.8) from Section 1.4.

Chapter 2

Riordan arrays

In this chapter, we make a brief introduction to the area of Riordan arrays, by presenting the definition of an ordinary proper Riordan array, and the production matrix which comes by analysing further the recursive formula of such array. Additionally, we make a reference to the Exponential and Double Riordan arrays, emphasising the differences among those different types of Riordan arrays, and showing some examples.

2.1 Ordinary and proper Riordan arrays

Definition 2.1.1. *[83] An **Ordinary Riordan array** is a lower triangular infinite matrix R, constructed by two fps*

$$g(z) = \sum_{n=0}^{\infty} g_n z^n, \quad \text{and} \quad f(z) = \sum_{n=1}^{\infty} f_n z^n,$$

where $g(z) \in \mathbb{F}_0$, $f(z) \in \mathbb{F}_1$ in such a way that the gf of the k^{th} column is $g(z)\big(f(z)\big)^k$, for all $k \geq 0$. We say that R is a Riordan array or Riordan matrix and we write $R = \big(g(z), f(z)\big)$.

At this point, we ought to mention that although both of the terms Riordan array, and Riordan matrix are used to express the same mathematical object that is presented in Definition 2.1.1, the appropriate term is chosen every time to emphasise, according to our approach.
Now, using Definition 2.1.1, we present the following examples.

Example 2.1.1. *One of the simplest examples of the above construction is the Riordan matrix which is produced by a modified form of Pascal's Triangle,*

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

According to the above definition, this matrix is generated by two fps, of which the first one is invertible. The sequences of numbers of the first two columns of matrix P *are* $1,1,1,1,1,\ldots$ *[OEIS, A000012] and* $0,1,2,3,4,\ldots$ *[OEIS, A007953], which correspond to the coefficients of the polynomials*

$$1 + 1z + 1z^2 + 1z^3 + 1z^4 + \cdots = \sum_{n=0}^{\infty} z^n,$$

and

$$0 + 1z + 2z^2 + 3z^3 + 4z^4 + \cdots = \sum_{n=0}^{\infty} nz^n,$$

respectively. The first polynomial represents a geometric series, hence its gf is $\frac{1}{1-z}$*, while the second gf is* $\frac{z}{(1-z)^2}$*, which comes from the multiplication* $\frac{1}{1-z}\frac{z}{1-z}$*. Similarly, the third column of the matrix* $0,0,1,3,6,\ldots$ *[OEIS, A161680] corresponds to the polynomial*

$$0 + 0z + 1z^2 + 3z^3 + 6z^4 + \cdots = \sum_{n=0}^{\infty} \frac{n(n-1)}{2} z^n$$

and the gf $\frac{z^2}{(1-z)^3}$*, and so on.*
Hence, by using the gf of each column, it can be represented as

$$\left(\frac{1}{1-z}, \frac{z}{(1-z)^2}, \frac{z^2}{(1-z)^3}, \frac{z^3}{(1-z)^4}, \cdots\right),$$

and the Riordan matrix has the form

$$P = (g(z), f(z)) = \left(\frac{1}{1-z}, \frac{z}{1-z}\right).$$

Now, using the rules of coefficient extraction from subsection 1.3.6, we have that

$$
\begin{aligned}
[z^n]\frac{1}{1-z}\left(\frac{z}{1-z}\right)^k &= [z^n]\frac{z^k}{(1-z)^{k+1}} \\
&= [z^{n-k}](1-z)^{-(k+1)} \\
&= [z^{n-k}]\sum_{j=0}^{\infty}\binom{-(k+1)}{j}(-1)^j z^j \\
&= [z^{n-k}]\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}(-1)^j(-1)^j z^j \\
&= [z^{n-k}]\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}(-1)^{2j} z^j \\
&= \binom{k+1+n-k-1}{n-k} \\
&= \binom{n}{n-k} \\
&= \binom{n}{k}
\end{aligned}
$$

Example 2.1.2. *The gf of the sequence of Fibonacci numbers 1,1,2,3,5,8,13,21,... [OEIS, A000045] is* $f(z) = \frac{z}{1-z-z^2}$*, using this as the second gf and choosing* $g(z) = 1$*, we have the Riordan matrix*

$$
\begin{aligned}
F = (g(z), f(z)) &= \left(1, \frac{z}{1-z-z^2}\right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 2 & 2 & 1 & 0 & 0 & \cdots \\
0 & 3 & 5 & 3 & 1 & 0 & \cdots \\
0 & 5 & 10 & 8 & 4 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\end{aligned}
$$

Combining Theorem 1.3.1 and Definition 2.1.1, we say that an invertible fps together with a non-invertible fps generate a Riordan matrix. However, this condition can be satisfied by a wide range of fps, as a non-invertible fps can be any possible function $f(z) \in \mathbb{F}_k$, where $k \in \mathbb{N}^*$. As an aftermath, choosing any non-invertible function $f(z)$, it might affect the triangular form of the matrix, as we see in the following example.

Example 2.1.3. *Let us compare the matrices* $G = (1+z, z)$ *and* $H = (1+z, z^2)$*. Their matrix forms are as follows.*

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We observe that in matrix H, the first non-zero entry in the second column is the coefficient of the term z^2.

Riordan arrays like the matrix H from Example 2.1.3, are called vertically stretched Riordan arrays [27]. To avoid such problems, we give an extra definition.

Definition 2.1.2. *[103] A **proper Riordan array** is a matrix $R = \big(g(z), f(z)\big)$, where $f'(0) \neq 0$.*

Riordan matrices which do not satisfy Definition 2.1.2 are called improper. At this point, we need to mention that we have exclusively limited our results into Ordinary Proper Riordan arrays, unless otherwise noted.
A typical ordinary proper Riordan matrix is of the form

$$\big(g(z), f(z)\big) = \begin{bmatrix} g_0 & 0 & 0 & 0 & \cdots \\ g_1 & g_0 f_1 & 0 & 0 & \cdots \\ g_2 & g_0 f_2 + g_1 f_1 & g_0 f_1^2 & 0 & \cdots \\ g_3 & g_0 f_3 + g_1 f_2 + g_2 f_1 & 2 g_0 f_1 f_2 + g_1 f_1^2 & g_0 f_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

2.1.1 The Production matrix of a Riordan array - A and Z sequences

We observe that the entries of a Riordan array follow a recursive formula. Each of the entries of a Riordan matrix comes as a linear combination of entries of the previous row.

Example 2.1.4. *The entries $1, 5, 10, 10, 5, 1$ of the fifth row of Pascal's Triangle from Example 2.1.1, are expressed by using the entries $1, 4, 6, 4, 1$ of the previous row, following the pattern of Fig. 2.1.*

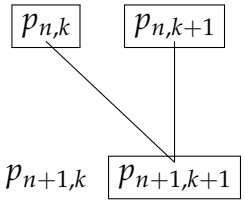


FIGURE 2.1: Diagram of recursive formula of Pascal's Triangle

More specifically, we see that each of the entry of Pascal's Triangle $(p_{n,k})_{n,k\geq 0}$*, except for the initial column, satisfy the recursive formula*

$$p_{n+1,k+1} = 1 \cdot p_{n,k} + 1 \cdot p_{n,k+1},$$

whereas, for the initial column, we have that

$$p_{n+1,0} = 1 \cdot p_{n,0}.$$

The coefficients of these two formulas, correspond to the sequences $1, 1, 0, 0, 0, 0, 0, ...$ *and* $1, 0, 0, 0, 0, 0, ...$*, respectively.*

Theorem 2.1.5. *[30] Let* $R = \big(g(z), f(z)\big) = (r_{n,k})_{n,k\geq 0}$ *be an infinite lower triangular matrix, where* n, k *are the numbers of the row and the column of each entry, respectively and* $r_{n,n} \neq 0$*. Then* R *is a proper Riordan matrix if and only if there exist unique sequences*

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, ...), \alpha_0 \neq 0, \text{ and } \zeta = (\zeta_0, \zeta_1, \zeta_2, \zeta_3, ...)$$

such that

1. *every element in column* 0 *can be expressed as a linear combination of all the elements in the preceding row, the coefficients being the element of the sequences* ζ*,*
$$r_{n+1,0} = \zeta_0 \cdot r_{n,0} + \zeta_1 \cdot r_{n,1} + \zeta_2 \cdot r_{n,2} + \zeta_3 \cdot r_{n,3} + \cdots;$$
2. *every element* $r_{n+1,k+1}$ *not lying in column* 0 *or row* 0*, can be expressed as a linear combination of the elements of the preceding row, starting from the preceding column, the coefficients being the elements of the sequence* α*,*
$$r_{n+1,k+1} = \alpha_0 \cdot r_{n,k} + \alpha_1 \cdot r_{n,k+1} + \alpha_2 \cdot r_{n,k+2} + \alpha_3 \cdot r_{n,k+3} + \cdots.$$

Definition 2.1.3. *[30] The sequences* α *and* ζ *of Theorem 2.1.5 will be called the* ***A-sequence*** *and the* ***Z-sequence*** *of the Riordan matrix* R*.*

These two sequences give rise to a square matrix which plays a vital role in the analysis of Riordan arrays.

Now, we are going to present the concept of an associated matrix which is generated by the recursive formulas of a Riordan array [29, 30, 31]. Let P_B be an infinite matrix, and r_0 be a row vector $r_0 = (1, 0, 0, ...)$. We define the row vector

$$r_i = r_{i-1} \cdot P_B, \text{ where } i \geq 1.$$

Stacking these rows we create another infinite matrix, which we denote by B. Then P_B is called the production matrix of B. More specifically, we have the following definition.

Definition 2.1.4. *[12] Let $R = \big(g(z), f(z)\big)$ be a Riordan array. The **production matrix** of this array is defined by*

$$P_R \quad = \quad R^{-1} \cdot \bar{R}, \tag{2.1}$$

where $\bar{R}$ denotes that we omit the first row of the array R.

Proposition 2.1.1. *[30] Let P_R be an infinite production matrix and let R be the matrix induced by P_R. Then R is a Riordan matrix if and only if P_R is of the form*

$$\begin{bmatrix} \zeta_0 & a_0 & 0 & 0 & 0 & \cdots \\ \zeta_1 & a_1 & a_0 & 0 & 0 & \cdots \\ \zeta_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ \zeta_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \zeta_4 & a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the sequences $(\zeta_0, \zeta_1, \zeta_2, ...)$ and $(a_0, a_1, a_2, ...)$ of the first two rows of the matrix P_R respectively, are the Z and A sequences of the Riordan matrix R.

Example 2.1.6. *[12] We have the Riordan matrix*

$$C = \big(c(z), zc(z)\big) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & \cdots \\ 14 & 14 & 9 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$c(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

is the gf of the Catalan numbers [OEIS, A000108].
By observation, the entries of matrix C follow the rules of Theorem 2.1.5, which give us the A-sequence $1, 1, 1, 1, ..$ and the Z-sequence $1, 1, 1, 1, ...$ Hence, the production matrix of C will be

$$P_C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which is equal to $C^{-1} \cdot \bar{C}$*, as*

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & -3 & 1 & 0 & \cdots \\ 0 & 0 & 3 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & \cdots \\ 14 & 14 & 9 & 4 & 1 & \cdots \\ 42 & 42 & 28 & 14 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Alternatively to Definition 2.1.4, we use $\bar{I} \cdot R$ instead of $\bar{R}$, where $\bar{I}$ stands for the modified square matrix of the Kronecker symbol

$$\bar{I} = (\delta_{i+1,j})_{i,j\geq 0} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.2}$$

Hence, eq 2.1 becomes $P_R = R^{-1} \cdot \bar{I} \cdot R$.
Additionally, by the definition of the production matrix, the A and Z sequences can be written in terms of the gfs of the matrix $\big(g(z), f(z)\big)$ [12] as

$$A(z) = \frac{z}{\bar{f}(z)} \quad \text{and} \quad Z(z) = \frac{1}{\bar{f}(z)}\left(1 - \frac{1}{g\big(\bar{f}(z)\big)}\right). \tag{2.3}$$

2.1.2 Riordan arrays and orthogonal polynomials

In this subsection we present some well-known results that link Riordan arrays, and orthogonal polynomials.
For $p_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k$, eq 1.8 becomes

$$\sum_{k=0}^{n+1} a_{n+1,k} z^k = (z - \alpha_n) \sum_{k=0}^{n} a_{n,k} z^k - \beta_n \sum_{k=0}^{n-1} a_{n-1,k} z^k$$

and,

$$a_{n+1,k} = a_{n,k-1} - \alpha_n a_{n,k} - \beta_n a_{n-1,k} \tag{2.4}$$

For α_n, β_n constants, we get a three-term recurrence formula that gives us the following proposition.

Proposition 2.1.2. *[9] Every Riordan array of the form*

$$\left(\frac{1}{1+rz+sz^2}, \frac{z}{1+rz+sz^2}\right)$$

is the coefficient array of a family of monic orthogonal polynomials.

For the general case, where α_n, β_n are not constant, and using the production matrix of a Riordan array, we have:

Corollary 2.1.7. *[9] If $L = \big(g(z), f(z)\big)$ is a Riordan array and its production matrix P is tri-diagonal, with*

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ b_1 & a & 1 & 0 & 0 & 0 & \cdots \\ 0 & b & a & 1 & 0 & 0 & \cdots \\ 0 & 0 & b & a & 1 & 0 & \cdots \\ 0 & 0 & 0 & b & a & 1 & \cdots \\ 0 & 0 & 0 & 0 & b & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.5}$$

then L^{-1} is the coefficient array of the family of orthogonal polynomials $p_n(z)$, where $p_0(z) = 1$, $p_1 = z - a_1$, and

$$p_{n+1}(z) = (z-a)p_n(z) - b_n p_{n-1}(z), \text{ for } n \geq 2,$$

where b_n is the sequence $0, b_1, b, b, b, ...$

This leads us to the following theorem.

Theorem 2.1.8. *[9] A Riordan array $L = \big(g(z), f(z)\big)$ is the inverse of the coefficient array of a family of orthogonal polynomials if and only if its production matrix P is tri-diagonal.*

2.2 Exponential Riordan arrays

Definition 2.2.1. *[31] An **exponential Riordan array** is a lower triangular infinite matrix E, constructed by two exponential gfs*

$$g_e(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!} \; ; \; f_e(z) = \sum_{n=1}^{\infty} f_n \frac{z^n}{n!},$$

where $g_\epsilon(z) \in \mathbb{F}_0$, $f_\epsilon(z) \in \mathbb{F}_1$, *in such a way that the gf of the* k^{th} *column of* E *is* $g_\epsilon(z)\frac{\left(f_\epsilon(z)\right)^k}{k!}$. *We say that* E *is an exponential Riordan array or matrix and we write*

$$E = \left[g_\epsilon(z), f_\epsilon(z)\right].$$

The $(n,k)^{th}$ element of an exponential Riordan array, is given by

$$\epsilon_{n,k} = \frac{n!}{k!}[z^n]g_\epsilon(z)f_\epsilon(z)^k. \tag{2.6}$$

A typical exponential Riordan matrix is of the form

$$\left[g_\epsilon(z), f_\epsilon(z)\right] = \begin{bmatrix} g_0 & 0 & 0 & 0 & \cdots \\ g_1 & g_0f_1 & 0 & 0 & \cdots \\ 2g_2 & 2g_1f_1 + g_0f_2 & g_0f_1^2 & 0 & \cdots \\ 6g_3 & 3(g_1f_2 + g_2f_1) + g_0f_3 & 3f_1(g_0f_2 + g_1f_1) & g_0f_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Example 2.2.1. *[12] Using the ordinary gfs of Pascal's triangle to create an exponential Riordan array, we have that*

$$T_\epsilon = \left[\frac{1}{1-z}, \frac{z}{1-z}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where each of the entries of the matrix comes from the formula

$$t_{n,k} = \frac{n!}{k!}[z^n]\frac{1}{1-z}\left(\frac{z}{1-z}\right)^k.$$

We note that a relationship between this matrix and its corresponding ordinary Riordan matrix, comes from the following factorization of T_ϵ *[12]*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 6 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 24 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 120 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{120} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

In Example 2.2.1, we saw how the ordinary gfs that produce the ordinary Riordan matrix of Pascal's triangle, generate a different exponential Riordan array. In the following example, we present the exponential gfs that corresponds to Pascal's triangle.

Example 2.2.2. *Pascal's Triangle is generated by the exponential Riordan matrix* $[e^z, z]$. *Hence,*

$$[e^z, z] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The corresponding A and Z sequences of the production matrix of an exponential Riordan array, A_ϵ and Z_ϵ respectively, are defined [12] by

$$A_\epsilon(z) = f'_\epsilon\big(\bar{f}_\epsilon(z)\big), \text{ and } Z_\epsilon(z) = \frac{g'_\epsilon\big(\bar{f}_\epsilon(z)\big)}{g_\epsilon\big(\bar{f}_\epsilon(z)\big)}, \tag{2.7}$$

where $A_\epsilon(z), Z_\epsilon(z) \in \mathbb{F}_0$. We owe it to mention that A_ϵ and Z_ϵ can also be found in bibliography as $r(z)$ and $c(z)$ [31], respectively. The bivariate gf $G_P(z, y)$ of the production matrix of an exponential Riordan array P_ϵ, [12] is given by

$$G_P(z, y) = e^{zy}\big(Z_\epsilon(z) + yA_\epsilon(z)\big). \tag{2.8}$$

In the case of an ordinary Riordan array, we see that its production matrix has a repeated pattern on its entries, except for the initial column. Although, the production matrix of an exponential Riordan array is generated similarly, the pattern of the entries is more complicated because of the definition of A_ϵ, and the $\frac{n!}{k!}$ factor.

By Definition 2.1.1 for the case of an exponential Riordan array $E = \big[g_\epsilon(z), f_\epsilon(z)\big]$, we get that

$$P_\epsilon = E^{-1} \cdot \overline{E}$$

and the entries of the production matrix $P_\epsilon = (p_{n,k})_{n,k\geq 0}$ satisfy the recursive formula

$$p_{n,k} = \frac{n!}{k!}(\zeta_{n-k} + k\alpha_{n-k+1}), \text{ where } \zeta_{-1} = 0 \quad [12, 31].$$

Hence, the production matrix of an exponential Riordan array is of the form

$$P_\epsilon = \begin{bmatrix} \zeta_0 & \alpha_0 & 0 & 0 & 0 & 0 & \cdots \\ 1!\zeta_1 & \frac{1!}{1!}(\zeta_0+\alpha_1) & \alpha_0 & 0 & 0 & 0 & \cdots \\ 2!\zeta_2 & \frac{2!}{1!}(\zeta_1+\alpha_2) & \frac{2!}{2!}(\zeta_0+2\alpha_1) & \alpha_0 & 0 & 0 & \cdots \\ 3!\zeta_3 & \frac{3!}{1!}(\zeta_2+\alpha_3) & \frac{3!}{2!}(\zeta_1+2\alpha_2) & \frac{3!}{3!}(\zeta_0+3\alpha_1) & \alpha_0 & 0 & \cdots \\ 4!\zeta_4 & \frac{4!}{1!}(\zeta_3+\alpha_4) & \frac{4!}{2!}(\zeta_2+2\alpha_3) & \frac{4!}{3!}(\zeta_1+3\alpha_2) & \frac{4!}{4!}(\zeta_0+4\alpha_1) & \alpha_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Example 2.2.3. *The production matrix of the exponential Riordan matrix in Example 2.2.1, is*

$$\begin{aligned} P_\epsilon &= T_\epsilon^{-1}.\overline{T_\epsilon} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & -4 & 1 & 0 & 0 & \cdots \\ -6 & 18 & -9 & 1 & 0 & \cdots \\ 24 & -96 & 72 & -16 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & \cdots \\ 120 & 600 & 600 & 200 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 5 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9 & 7 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 16 & 9 & 1 & \cdots \\ 0 & 0 & 0 & 00 & 25 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

Additionally, Corollary 2.1.7 and Theorem 2.1.8 that link the tri-diagonal production matrix of an Ordinary Riordan array with a family of orthogonal polynomials, can also be applied for tri-diagonal matrices of Exponential Riordan arrays with a few alterations in the recurrence formula [9].

2.3 Double Riordan arrays

According to the Definition 2.1.1 of an ordinary Riordan array $\big(g(z), f(z)\big)$, the power of the multiplier function determines the entries of each column of

a Riordan matrix. Now, suppose that $g(z)$ is the gf of the zero column and we use two multipliers $f_1(z)$ and $f_2(z)$, instead of one. Those gfs generate a Riordan matrix, according to the following alternating rule. The gf $g(z)f_1(z)$ generates the second column of the matrix, the gf $g(z)f_1(z)f_2(z)$ the third one, the gf $g(z)f_1(z)f_2(z)f_1(z)$ the forth one, and so on.

Definition 2.3.1. *[28] Let $g(z) = 1 + \sum_{k=1}^{\infty} g_k z^k$, and $f_1(z) = \sum_{k=0}^{\infty} f_{1k} z^k$, $f_2(z) = \sum_{k=0}^{\infty} f_{2k} z^k$, where $g \in \mathbb{F}_0$, $f_1, f_2 \in \mathbb{F}_1$ then the **double Riordan array** (or matrix) of $g(z)$, $f_1(z)$ and $f_2(z)$, denoted by $\big(g(z); f_1(z), f_2(z)\big)$ has column vectors*

$$\Big(g, gf_1, gf_1f_2, gf_1^2f_2, gf_1^2f_2^2, gf_1^3f_2^2, \dots\Big).$$

The set of all double Riordan matrices is denoted as $d\mathcal{R}$.

According to the column vectors of Definition 2.3.1, a typical double Riordan matrix is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ g_1 & f_{11} & 0 & 0 & \cdots \\ g_2 & f_{21} + g_1 f_{1_1} & f_{11}f_{12} & 0 & \cdots \\ g_3 & f_{31} + g_1 f_{21} + g_2 f_{11} & f_{11}f_{22} + f_{21}f_{12} + g_1 f_{11} f_{12} & f_{11}^2 f_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where g_k, f_{1k}, and f_{2k} are the coefficients of the gf g, f_1 and f_2, for $k \in \mathbb{N}$, respectively.

Example 2.3.1. *Let $g(z) = \frac{1}{1-z^2}$, $f_1(z) = \frac{z}{1-z^2}$ and $f_2(z) = z$. These three gfs give rise to the double Riordan array*

$$\begin{aligned} D &= (g; f_1, f_2) \\ &= \left(\frac{1}{1-z^2}; \frac{z}{1-z^2}, z\right) \\ &= \left(\frac{1}{1-z^2}, \frac{z}{(1-z^2)^2}, \frac{z^2}{(1-z^2)^2}, \frac{z^3}{(1-z^2)^3}, \frac{z^4}{(1-z^2)^3}, \dots\right) \end{aligned}$$

which is equal to the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 2 & 0 & 1 & 0 & \cdots \\ 0 & 3 & 0 & 3 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Chapter 3

Algebraic Structures of Riordan arrays

In this chapter, we present the main results of the Riordan group theory that our research is based on. This contains properties of the Riordan group, all known Riordan subgroups, different types of Riordan arrays and some of the latest results which are related to our research.

3.1 The Ordinary Riordan group

Before we proceed to the definition of the operation of the Riordan group, we need to refer to a theorem which is known as the **Fundamental Theorem of Riordan arrays** (FTRA)[86].

Theorem 3.1.1. *(FTRA) [86] Let $R = \big(g(z), f(z)\big)$ be a Riordan matrix and P, Q are two column vectors, where their gfs are $p(z)$ and $q(z)$, such that*

$$R \cdot P = Q.$$

This relation holds if and only if the following relation among the gfs is true

$$\begin{aligned} \big(g(z), f(z)\big) \cdot p(z) &= q(z) \\ \Leftrightarrow g(z)p\big(f(z)\big) &= q(z). \end{aligned}$$

FTRA simply says that the product $R \cdot P$ between a Riordan matrix R and a vector $P = [p_0, p_1, p_2, \dots]^T$ has generating series $g(z)p\big(f(z)\big)$, if $R = \big(g(z), f(z)\big)$, and $p(z) = \sum_{n\geq 0} p_n z^n$.
Using FTRA, and the fact that $\frac{1}{1-z}$ is the gf which has coefficient sequence of all ones, we have the following relation for the **gf of the row sums of a Riordan matrix** [85]

$$\big(g(z), f(z)\big)\frac{1}{1-z} = \frac{g(z)}{1-f(z)}. \tag{3.1}$$

Now, the operation $\cdot$ of the Riordan group, combining the gfs of the matrices, is defined as follows. Suppose that we have two Riordan matrices

$$\Lambda = \big(g(z), f(z)\big), \text{ and } K = \big(h(z), k(z)\big)$$

then we define the product $\Lambda \cdot K$ as

$$\begin{aligned} \Lambda \cdot K &= \big(g(z), f(z)\big) \cdot \big(h(z), k(z)\big) \\ &= \Big(g(z)h\big(f(z)\big), k\big(f(z)\big)\Big). \end{aligned}$$

The product of two Riordan matrices is equal to the ordinary matrix multiplication, since for the Riordan matrices

$$R_1 = [r'_{n,k}] = \big(g(z), f(z)\big) \text{ , and } R_2 = [r''_{n,k}] = \big(h(z), k(z)\big)$$

we have

$$R = R_1 \cdot R_2 = \Big(g(z)h\big(f(z)\big), k\big(f(z)\big)\Big).$$

Now, for $R = [r_{n,k}]$ we have:

$$\begin{aligned} r_{n,k} &= [z^n]g(z)h\big(f(z)\big)k\big(f(z)\big)^k \\ &= [z^n]g(z)\sum_{i\geq 0} r''_{i,k} f(z)^i \\ &= [z^n]\sum_{i\geq 0} r''_{i,k} g(z) f(z)^i \\ &= [z^n]\sum_{i\geq 0} r''_{i,k} \sum_{n\geq 0} r'_{n,i} z^n \\ &= [z^n]\sum_{n\geq 0} \left(\sum_{i\geq 0} r'_{n,i} r''_{i,k}\right) z^n \\ &= \sum_{i\geq 0} r'_{n,i} r''_{i,k} \\ &= \sum_{i=k} r'_{n,i} r''_{i,k}. \end{aligned}$$

We note that the product is well defined, since the matrices are lower triangular.
This product can be shown to be associative, with $I = (1, z)$ being the identity element. i.e.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Additionally, the inverse element is given by

$$\big(g(z), f(z)\big)^{-1} = \left(\frac{1}{g\big(\bar{f}(z)\big)}, \bar{f}(z)\right),$$

which leads us to the following definition.

Definition 3.1.1. *[83] The set $\mathcal{R}$ of all Riordan arrays together with the above operation, form the **Riordan group**, $\langle \mathcal{R}, \cdot \rangle$.*

The order of this group is infinite, while regarding the order of the Riordan elements, we know that any element with integer entries having finite order must have order 1 or 2 [84]. The only Riordan element which has order 1 is the identity, whereas there is more than one element of order 2.

Definition 3.1.2. *[49] Let $\Lambda = \big(g(z), f(z)\big)$ be a Riordan matrix. If $\Lambda \cdot \Lambda = I$, then Λ is called an **involution**. i.e. Riordan elements of order 2 are called involutions.*

Example 3.1.2. *The element*

$$(1, -z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is an involution, as

$$\begin{aligned} (1, -z) \cdot (1, -z) &= \big(1, -(-z)\big) \\ &= (1, z). \end{aligned}$$

Proposition 3.1.1. *[50] The Riordan array $\big(g(z), f(z)\big)$ is an involution if and only if*

$$g(z) = \frac{1}{g\big(\bar{f}(z)\big)}, \text{ and } f(z) = \bar{f}(z).$$

We note that the relationship $f(z) = \bar{f}(z)$ emanates from the Babbage functional equation [3, 4]

$$f(z)^{\circ k} = f(z) \underbrace{\circ \cdots \circ}_{k-times} f(z) = z \tag{3.2}$$

for $k = 1$. So, for the self-compositional inverse function $f(z)$, where $f\big(f(z)\big) = z$, we have $f(z) = \bar{f}(z)$.

Definition 3.1.3. *[16] Let K be a Riordan matrix and $M = (1, -z)$. If $K \cdot M$ is an involution, then we call K a **pseudo-involution**. i.e. $(K \cdot M)^2 = I$ i.e $K \cdot M$ has order 2.*

Example 3.1.3. *The Pascal's triangle that we described earlier in Example 2.1.1 is a pseudo-involution as*

$$\begin{aligned} P \cdot M &= \left(\frac{1}{1-z}, \frac{z}{1-z}\right) \cdot (1, -z) \\ &= \left(\frac{1}{1-z}, -\frac{z}{1-z}\right), \end{aligned}$$

and

$$\begin{aligned} (P \cdot M)^2 &= \left(\frac{1}{1-z}, -\frac{z}{1-z}\right) \cdot \left(\frac{1}{1-z}, -\frac{z}{1-z}\right) \\ &= \left(\frac{1}{1-z}\frac{1}{1+\frac{z}{1-z}}, \frac{\frac{z}{1-z}}{1+\frac{z}{1-z}}\right) \\ &= (1, z). \end{aligned}$$

At this point, we need to mention that although the identity element satisfies the condition $I \cdot I = I$, we consider it as a trivial case of involution, therefore we usually avoid it. Nevertheless, some other authors think of it as an example of a pseudo-involution [49].

Proposition 3.1.2. *[17] A sufficient and necessary condition for a Riordan element $K = \big(g(z), f(z)\big)$ to be a pseudo-involution is to have*

$$-f\big(-f(z)\big) = z \text{ , and } \quad g(z) = \frac{1}{g\big(-f(z)\big)}.$$

3.2 Ordinary Riordan subgroups

Shortly after the publication of Shapiro's historical paper, the first few Riordan subgroups appeared, by choosing the appropriate gfs, according to the rules of the Riordan multiplication.
In personal discussions with L. Shapiro (October 2017, and June 2018), about the origins of the known Riordan subgroups, he informed us that the names of the Associated and the Appell subgroups came from the Umbral Calculus and possibly from G-C. Rota and S.M. Roman. Furthermore, he notified us that there was a scientific team led by himself at Howard University, USA. One of their purposes was to name objects related to this new found field. Thus, except for the name of Riordan arrays, they also came up with the terms of Bell, Power-Bell, Hitting-time, Derivative, Checkerboard, Derivative, and Stochastic subgroups.
In this section, we present all Riordan subgroups that have been defined to date, together with the A and Z sequences of each subgroup. Whenever it is possible, we express Z sequences in terms of A sequences. We finally present a more detailed form of the Stieltjes matrices of (some of) the subgroups. For the rest of the section, let $1, a_1, a_2, ...$ be the A sequence, and $1, g_1, g_2, ...$ be the corresponding sequence of the first generating function of a Riordan matrix $\big(g(z), f(z)\big)$.

3.2.1 The Associated subgroup

One of the simplest forms of Riordan subgroups, is the **Associated** (or Lagrange) subgroup (Assoc). It contains Riordan elements of the form $\big(1, f(z)\big)$, and it is the only Riordan subgroup that contains a constant as a gf. The Associated Riordan subgroup is isomorphic to the group of fps under composition [41], and it is also the stabilizer of $(k, 0, 0, \dots)^T, \forall k \in \mathbb{Z}^*$, as by applying the FTRA, we get

$$\big(1, f(z)\big) \cdot \begin{bmatrix} k \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The A and Z sequences of an Associated matrix [12] are

$$A(z) = \frac{z}{\bar{f}(z)}, \text{ and } Z(z) = 0.$$

The production matrix of any Associated Riordan array is of the form

$$P_{Assoc} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & a_1 & 1 & 0 & \cdots \\ 0 & a_2 & a_1 & 1 & \cdots \\ 0 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

3.2.2 The Bell subgroup

The **Bell** (a.k.a. Renewal or Rogers) subgroup contains Riordan elements of the form $\big(g(z), f(z)\big)$, where $f(z) = zg(z)$. Alternatively, Bell elements can be written as $\left(\frac{f(z)}{z}, f(z)\right)$. This ability of the Bell subgroup to be presented in two ways, solely in terms of $g(z)$ or in terms of $f(z)$ functions, will be used later on. An example of a Riordan element of this subgroup, is Pascal's Triangle, for $f(z) = \frac{z}{1-z}$, or $g(z) = \frac{1}{1-z}$.
The A and Z sequences of a Bell matrix [12] are

$$A(z) = \frac{z}{\bar{f}(z)}\text{, and } Z(z) = \frac{A(z)-1}{z}, \tag{3.3}$$

and its production matrix is

$$P_{Bell} = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots \\ a_2 & a_1 & 1 & 0 & \cdots \\ a_3 & a_2 & a_1 & 1 & \cdots \\ a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Moreover, as an extension of the original subgroup, we have

$$c\text{-Bell} = \left\{ \left(\frac{f(z)}{z}, cf(z) \right) \middle| c \neq 0 \right\}, \tag{3.4}$$

for any distinct value of c [85].

3.2.3 The Appell subgroup

The **Appell** (or Toeplitz) subgroup contains Riordan elements of the form $\big(g(z), z\big)$. The Appell subgroup (App) is isomorphic to the group of invertible fps under multiplication [41]. This is the only known abelian Riordan subgroup, and it is also the only Riordan subgroup that is $\big(g(z), z\big) \triangleleft \mathcal{R}$, i.e. it is normal in $\mathcal{R}$. By using the latter property, we are allowed to present the Riordan group as a semi-direct product of the Appell and the Associated subgroups [67], as

$$\big(g(z), f(z)\big) = \big(g(z), z\big) \cdot \big(1, f(z)\big), \tag{3.5}$$

and as a semi-direct product of the Appell and the Bell subgroups [16], with the appropriate form of the Appell subgroup, as

$$\big(g(z), f(z)\big) = \left(\frac{zg(z)}{f(z)}, z\right) \cdot \left(\frac{f(z)}{z}, f(z)\right). \tag{3.6}$$

Proposition 3.2.1. *The Appell subgroup cannot be a stabilizer of any vector.*

Proof. Let $h(z) \in \mathbb{F}_k$. Applying FTRA to any Riordan element of the Appell subgroup, we get $g(z) = 1$, which corresponds to the identity matrix. □

The A and Z-sequence of an Appell matrix [12] are

$$A(z) = 1 \text{ , and } Z(z) = \frac{g(z) - 1}{g(z)z}.$$

The production matrix of an Appell Riordan array is of the form

$$P_{App} = \begin{bmatrix} g_1 & 1 & 0 & 0 & \cdots \\ g_2 - g_1^2 & 0 & 1 & 0 & \cdots \\ g_3 - 2g_1g_2 + g_1^3 & 0 & 0 & 1 & \cdots \\ g_4 - 2g_1g_3 - g_2^2 + 3g_1^2g_2 - g_1^4 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

More Appell subgroups can also be defined, as extensions of the original subgroup [85],

$$c\text{-Appell} = \big\{ \big(g(z), cz\big) \big| c \neq 0 \big\}. \tag{3.7}$$

for any distinct value of c.

3.2.4 The Derivative subgroup

The **Derivative** (or Co-Lagrange) subgroup (Der) contains Riordan elements of the form $\big(f'(z), f(z)\big)$, where $f \in \mathbb{F}_1$. It is proven that the Derivative, the Associated and the Bell subgroups are isomorphic [49].
The A and the Z-sequences of a Derivative matrix [12] are

$$A(z) = \frac{z}{\bar{f}(z)} \text{ , and } Z(z) = \frac{A(z) - 1}{z} + \frac{A'(z)}{A(z)}.$$

The production matrix of a Derivative Riordan array is

$$P_{Der} = \begin{bmatrix} 2a_1 & 1 & 0 & 0 & \cdots \\ 3a_2 - a_1^2 & a_1 & 1 & 0 & \cdots \\ 4a_3 - 3a_1a_2 + a_1^3 & a_2 & a_1 & 1 & \cdots \\ 5a_4 - 4a_1a_3 + 4a_1^2a_2 - 2a_2^2 + a_1^4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

3.2.5 The Hitting-time subgroup

The **Hitting-time** subgroup (H-t) contains Riordan elements of the form $\left(\frac{zf'(z)}{f(z)}, f(z)\right)$. It was introduced in 2000 [71], and it became the second subgroup which involves derivatives.
The name of the subgroup came from Stochastic Processes, where a hitting time is the first time at which a given process "hits" a given subset of the state space [94].
Every Hitting-time Riordan matrix satisfies a divisibility property, according to which, every Hitting-time Riordan matrix $(m_{n,k})_{n,k\geq 0}$ satisfies the property

$$n/km_{n,k}, \text{ whenever } 0 < k < n, \text{ for } k, n \in \mathbb{N}.$$

It is also proven that the Hitting-time subgroup is isomorphic to the Derivative subgroup [42].
The A and the Z-sequences of a Hitting-time matrix [12] are

$$A(z) = \frac{z}{\bar{f}(z)}, \text{ and } Z(z) = A'(z)$$

Hence, the production matrix of the Hitting-time subgroup is of the form

$$P_{H-t} = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots \\ 2a_2 & a_1 & 1 & 0 & \cdots \\ 3a_3 & a_2 & a_1 & 1 & \cdots \\ 4a_4 & a_3 & a_2 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

3.2.6 The Checkerboard subgroup

The **Checkerboard** subgroup (Checkb) contains Riordan elements of the form $(g_e(z), f_o(z))$, where g_e is an even function, and f_o is an odd function. The Checkerboard subgroup is the centralizer of $M = (1, -z)$ [49], as

$$\big(g_e(z), f_o(z)\big) \cdot (1, -z) = (1, -z) \cdot \big(g_e(z), f_o(z)\big).$$

A typical element of this Riordan subgroup, follows a "black and white" pattern, as follows

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & \cdots \\ 0 & * & 0 & 0 & 0 & \cdots \\ * & 0 & * & 0 & 0 & \cdots \\ 0 & * & 0 & * & 0 & \cdots \\ * & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where " $*$ " represents a non-zero entry. Such arrays are called aerated [28].

Example 3.2.1. *The gfs* $g(z) = \frac{1}{\sqrt{1-4z^2}}$, *and* $f(z) = \frac{1-\sqrt{1-4z^2}}{2z}$ *are even and odd functions, respectively. They generate the Riordan matrix*

$$\big(g(z), f(z)\big) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 1 & 0 & \cdots \\ 6 & 0 & 4 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

3.2.7 The Stabilizer subgroup

The **Stabilizer** subgroup (Stab) [49] contains Riordan elements of the form $\left(\frac{h(z)}{h\big(f(z)\big)}, f(z)\right)$. This is the only known subgroup that uses composition on its gfs . It stabilizes a column vector $h(z)$,

$$\big(g(z), f(z)\big) \cdot h(z) = h(z) \tag{3.8}$$

and by applying FTRA, we get

$$g(z) \cdot h\big(f(z)\big) = h(z),$$

which gives

$$g(z) = \frac{h(z)}{h\big(f(z)\big)}.$$

We usually denote the stabilizer of $h = h(z)$, as S_h [43] so that

$$S_h = \big\{ \big(g(z), f(z)\big) \big| \big(g(z), f(z)\big) \cdot h(z) = h(z) \big\}. \tag{3.9}$$

The A and Z-sequences of a Stabilizer matrix [12] are

$$A(z) = \frac{z}{\bar{f}(z)}\text{, and } Z(z) = \frac{1}{\bar{f}(z)}\left(1 - \frac{h(z)}{h(\bar{f}(z))}\right).$$

The Stabilizer subgroup $\left(\frac{h(z)}{h\left(f(z)\right)}, f(z)\right)$ was defined, without providing any information about the characteristics of the arbitrary function $h(z)$ [49]. For the two gfs of a Stabilizer Riordan array we need

$$\frac{h(z)}{h\left(f(z)\right)} \in \mathbb{F}_0, \text{ and } f(z) \in \mathbb{F}_1.$$

Now, let us take an arbitrary function $h(z) \in \mathbb{F}_k$, for $k \in \mathbb{N}$, such that

$$h(z) = h_k z^k + h_{k+1} z^{k+1} + h_{k+2} z^{k+2} + \cdots,$$

and an $f(z) \in \mathbb{F}_1$, so

$$f(z) = f_1 z + f_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots$$

Then

$$\begin{aligned} h\left(f(z)\right) &= h_k(f_1 z + f_2 z^2 + \cdots)^k + h_{k+1}(f_1 z + f_2 z^2 + \cdots)^{k+1} \\ &\qquad + (f_1 z + f_2 z^2 + \cdots)^{k+2} + \cdots \\ &= h_k f_1^k z^k + \cdots \in \mathbb{F}_k. \end{aligned}$$

Hence, we have that $\frac{h(z)}{h\left(f(z)\right)}$ is the division of two formal power series in $\mathbb{F}_k$, thus having a non-zero constant as its first term. We are going to examine the above observation further in Chapter 7.

3.2.8 The Stochastic subgroup

The **Stochastic** subgroup (Stoch) contains Riordan elements of the form $\left(\frac{f(z)-1}{z-1}, f(z)\right)$ [85]. Every Stochastic matrix has row sums equal to one. Alternatively, the Stochastic subgroup is the stabilizer of the column vector $(1, 1, 1, ..)^T$ [85], as

$$\left(\frac{f(z)-1}{z-1}, f(z)\right) \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{1}{1-z} \tag{3.10}$$

The A and Z-sequences of a Stochastic matrix [12] are

$$A(z) = \frac{z}{\bar{f}(z)}\text{, and } Z(z) = \frac{A(z)-1}{z-1}.$$

The production matrix of the Stochastic subgroup is of the form

$$P_{Stoch} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ -a_1 & a_1 & 1 & 0 & \cdots \\ -a_1-a_2 & a_2 & a_1 & 1 & \cdots \\ -a_1-a_2-a_3 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We notice that the row sums of its production matrix are also equal one. Additionally, we observe that the gfs of the production matrix are analogous to the gfs of the subgroup.
Expanding the stabilizing property (3.10) of the Stochastic subgroups, Riordan subgroups are defined as a stabilizer S_h, where $h(z) = \frac{1}{1-kz}$, for $k \in \mathbb{Z}^*$ [43].

Proposition 3.2.2. *[43] The set of Riordan arrays*

$$S_{\frac{1}{1-kz}} = \left\{ \left(\frac{kf(z)-1}{kz-1}, f(z) \right) \middle| f(z) \in \mathbb{F}_1 \right\},$$

forms a subgroup.

Proof. Consider

$$S_1 = \left(\frac{kf(z)-1}{kz-1}, f(z) \right), \text{ and } S_2 = \left(\frac{kg(z)-1}{kz-1}, g(z) \right)$$

as two elements of $S_{\frac{1}{1-kz}}$. We have

$$\begin{aligned} S_1 \cdot S_2 &= \left(\frac{kf(z)-1}{kz-1}, f(z) \right) \cdot \left(\frac{kg(z)-1}{kz-1}, g(z) \right) \\ &= \left(\frac{kf(z)-1}{kz-1} \frac{kg(f(z))-1}{kf(z)-1}, g(f(z)) \right) \\ &= \left(\frac{kg(f(z))-1}{kz-1}, g(f(z)) \right) \end{aligned}$$

This shows closure. The inverse element will be

$$\begin{aligned}\left(\frac{kf(z)-1}{kz-1}, f(z)\right)^{-1} &= \left(\frac{1}{\frac{kf(\bar{f}(z))-1}{k\bar{f}(z)-1}}, \bar{f}(z)\right)\\ &= \left(\frac{k\bar{f}(z)-1}{kz-1}, \bar{f}(z)\right).\end{aligned}$$

Hence, $S_{\frac{1}{1-kz}}$ is a Riordan subgroup. □

3.2.9 The Cheon subgroups

The **Cheon** subgroups contain Riordan elements of the form $\left(g(z), zf(z^m)\right)$, where $g, f \in \mathbb{F}_0$ and $m \in \mathbb{N}$ [21], and we denote as

$$H_m = \left\{ \left(g(z), zf(z^m)\right) \middle| g, f \in \mathbb{F}_0 \text{ and } m \in \mathbb{N} \right\}.$$

This family of subgroups contains the Checkerboard subgroup, and satisfies that H_k is a subgroup of H_m if and only if k is a multiple of m [21].
The A-sequence of a Cheon matrix for $m \in \mathbb{N}$, is

$$\begin{aligned}A(z) &= \frac{z}{z\bar{f}(z^m)}\\ &= \frac{1}{\bar{f}(z^m)}.\end{aligned}$$

For its Z-sequence we have that

$$\begin{aligned}Z(z) &= \frac{1}{z\bar{f}(z^m)}\left(1 - \frac{1}{g(z\bar{f}(z^m))}\right)\\ &= \frac{A(z)}{z}\left(1 - \frac{1}{g\left(\frac{z}{A(z)}\right)}\right),\end{aligned}$$

which can be written as

$$\frac{zZ(z)}{A(z)} + \frac{1}{g\left(\frac{z}{A(z)}\right)} = 1.$$

3.2.10 The family of Power-Bell subgroups

The **Power-Bell** subgroups contain Riordan elements of the form $\left(g(z), zg(z)^r\right)$, where r is a fixed real number [49] for $g_0 = 1$, while for $r = 0$ and $r = 1$, we have the Appell and the Bell subgroups, respectively. The A and the Z-sequences of a power-Bell matrix are

$$A(z) = \frac{z}{\bar{f}(z)}\text{, and } Z(z) = \frac{A(z)^n - 1}{zA(z)^{n-1}}.$$

Since the production matrix of a power-Bell subgroup depends on the value of its power n, we present the matrices of the cases for $n = 2$ and $n = 3$,

$$\left(\left(\frac{f(z)}{z}\right)^2, f(z)\right), \text{ and } \left(\left(\frac{f(z)}{z}\right)^3, f(z)\right),$$

as

$$P_{power-Bell(2)} = \begin{bmatrix} 2a_1 & 1 & 0 & 0 & \cdots \\ 2a_2 - a_1^2 & a_1 & 1 & 0 & \cdots \\ 2a_3 - 2a_1a_2 + a_1^3 & a_2 & a_1 & 1 & \cdots \\ 2a_4 - a_2^2 + 3a_1^2a_2 - 2a_1a_3 - a_1^4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and

$$P_{power-Bell(3)} = \begin{bmatrix} 3a_1 & 1 & 0 & 0 & \cdots \\ 3a_2 - 3a_1^2 & a_1 & 1 & 0 & \cdots \\ 4a_1^3 + 3a_3 - 6a_1a_2 & a_2 & a_1 & 1 & \cdots \\ -5a_1 - 6a_1a_3 - 12a_1^2a_2 - 3a_2^2 + 3a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

respectively.

3.2.11 A special family of Riordan subgroups

An important addition to the area of Riordan subgroups was made in 2014, where a general form of subgroups, presented as a new Riordan family of subgroups by Ana Luzon et al. [58], by combining the gfs of some of the subgroups. They first defined the family as

$$H[r,s] = \left\{ \left(\left(\frac{f(z)}{z} \right)^r f'(z)^s, f(z) \right) \middle| f \in \mathbb{F}_1, (r,s) \in \mathbb{Z}^2 \right\}, \tag{3.11}$$

and then, by using this notation, they expressed those Riordan subgroups, according to the powers r and s, as in the following table.

Name of the subgroup	$H[r,s]$
Associated	$H[0,0]$
Derivative	$H[0,1]$
Bell	$H[1,0]$
Hitting-time	$H[-1,1]$

TABLE 3.1: H-notation of some Riordan subgroups

3.2.12 Other Riordan subgroups

Searching for other Riordan subgroups, we know that the set of the Riordan pseudo-involutions in general, do not form a subgroup [17], while on the other hand the set of Riordan involutions can be characterised by general recurrence formulas by the Theorem of the Riordan Involutions Formula [59]. In addition, Riordan matrices with 1's in the main diagonal form a group, according to the following theorem.

Theorem 3.2.2. *[60] The commutator subgroup of $\mathcal{R}$, denoted by $[\mathcal{R},\mathcal{R}]$, is formed by all Riordan matrices with 1's in the main diagonal. That is*

$$[\mathcal{R},\mathcal{R}] = \{ (g(z), f(z)) | g_0 = 1, f_1 = 1 \}.$$

A different notation of Riordan arrays, based on a parametrization which was first presented in [55], is used as it is more efficient than the classical one, for the purpose of this research. Suppose that

$$D = (g(z), f(z)) = (d_{i,j})_{i,j \geq 0}$$

is a Riordan matrix. Then we have

$$\begin{aligned}(g(z), f(z)) &= T\left(\frac{zg(z)}{f(z)}\middle|\frac{z}{f(z)}\right)\\ &= \left(\frac{d(z)}{h(z)}, \frac{z}{h(z)}\right)\\ &= T\big(d(z)\big|h(z)\big).\end{aligned}$$

The product of this notation is

$$T\big(d(z)\big|h(z)\big)\cdot\big(l(z)\big|m(z)\big) = T\left(d(z)l\left(\frac{m(z)}{h(z)}\right)\middle|h(z)m\left(\frac{z}{h(z)}\right)\right)$$

and the inverse is

$$\Big(T\big(d(z)\big|h(z)\big)\Big)^{-1} \equiv T^{-1}\big(d(z)\big|h(z)\big) = T\left(\frac{1}{d\left(\frac{z}{A(z)}\right)}\middle|A(z)\right).$$

where $A(z)$ is the A-sequence. So, if $T\big(d(z)\big|h(z)\big)$ is an involution, then $h(z) = A(z)$ [57]. The following propositions provide us Riordan subgroups of involutions, according to 0 coefficients of $h(z)$ function.

Proposition 3.2.3. *[59, 60] If* $\Omega_0 = \big\{T\big(d(z)\big|h(z)\big) \in \mathcal{R}\big|h_2 = 0\big\}$, *then* Ω_0 *is a subgroup of* $\mathcal{R}$.

Proposition 3.2.4. *[59]*

- *The set* $\big\{T\big(d(z)\big|h(z)\big) \in \mathcal{R}\big|h_1 = 0\big\}$ *is a subgroup* $(h_1 = a_1 = 0)$.
- *Given* $m \geq 3$, $m \in \mathbb{N}$, *the set* $\big\{T\big(d(z)\big|h(z)\big) \in \mathcal{R}\big|h_m = 0\big\}$ *is not a subgroup.*
- *Given* $k \in \mathbb{N}$, $k \geq 1$, *the set* $\big\{T\big(d(z)\big|h(z)\big) \in \mathcal{R}\big|h_1 = 1$ *and* $h_1 = h_2 = ... = h_k = 0\big\}$ *is a normal subgroup.*

3.3 The Exponential Riordan group

By Definition 2.1.1, an Ordinary Riordan array is constructed by two fps $g(z)$ and $f(z)$. In other words, every pair of functions that can be written as fps of the forms $\sum_{n=0}^{\infty} a_n z^n$, $\sum_{n=1}^{\infty} b_n z^n$, for two sequences a_n and b_n, $n \in \mathbb{Z}$, respectively, and satisfy the restrictions of this definition, allow us to build a Riordan matrix. Those matrices form the Riordan group. In the current section, we present an analogous definition for those Riordan arrays that are generated by using

exponential Riordan arrays, and the group made by these matrices that we presented in Section 2.2, called the exponential Riordan group.
The **Fundamental Theorem of Exponential Riordan arrays** (FTeRA) is similar to FTRA 3.1.1 for the egf $g_\epsilon(z)$, and $f_\epsilon(z)$.

Theorem 3.3.1. *(FTeRA) [12] Let $E = \left[g_\epsilon(z), f_\epsilon(z)\right]$ be an exponential Riordan matrix and R, S are two column vectors, where their gfs are $A(z)$ and $B(z)$ respectively, such that*

$$E \cdot R = S.$$

This relation holds if and only if the following relation among the gfs is true

$$\begin{aligned} \left[g_\epsilon(z), f_\epsilon(z)\right] \cdot A(z) &= B(z) \\ \Leftrightarrow g_\epsilon(z) \cdot A\big(f_\epsilon(z)\big) &= B(z). \end{aligned}$$

Now, since e^z is the egf of the coefficient sequence of all ones, the generating function of the row sums of $\left[g(z), f(z)\right]$ is given by

$$\left[g(z), f(z)\right] \cdot e^z = g(z) \cdot e^{f(z)}.$$

Definition 3.3.1. *[12] The set of all exponential Riordan arrays $\left[g_\epsilon(z), f_\epsilon(z)\right]$, where $g_\epsilon(z) \in \mathbb{F}_0$ and $f_\epsilon(z) \in \mathbb{F}_1$, together with the product*

$$\left[g_\epsilon(z), f_\epsilon(z)\right] \cdot \left[d_\epsilon(z), e_\epsilon(z)\right] = \left[g_\epsilon(z) d_\epsilon\big(f_\epsilon(z)\big), e_\epsilon\big(f_\epsilon(z)\big)\right]$$

*where $d_\epsilon(z) \in \mathbb{F}_0$ and $e_\epsilon(z) \in \mathbb{F}_1$, define the **exponential Riordan group**, $\epsilon\mathcal{R}$.*

3.3.1 Exponential Riordan subgroups

Some types of subgroups of the Ordinary Riordan group are also defined in the Exponential Riordan group. In the following table we present the Z_ϵ sequences of these exponential Riordan subgroups, together with the Z sequences of the ordinary Riordan subgroups to compare their differences.

Riordan subgroup	Ordinary form	Exponential form
Associate	$Z(z) = 0$	$Z_\epsilon(z) = 0$
Bell	$Z(z) = \frac{A(z)}{z} - \frac{1}{z}$	$Z_\epsilon(z) = \frac{A_\epsilon(z)}{z} + \frac{1}{\bar{f}(z)}$
Power-Bell	$Z(z) = \frac{A(z)}{z} - \frac{1}{zA(z)^{n-1}}$	$Z_\epsilon(z) = \frac{nA_\epsilon(z)}{z} - \frac{n}{\bar{f}(z)}$
Derivative	$Z(z) = \frac{A(z)}{z} - \frac{1}{z} + \frac{A'(z)}{A(z)}$	$Z_\epsilon(z) = \frac{A'_\epsilon(z)}{A_\epsilon(z)}$
Hitting-time	$Z(z) = A'(z)$	$Z_\epsilon(z) = \frac{1}{\bar{f}(z)} - \frac{A_\epsilon(z)}{z} - \frac{A'_\epsilon(z)}{A_\epsilon(z)}$
Stochastic	$Z(z) = \frac{A(z)-1}{z-1}$	$Z_\epsilon(z) = \frac{A_\epsilon}{z-1} - \frac{1}{\bar{f}(z)-1}$
Appell	$Z(z) = \frac{g(z)-1}{zg(z)}$, $A(z) = 1$	$Z_\epsilon(z) = \frac{g'(z)}{g(z)}$, $A_\epsilon(z) = 1$

TABLE 3.2: Z and Z_ϵ sequences for Ordinary and Exponential Riordan subgroups

3.4 The Double Riordan group

In general, the set of double Riordan arrays is not closed under multiplication. Nevertheless, if we require that $g(z)$ be an even function, and $f_1(z)$ and $f_2(z)$ be odd functions we can develop an analog of FTRA for the double Riordan arrays, and thus obtain a group structure [28].

Theorem 3.4.1. *[28] (**Fundamental Theorem of Double Riordan arrays**) Let*

$$g(z) = \sum_{k=0}^{\infty} g_{2k} z^{2k}, f_1(z) = \sum_{k=0}^{\infty} f_{1,2k+1} z^{2k+1} \text{ and, } f_2(z) = \sum_{k=0}^{\infty} f_{2,2k+1} z^{2k+1}$$

and $A(z), B(z)$ are two column vectors where

$$(g; f_1, f_2) \cdot A = B. \tag{3.12}$$

- *If*

$$A(z) \;=\; (a_0, 0, a_2, 0, a_4, \ldots)^T = \sum_{k=0}^{\infty} a_{2k} z^{2k}$$

and,

$$B(z) \;=\; (b_0, 0, b_2, 0, b_4, \ldots)^T = \sum_{k=0}^{\infty} b_{2k} z^{2k},$$

then eq 3.12 is satisfied if and only if

$$B(z) = g(z) \cdot A\left(\sqrt{f_1(z) f_2(z)}\right).$$

- *If*

$$A(z) = (0, a_1, 0, a_3, 0, \ldots)^T = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}$$

and,

$$B(z) = (0, b_1, 0, b_3, 0, \ldots)^T = \sum_{k=0}^{\infty} b_{2k+1} z^{2k+1},$$

then eq 3.12 is satisfied if and only if

$$B(z) = g(z) \cdot \sqrt{f_1(z) f_2(z)} \cdot A\left(\sqrt{f_1(z) f_2(z)}\right).$$

In addition [28], the row sum $\Sigma(z)$ of a double Riordan array $D = \big(g(z); f_1(z), f_2(z)\big)$ is given by the formula

$$\Sigma(z) = \frac{g(z)\big(1 + f_1(z)\big)}{1 - f_1(z) f_2(z)}.$$

A binary operation $\cdot$ analogous to the one of single Riordan arrays is defined as follows.

Proposition 3.4.1. *[28] Let $(g; f_1, f_2)$ and $(G; F_1, F_2)$ be elements of dR. Then*

$$(g; f_1, f_2) \cdot (G; F_1, F_2) = \left(gG\left(\sqrt{f_1 f_2}\right); \sqrt{\frac{f_1}{f_2}} F_1\left(\sqrt{f_1 f_2}\right), \sqrt{\frac{f_2}{f_1}} F_2\left(\sqrt{f_1 f_2}\right)\right).$$

This operation can be shown to be associative, while the matrix $(1; z, z)$ is the double Riordan identity. Let $(g; f_1, f_2)$ be a double Riordan element and let $h = \sqrt{f_1 f_2}$ be the geometric mean of the multiplier functions f_1 and f_2, where $\bar{h}$ is the compositional inverse of h. Then

$$(g; f_1, f_2)^{-1} = \left(\frac{1}{g(\bar{h})}; \frac{z\bar{h}}{f_1(\bar{h})}, \frac{z\bar{h}}{f_2(\bar{h})}\right),$$

is the inverse of $(g; f_1, f_2)$, which leads us to the following definition.

Definition 3.4.1. *[28] The set $d\mathcal{R}$ together with the operation $\cdot$ form the **double Riordan group**, denoted as $\langle d\mathcal{R}, \cdot \rangle$.*

As we have already mentioned, some of the conditions that need to be satisfied in order to define double Riordan arrays is to have g even and f_1, f_2 odd

functions. As an aftermath, a typical double Riordan array is aerated as an ordinary Checkerboard Riordan matrix, which is also generated by an odd and an even function. Additionally, if $f_1 = f_2 = f$, then there is a mapping

$$(g; f) \to (g; f, f)$$

from the Checkerboard Riordan subgroup to $d\mathcal{R}$, which is an isomorphism. Hence,

$$Checkb \simeq d\mathcal{R}$$

In Fig 3.1, we present this relationship between the Riordan group and the Double Riordan group.

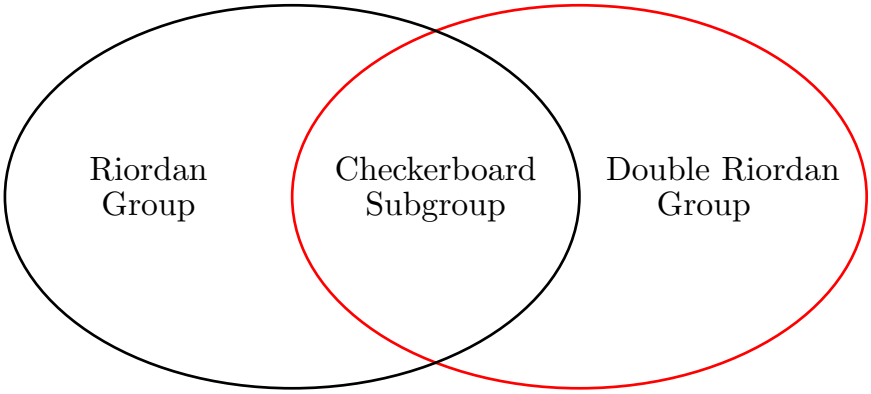


FIGURE 3.1: The Double Riordan group

We should also mention that the question if there is a subgroup of $d\mathcal{R}$ which is isomorphic to the Riordan group, remains open [28].
Some of the subgroups of $d\mathcal{R}$, based on the Associated, the Appell and the Bell (in two types) subgroups of the Riordan group are

$$\begin{aligned} dAssoc &= \{(g; f_1, f_2) \in d\mathcal{R} : g = 1\} = \{(1; f_1, f_2) \in d\mathcal{R}\}, \\ dApp &= \{(g; f_1, f_2) \in d\mathcal{R} : f_1 = f_2 = z\} = \{(g; z, z) \in d\mathcal{R}\}, \\ dB_1 &= \{(g; f_1, f_2) \in d\mathcal{R} : f_1 = zg\} = \{(g; zg, f_2) \in d\mathcal{R}\}, \\ dB_2 &= \{(g; f_1, f_2) \in d\mathcal{R} : f_2 = zg\} = \{(g; f_1, zg) \in d\mathcal{R}\}, \end{aligned}$$

respectively.

Theorem 3.4.2. *[28] The double Riordan subgroup $dApp$ is a normal subgroup of dR, and $d\mathcal{R}$ is the semi-direct product of $dApp$ and $dAssoc$. Hence,*

$$dApp \cdot dAssoc = d\mathcal{R}.$$

Similarly, a triple Riordan group is defined by the functions $g, \frac{f_1}{z}, \frac{f_2}{z}$ and $\frac{f_3}{z}$, while the geometric mean in this case is $h = (f_1 f_2 f_3)^{\frac{1}{3}}$. Additionally, for each positive integer k, and with the appropriate alterations, we define the k-tuple Riordan group [28].

Chapter 4

Properties of the Ordinary Riordan subgroups

In the current chapter, we present our research on the group structure of Riordan arrays. We show new properties and relations among the Riordan subgroups, while we also present new Riordan subgroups as intersections of the known ones, and a family of Riordan subgroups.

4.1 A class of Riordan arrays - RC_6

As we saw earlier in Chapter 4, the Bell subgroup can be written in terms of only one gf, instead of two. By searching other Riordan subgroups which behave similarly, we found that six of the known Riordan subgroups can be exclusively written in terms of the first or the second gf. In the following table, we present those six subgroups. At this point, we need to mention that even if the form of the Associated subgroup can not be written in terms of $g(z)$, we still include it here because of its simplicity. Another observation is that although the general form in terms of $f(z)$ of the power-Bell subgroup is $\left(\left(\frac{f(z)}{z}\right)^{\frac{1}{r}}, f(z)\right)$, we set the power $n = \frac{1}{r}$ for the sake of simplicity.

Name	RAs in terms of $g(z)$	RAs in terms of $f(z)$
Associated	-	$\big(1, f(z)\big)$
Bell	$\big(g(z), zg(z)\big)$	$\left(\frac{f(z)}{z}, f(z)\right)$
Derivative	$\big(g(z), \int g(z)\,dz\big)$	$\big(f'(z), f(z)\big)$
Stochastic	$\big(g(z), g(z)(z-1)+1\big)$	$\left(\frac{f(z)-1}{z-1}, f(z)\right)$
Hitting-time	$\big(g(z), e^{\int \frac{g(z)}{z}\,dz}\big)$	$\left(\frac{zf'(z)}{f(z)}, f(z)\right)$
Power-Bell	$\big(g(z), zg(z)^r\big)$	$\left(\left(\frac{f(z)}{z}\right)^n, f(z)\right)$

TABLE 4.1: Class of six subgroups

Hence, these six subgroups that were found to this point, form a class of Riordan subgroups. Additionally, we prove that there exists a homomorphism between the Associated and the Stochastic subgroups, which is also a bijection. More specifically, we prove that.

Proposition 4.1.1. *The Associated and the Stochastic Riordan subgroups are isomorphic.*

Proof. Define

$$\phi : \big(1, f(z)\big) \to \left(\frac{f(z)-1}{z-1}, f(z)\right)$$

to be a mapping between the two subgroups, and suppose that

$$\big(1, f(z)\big), \text{ and } \big(1, h(z)\big)$$

are two elements of the Associated subgroup. Then we have

$$\begin{aligned}
\phi\Big(\big(1, f(z)\big)\cdot\big(1, h(z)\big)\Big) &= \phi\Big(1, h\big(f(z)\big)\Big) \\
&= \left(\frac{h\big(f(z)\big)-1}{z-1}, h\big(f(z)\big)\right) \\
&= \left(\frac{f(z)-1}{z-1}\frac{h\big(f(z)\big)-1}{f(z)-1}, h\big(f(z)\big)\right) \\
&= \left(\frac{f(z)-1}{z-1}, f(z)\right)\cdot\left(\frac{h(z)-1}{z-1}, h(z)\right) \\
&= \phi\big(1, f(z)\big)\cdot\phi\big(1, h(z)\big),
\end{aligned}$$

which means that ϕ is a homomorphism. The mapping ϕ is also an injection if

$$Ker(\phi) = \{(1,z)|z \in \mathbb{C}\}.$$

So, we have

$$Ker(\phi) \quad = \quad \{(1,f(z)) \in \text{ Associated } |\phi(1,f(z)) = (1,z)\},$$

which leads us to the equation

$$\left(\frac{f(z)-1}{z-1}, f(z)\right) \quad = \quad (1,z),$$

and to the simultaneous equations

$$\frac{f(z)-1}{z-1} = 1 \, ; f(z) = z.$$

Therefore, the only solution is $f(z) = z$. Hence, ϕ is an injection. It is also clear from the Stochastic entry in the last column of Table 4.1 that ϕ is onto. Hence, $Assoc \simeq Stoch$. □

Hence, by [42] and [49], as we referred to earlier in Section 3.2 and by Proposition 4.1.1, we have the following

Proposition 4.1.2. *The Associated, the Bell, the Derivative, the Stochastic, the Hitting-time, and the Power-Bell subgroups are isomorphic.*

Using the general form of each of the Riordan subgroups of the class, in terms of the multiplier function f, we have the following.

Corollary 4.1.1. *Every $f \in \mathbb{F}_1$ generates a $g \in \mathbb{F}_0$ function, for every $(g(z), f(z))$ Riordan subgroup of the class.*

As long as the first gf $g(z)$ of each of the Riordan subgroups of the class depends on the multiplier function $f(z)$, we also have the following.

Corollary 4.1.2. *A Riordan element $(g(z), f(z))$ of the class is an involution if and only if $f(z) = \bar{f}(z)$.*

Now, let us denote the set of Riordan matrices of the class as RC_6 and the set of the involutions of the class as $I[RC_6]$. By Corollary 4.1.1, we have that $f \in \mathbb{F}_1$ characterises an entire Riordan class of subgroups. Hence, a subset of $I[RC_6]$, for a fixed $f(z)$, is denoted by $I[RC_6(f)]$. Obviously, we have that

$$I[RC_6(f)] \subseteq I[RC_6] \subseteq RC_6 \text{ , and } \bigcup I[RC_6(f)] = I[RC_6],$$

for all $f(z) = \bar{f}(z) \in \mathbb{F}_1$.

Proposition 4.1.3. *For a fixed $f(z)$, if*

$$A = \big(g(z), f(z)\big), \text{ and } B = \big(h(z), f(z)\big)$$

are two Riordan matrices such that $A, B \in I[RC_6(f)]$, where $g(z) \neq h(z)$, then

$$A \cdot B = \big(g(z), f(z)\big) \cdot \big(h(z), f(z)\big) = \left(\frac{g(z)}{h(z)}, z\right).$$

Proof. We have

$$\begin{aligned} \big(g(z), f(z)\big) \cdot \big(h(z), f(z)\big) &= \Big(g(z)h\big(f(z)\big), f\big(f(z)\big)\Big) \\ &= z\Big(g(z)h\big(f(z)\big), z\Big). \end{aligned} \tag{4.1}$$

The Riordan element $\big(h(z), f(z)\big)$ is an involution in RC_6, hence it satisfies the equation

$$\begin{aligned} \big(h(z), f(z)\big) \cdot \big(h(z), f(z)\big) &= \Big(h(z)h\big(f(z)\big), f\big(f(z)\big)\Big) \\ &= \Big(h(z)h\big(f(z)\big), z\Big) \\ &= (1, z). \end{aligned}$$

Hence, we have that

$$h(z)h\big(f(z)\big) = 1$$

So,

$$h\big(f(z)\big) = \frac{1}{h(z)},$$

and the RHS of eq 4.1 becomes $\left(\frac{g(z)}{h(z)}, z\right)$. □

4.2 A family of RC_6 subgroups

Considering the family of subgroups as defined in eq 3.11 in subsection 3.2.11, which contains five of the subgroups of our class, we extend the definition by adding one extra parameter, which corresponds to the p^{th} power of the first gf of the Stochastic group, as follows:

$$\Upsilon[r, s, p] = \left\{ \left(\left(\frac{f(z)}{z}\right)^r \big(f'(z)\big)^s \left(\frac{f(z)-1}{z-1}\right)^p, f(z) \right) \middle| f \in \mathbb{F}_1, f_1 = 1, (r, s, p) \in \mathbb{Q}^3 \right\}. \tag{4.2}$$

In order to avoid any possible confusion, we owe to explain the terms that we are going to use from now on, which are based on this family of subgroups. So, the Greek letters ρ, σ and π, instead of the Latin r, s and p will be used for fixed parameters, and the index f will be used to declare the dependence of the algebraic structure that we are referring to, on a function $f(z) \in \mathbb{F}_1$. Hence, we have that

$$Y[r,s,p] \supseteq Y[\rho,\sigma,\pi] \supseteq Y_f[\rho,\sigma,\pi],$$

where $Y[\rho,\sigma,\pi]$ is a Riordan subgroup, and $Y_f[\rho,\sigma,\pi]$ is a Riordan element.

Now, every Riordan subgroup of the RC_6 class can be written in terms of $Y[r,s,p]$, as shown in the following table.

Name	$Y[r,s,p]$
Associated	$Y[0,0,0]$
Derivative	$Y[0,1,0]$
Bell	$Y[1,0,0]$
Hitting-time	$Y[-1,1,0]$
Stochastic	$Y[0,0,1]$
Power-Bell	$Y[n,0,0]$

TABLE 4.2: New notation of Riordan subgroups of the RC_6 class

Proposition 4.2.1. *The Riordan family $Y[r,s,p]$, represents a subgroup of the Riordan group, for each triple $(\rho,\sigma,\pi) \in \mathbb{Q}^3$.*

Proof. Let

$$Y_{f_1}[\rho,\sigma,\pi] = \left(\left(\frac{f_1(z)}{z}\right)^{\rho} (f_1'(z))^{\sigma} \left(\frac{f_1(z)-1}{z-1}\right)^{\pi}, f_1(z) \right)$$

and

$$Y_{f_2}[\rho,\sigma,\pi] = \left(\left(\frac{f_2(z)}{z}\right)^{\rho} (f_2'(z))^{\sigma} \left(\frac{f_2(z)-1}{z-1}\right)^{\pi}, f_2(z) \right)$$

be two Riordan elements of $Y[\rho,\sigma,\pi]$, where $f_1, f_2 \in \mathbb{F}_1$. We have that

$$Y_{f_1}[\rho,\sigma,\pi]\cdot Y_{f_2}[\rho,\sigma,\pi] = \left(\left(\frac{f_1(z)}{z}\right)^{\rho}\left(f_1'(z)\right)^{\sigma}\left(\frac{f_1(z)-1}{z-1}\right)^{\pi}, f_1(z)\right) \cdot \left(\left(\frac{f_2(z)}{z}\right)^{\rho}\left(f_2'(z)\right)^{\sigma}\left(\frac{f_2(z)-1}{z-1}\right)^{\pi}, f_2(z)\right),$$

which becomes

$$\left(\left(\frac{f_2(f_1(z))}{z}\right)^{\rho}\left(f_1'(z)\right)^{\sigma}\left(f_2'(f_1(z))\right)^{\sigma}\left(\frac{f_2(f_1(z))-1}{z-1}\right)^{\pi}, f_2(f_1(z))\right)$$

and,

$$\left(\left(\frac{f_2(f_1(z))}{z}\right)^{\rho}\left(d\Big(f_2(f_1(z))\Big)\right)^{\sigma}\left(\frac{f_2(f_1(z))-1}{z-1}\right)^{\pi}, f_2(f_1(z))\right),$$

where $d\Big(f_2(f_1(z))\Big)$ is the derivative of $f_2(f_1(z))$. This shows closure. Now, the inverse of an element $Y_f[\rho,\sigma,\pi]$ will be

$$\begin{aligned} Y_f[\rho,\sigma,\pi]^{-1} &= \left(\left(\frac{f(z)}{z}\right)^{\rho}\left(f'(z)\right)^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z)\right)^{-1} \\ &= \left(\frac{1}{\left(\frac{z}{\bar{f}(z)}\right)^{\rho}\left(f'(\bar{f}(z))\right)^{\sigma}\left(\frac{z-1}{\bar{f}(z)-1}\right)^{\pi}}, \bar{f}(z)\right) \\ &= \left(\left(\frac{\bar{f}(z)}{z}\right)^{\rho}\left(\frac{1}{f'(\bar{f}(z))}\right)^{\sigma}\left(\frac{\bar{f}(z)-1}{z-1}\right)^{\pi}, \bar{f}(z)\right). \end{aligned}$$

Differentiating the equation

$$f(\bar{f}(z)) = z, \tag{4.3}$$

we get that

$$f'(\bar{f}(z))\bar{f}'(z) = 1,$$

and

$$\bar{f}'(z) = \frac{1}{f'(\bar{f}(z))} \tag{4.4}$$

So,

$$Y_f[\rho,\sigma,\pi]^{-1} = \left(\left(\frac{\bar{f}(z)}{z}\right)^{\rho} \left(\bar{f}'(z)\right)^{\sigma} \left(\frac{\bar{f}(z)-1}{z-1}\right)^{\pi}, \bar{f}(z) \right).$$

Hence, every $Y[\rho,\sigma,\pi]$ for different values of the variables ρ,σ,π is a Riordan subgroup.

□

Proposition 4.2.2. *All Riordan subgroups $Y[\rho,\sigma,\pi]$ for distinct triples $(\rho,\sigma,\pi) \in \mathbb{Q}^3$, where $f(z) \in \mathbb{F}_1$ is fixed, are isomorphic.*

Proof. Let

$$Y[\rho,\sigma,\pi] = \left(\left(\frac{f(z)}{z}\right)^{\rho} \left(f'(z)\right)^{\sigma} \left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z) \right), \text{ and}$$

$$Y[\rho',\sigma',\pi'] = \left(\left(\frac{f(z)}{z}\right)^{\rho'} \left(f'(z)\right)^{\sigma'} \left(\frac{f(z)-1}{z-1}\right)^{\pi'}, f(z) \right)$$

be two arbitrary Riordan subgroups of $Y[r,s,p]$. We will prove that there is a mapping between these two subgroups, which is an isomorphism. Now, let ψ be a mapping between the Associated subgroup, and $Y[\rho,\sigma,\pi]$, such that

$$\psi : Y[0,0,0] \to Y[\rho,\sigma,\pi].$$

Suppose that $Y_f[0,0,0]$, and $Y_h[0,0,0]$ are two Riordan elements of the Associated subgroup, for the fixed functions $f(z), h(z) \in \mathbb{F}_1$. Then we have that

$$\begin{aligned}
\psi\Big(Y_f[0,0,0] \cdot Y_h[0,0,0]\Big) &= \psi\Big((1,f(z)) \cdot (1,h(z))\Big) \\
&= \left(\left(\frac{h(f(z))}{z}\right)^{\rho} \left(\Big(h(f(z))\Big)'\right)^{\sigma} \left(\frac{h(f(z))-1}{z-1}\right)^{\pi}, h(f(z)) \right) \\
&= \left(\left(\frac{f(z)}{z}\right)^{\rho} \left(f'(z)\right)^{\sigma} \left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z) \right) \\
&\cdot \left(\left(\frac{h(z)}{z}\right)^{\rho} \left(h'(z)\right)^{\sigma} \left(\frac{h(z)-1}{z-1}\right)^{\pi}, h(z) \right) \\
&= \psi(1,f(z)) \cdot \psi(1,h(z)) \\
&= \psi\big(Y_f[0,0,0]\big) \cdot \psi\big(Y_h[0,0,0]\big).
\end{aligned}$$

Hence, ψ is a homomorphism. The homomorphism ψ is also an epimorphism as

$$\begin{aligned} Im(\psi) &= \left\{\psi\left(Y_f[0,0,0]\right) \middle| Y_f[0,0,0] \in Y[0,0,0]\right\} \\ &= Y[\rho,\sigma,\pi], \end{aligned} \tag{4.5}$$

and it is an injection as

$$Ker(\psi) = \left\{Y_f[0,0,0] \middle| \psi\left(Y_f[0,0,0]\right) = I\right\},$$

which means that

$$\left(\left(\frac{f(z)}{z}\right)^{\rho}\left(f'(z)\right)^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z)\right) = (1,z). \tag{4.6}$$

This is true only for $f(z) = z$. So,

$$Ker(\psi) = \{(1,z) \mid z \in \mathbb{C}\}.$$

Hence, ψ is an isomorphism, which means that the Associated subgroup is isomorphic to the arbitrary subgroup $Y[\rho,\sigma,\pi]$ of $Y[r,s,p]$. Now, using the inverse mapping of a similar isomorphism from the Associated subgroup to the arbitrary Riordan subgroup

$$Y[\rho',\sigma',\pi'] = \left(\left(\frac{f(z)}{z}\right)^{\rho'}\left(f'(z)\right)^{\sigma'}\left(\frac{f(z)-1}{z-1}\right)^{\pi'}, f(z)\right),$$

we get the following commutative diagram.

$$\begin{array}{ccc} Y[\rho',\sigma',\pi'] & \xrightarrow{\quad\theta\quad} & Y[\rho,\sigma,\pi] \\ & \searrow^{\psi'^{-1}} & \uparrow{\scriptstyle\psi} \\ & & Y[0,0,0] \end{array}$$

where ψ' is defined as

$$\psi' : Y[0,0,0] \to Y[\rho',\sigma',\pi'],$$

and we finally have $\theta = \psi \circ \psi'^{-1}$. Hence, two arbitrary Riordan subgroups of $Y[r,s,p]$ are isomorphic.

□

Using the general form of a Riordan subgroup of $Y[r,s,p]$, we now present the following results.

Proposition 4.2.3. *An arbitrary Riordan element $Y_f[\rho,\sigma,\pi] \in Y[r,s,p]$ is an involution if and only if $f(z) = \bar{f}(z)$.*

Proof. Let

$$Y_f[\rho,\sigma,\pi] = \left(\left(\frac{f(z)}{z}\right)^{\rho} (f'(z))^{\sigma} \left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z) \right)$$

and $f(z) = \bar{f}(z)$. Then we have that

$$Y_f[\rho,\sigma,\pi] \cdot Y_f[\rho,\sigma,\pi] = \left(\left(\frac{f(z)}{z}\right)^{\rho} (f'(z))^{\sigma} \left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z) \right) \cdot \left(\left(\frac{f(z)}{z}\right)^{\rho} (f'(z))^{\sigma} \left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z) \right).$$

which becomes

$$\left(\left(\frac{f(z)}{z}\right)^{\rho} (f'(z))^{\sigma} \left(\frac{f(z)-1}{z-1}\right)^{\pi} \left(\frac{f(f(z))}{f(z)}\right)^{\rho} \left(f'(f(z))\right)^{\sigma} \left(\frac{f(f(z))-1}{f(z)-1}\right)^{\pi}, f(f(z)) \right) \tag{4.7}$$

Differentiating the equation

$$f(f(z)) = z, \tag{4.8}$$

we get that

$$f'(f(z))f'(z) = 1,$$

and

$$f'(f(z)) = \frac{1}{f'(z)} \tag{4.9}$$

Using eq 4.8, and applying eq 4.9 in eq 4.7, the latter becomes

$$\left(\left(\frac{f(z)}{z}\right)^{\rho} (f'(z))^{\sigma} \left(\frac{f(z)-1}{z-1}\right)^{\pi} \left(\frac{z}{f(z)}\right)^{\rho} \left(\frac{1}{f'(z)}\right)^{\sigma} \left(\frac{z-1}{f(z)-1}\right)^{\pi}, z \right), \tag{4.10}$$

which is equal to $(1,z)$. Hence, $Y_f[\rho,\sigma,\pi]$ is an involution.
Now, let us assume that $Y_f[\rho,\sigma,\pi]$ is an involution. So, we have that

$$Y_f[\rho,\sigma,\pi] \cdot Y_f[\rho,\sigma,\pi] = (1,z),$$

and the equation

$$\left(\left(\frac{f(f(z))}{z}\right)^{\rho}\left(\left(f(f(z))\right)'\right)^{\sigma}\left(\frac{f(f(z))-1}{z-1}\right)^{\pi}, f(f(z))\right) = (1,z)$$

which is satisfied if $f(z) = \bar{f}(z)$.

□

Proposition 4.2.4. *An element $Y_f[\rho,\sigma,\pi] \in Y[r,s,p]$ is a pseudo-involution if $-f(-f(z)) = z$ and $\pi = 0$.*

Proof. Pseudo-involutions of $Y[r,s,p]$ satisfy the following

$$\begin{aligned} Y_f[\rho,\sigma,\pi]\cdot(1,-z) &= \left(\left(\frac{f(z)}{z}\right)^{\rho}(f'(z))^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z)\right)\cdot(1,-z) \\ &= \left(\left(\frac{f(z)}{z}\right)^{\rho}(f'(z))^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}, -f(z)\right) \end{aligned}$$

Now, $\left(Y_f[\rho,\sigma,\pi]\cdot(1,-z)\right)\cdot\left(Y_f[\rho,\sigma,\pi]\cdot(1,-z)\right)$ becomes

$$\left(\left(\frac{f(z)}{z}\right)^{\rho}(f'(z))^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}\left(\frac{f(-f(z))}{-f(z)}\right)^{\rho}\left(f'(-f(z))\right)^{\sigma}\left(\frac{f(-f(z))-1}{-f(z)-1}\right)^{\pi}, -f(-f(z))\right) \tag{4.11}$$

To show that this is a pseudo-involution, we firstly need

$$-f(-f(z)) = z,$$

and (4.11) becomes

$$\left(\left(\frac{f(z)}{z}\right)^{\rho}\left(\frac{-z}{-f(z)}\right)^{\rho}(f'(z))^{\sigma}\left(f'(-f(z))\right)^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}\left(\frac{-z-1}{-f(z)-1}\right)^{\pi}, z\right),$$

and

$$\left(\left(f'(z)f'(-f(z))\right)^{\sigma}\left(\frac{(-z-1)(f(z)-1)}{(-f(z)-1)(z-1)}\right)^{\pi}, z\right). \tag{4.12}$$

Differentiating the expression $f(-f(z)) = -z$, we get

$$\Big(f\big(-f(z)\big)\Big)' = (-z)' \Rightarrow f'\big(-f(z)\big) = \frac{1}{f'(z)}$$

Substituting this result to (4.12), we get

$$\left(\left(\frac{(z+1)\big(f(z)-1\big)}{\big(f(z)+1\big)(z-1)}\right)^{\pi}, z\right)$$

Only when $f(z) = z$, the fraction

$$\frac{(z+1)\big(f(z)-1\big)}{\big(f(z)+1\big)(z-1)} = 1,$$

which gives us $Y_z[\rho, \sigma, \pi] = 1$, hence $\pi = 0$. □

4.3 Common Riordan elements in subgroups of the RC_6 class

Returning to RC_6, we will now focus on the structure of those six Riordan subgroups. Pascal's triangle is usually referred in the literature as a Riordan element of the Bell subgroup [17, 49, 85] and at some other times as an element of the Hitting-time subgroup [49, 71]. Hence, a natural question to ask is "What is the relationship between these two Riordan subgroups?" or even more generally "Under what condition does a Riordan element belong to more than one subgroup?" As both of the above mentioned subgroups belong to RC_6 class, we started our research on these six Riordan subgroups.
Here we note that the intersection of any collection of subgroups of a given group will be a subgroup of that group, so if the intersection of any pair of Riordan subgroups is non-trivial (i.e. contains more than the identity), then we will have found a non-trivial subgroup of the Riordan group.
One of the earliest of our results was the discovery of a new Riordan subgroup, the intersection of the Bell and the Hitting-time subgroups. The solution of the differential equation

$$\frac{f(z)}{z} = \frac{zf'(z)}{f(z)},$$

yields $f(z) = \frac{z}{1-cz}$, where c is a constant. Hence, elements that belong in both subgroups are of the form

$$\left(\frac{1}{1-cz}, \frac{z}{1-cz}\right).$$

Definition 4.3.1. *The intersection of the Bell and Hitting-time subgroups is given by the subset*

$$P_c = \left\{ \left(\frac{1}{1-cz}, \frac{z}{1-cz} \right) \middle| c \in \mathbb{R} \right\}.$$

Proposition 4.3.1. *All elements in the intersection of the Bell and Hitting-time subgroups are pseudo-involutions.*

Proof. We have

$$\left(\frac{1}{1-cz}, \frac{z}{1-cz} \right) \cdot (1, -z) = \left(\frac{1}{1-cz}, -\frac{z}{1-cz} \right),$$

and

$$\begin{aligned} \left(\frac{1}{1-cz}, -\frac{z}{1-cz} \right) \cdot \left(\frac{1}{1-cz}, -\frac{z}{1-cz} \right) &= \left(\frac{1}{1-cz} \frac{1}{1+\frac{cz}{1-cz}}, \frac{\frac{z}{1-cz}}{1+\frac{cz}{1-cz}} \right) \\ &= \left(1, \frac{z}{1-cz+cz} \right) \\ &= (1, z). \end{aligned}$$

□

Corollary 4.3.1. *The intersection of the Bell and the Hitting-time subgroups contains only trivial involutions.*

Proposition 4.3.2. *The subset P_c is an abelian Riordan subgroup.*

Proof. We could directly say that the intersection of two subgroups of a group is itself a subgroup, by Proposition A.1.1 of the appendix. However, we are going to prove this result by using the operation $\cdot$ and the fact that P_c is a subset of pseudo-involutions.
Let

$$a = \left(\frac{1}{1-c_1 z}, \frac{z}{1-c_1 z} \right) \text{ and } b = \left(\frac{1}{1-c_2 z}, \frac{z}{1-c_2 z} \right)$$

be two elements of P_c, where $c_1, c_2 \in \mathbb{R}$. Then

$$\begin{aligned} a \cdot b &= \left(\frac{1}{1-c_1 z}, \frac{z}{1-c_1 z}\right) \cdot \left(\frac{1}{1-c_2 z}, \frac{z}{1-c_2 z}\right) \\ &= \left(\frac{1}{1-c_1 z} \frac{1}{1-c_2 \dfrac{z}{1-c_1 z}}, \frac{\dfrac{z}{1-c_1 z}}{1-c_2 \dfrac{z}{1-c_1 z}}\right) \\ &= \left(\frac{1}{1-(c_1+c_2)z}, \frac{z}{1-(c_1+c_2)z}\right). \end{aligned} \tag{4.13}$$

This shows closure.
Now, according to Proposition 4.3.1, a is a pseudo-involution, therefore by Proposition 2 of [49], we get

$$a^{-1} = (g(-z), -f(-z)).$$

So,

$$a^{-1} = \left(\frac{1}{1+cz}, \frac{z}{1+cz}\right).$$

Hence, by Lemma A.1.6 in Subsection A.1 of the appendix, P_c is a subgroup. By (4.13), P_c satisfies commutativity, as $a \cdot b = b \cdot a$. Hence, P_c is abelian. □

This Riordan subgroup was first described as a class of generalized Pascal's Triangles [17] and later on, as a larger subset of Riordan matrices [49]. Additionally, G.-S. Cheon et al. have shown in Lemma 4.1 in [22] that P_c belongs to a family of cyclic subgroups, which came as an intersection of the family of power-Bell subgroups and the Hitting-time subgroup. P_c is a special case of this intersection. While, another reference of this can be found in Proposition 5 of [57], where it is written in its $T\left(d(z)\middle|h(z)\right)$ form, as presented in Subsection 3.2.12.
Searching for other common Riordan elements that belongs to more than one subgroup of RC_6, we have found three new Riordan subgroups.

Considering the Stochastic and Hitting-time subgroups yields the differential equation

$$\frac{f(z)-1}{z-1} = \frac{zf'(z)}{f(z)},$$

which gives us the solution $f(z) = \frac{z}{e^k+(1-e^k)z}$, where $k \neq 0$. Thus, the first gf will be $g(z) = \frac{e^k}{e^k+(1-e^k)z}$.
Hence, elements of the form

$$\left(\frac{e^k}{e^k+(1-e^k)z}, \frac{z}{e^k+(1-e^k)z}\right)$$

belong to both subgroups.
We set $e^k = A$. Then, elements of this form can be written as

$$\left(\frac{A}{A+(1-A)z}, \frac{z}{A+(1-A)z}\right),$$

where $A \neq 0, 1$. To avoid the A factor in the numerator of the first gf, we have

$$\begin{aligned}\frac{A}{A+(1-A)z} &= \frac{1}{1+\frac{1-A}{A}z} \\ &= \frac{1}{1+\left(\frac{1}{A}-1\right)z}.\end{aligned}$$

For $\frac{1}{A}-1 = c$, we have $A = \frac{1}{1+c}$. Substituting this for A in the second gf, we get

$$\frac{z}{\frac{1}{1+c}+\left(1-\frac{1}{1+c}\right)z} = (1+c)\frac{z}{1+cz}.$$

Hence, we have the following.

Definition 4.3.2. *The intersection of the Stochastic and the Hitting-time subgroups is given by the subset*

$$P_{c,c+1} = \left\{ \left(\frac{1}{1+cz}, (1+c)\frac{z}{1+cz}\right) \middle| c \neq -1 \right\}.$$

Proposition 4.3.3. *The set $P_{c,c+1}$ is an abelian Riordan subgroup.*

Proof. We can easily show closure and its commutative property. It suffices to show that the inverse element belongs to the $P_{c,c+1}$ subset. Now, let

$$\left(\frac{1}{1+cz}, (1+c)\frac{z}{1+cz}\right)^{-1}$$

be the inverse element, and suppose that

$$f(z) = (1+c)\frac{z}{1+cz},$$

then

$$\begin{aligned}(f\circ\bar{f})(z)=z &\Rightarrow f(\bar{f}(z))=z\\ &\Rightarrow (1+c)\frac{\bar{f}(z)}{1+c\bar{f}(z)}=z\\ &\Rightarrow (1+c)\bar{f}(z)=z+cz\bar{f}(z)\\ &\Rightarrow (1+c-cz)\bar{f}(z)=z\\ &\Rightarrow \bar{f}(z)=\frac{z}{1+c-cz}.\end{aligned}$$

By the definition of the inverse, we have:

$$\begin{aligned}\left(\frac{1}{1+cz},(1+c)\frac{z}{1+cz}\right)^{-1} &= \left(\frac{1}{\dfrac{1}{1+\dfrac{cz}{1+c-cz}}},\frac{z}{1+c-cz}\right)\\ &= \left(\frac{1}{\dfrac{1+c-cz}{1+c}},\frac{z}{1+c-cz}\right)\\ &= \left(\frac{1+c}{1+c-cz},\frac{z}{1+c-cz}\right),\end{aligned}$$

which for $1+c=A$ can also be transformed into an element of $P_{c,c+1}$. □

Proposition 4.3.4. *The row sums of a matrix of the Riordan subgroup $P_{c,c+1}$ are equal to one.*

Proof. Let $\left(\frac{1}{1+kz},(k+1)\frac{z}{1+kz}\right)$ be an arbitrary Riordan matrix of the subgroup $P_{c,c+1}$, for $k\in\mathbb{Z}\setminus\{-1\}$. Using the formula of eq 3.1, we have

$$\begin{aligned}\frac{\dfrac{1}{1+kz}}{1-(k+1)\dfrac{z}{1+kz}} &= \frac{\dfrac{1}{1+kz}}{\dfrac{1+kz-kz-z}{1+kz}}\\ &= \frac{1}{1-z}.\end{aligned}$$

□

Example 4.3.2. *For $c=3$, we get the Riordan matrix*

$$
\begin{aligned}
P_{3,4} &= \left(\frac{1}{1+3z}, \frac{4z}{1+3z}\right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
-3 & 4 & 0 & 0 & 0 & \cdots \\
9 & -24 & 16 & 0 & 0 & \cdots \\
-27 & 108 & -144 & 64 & 0 & \cdots \\
81 & -432 & 864 & -768 & 256 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\end{aligned}
$$

which has row sums equal one.

Proposition 4.3.5. *The only element of $P_{c,c+1}$, $c \in \mathbb{Z}$ which is a non-trivial involution is $\left(\frac{1}{1-2z}, -\frac{z}{1-2z}\right)$.*

Proof. Let us try first to find any possible involutions in this group. For the arbitrary element $\left(\frac{1}{1+kz}, (k+1)\frac{z}{1+kz}\right)$ of $P_{c,c+1}$, which is an involution, we have that

$$
\begin{aligned}
(1,z) &= \left(\frac{1}{1+kz}, (k+1)\frac{z}{1+kz}\right) \cdot \left(\frac{1}{1+kz}, (k+1)\frac{z}{1+kz}\right) \\
&= \left(\frac{1}{1+kz}\frac{1}{1+\frac{k(k+1)z}{1+kz}}, \frac{\frac{(k+1)^2 z}{1+kz}}{1+\frac{k(k+1)z}{1+kz}}\right) \\
&= \left(\frac{1}{1+k(k+2)z}, (k+1)^2\frac{z}{1+k(k+2)z}\right),
\end{aligned}
$$

has solutions

$$k = 0, \text{ and } k = -2.$$

For $k = 0$ we get the identity element, while for $k = -2$, we get $\left(\frac{1}{1-2z}, \frac{-z}{1-2z}\right)$. □

Proposition 4.3.6. *The Riordan subgroup $P_{c,c+1}$, $c \in \mathbb{Z}$ does not contain non-trivial pseudo-involutions.*

Proof. For the arbitrary element $\left(\frac{1}{1+kz}, (k+1)\frac{z}{1+kz}\right)$ of $P_{c,c+1}$, we have

$$\left(\frac{1}{1+kz}, (k+1)\frac{z}{1+kz}\right) \cdot (1,-z) = \left(\frac{1}{1+kz}, -(k+1)\frac{z}{1+kz}\right).$$

Now,

$$\begin{aligned}
&\left(\frac{1}{1+kz}, -(k+1)\frac{z}{1+kz}\right) \cdot \left(\frac{1}{1+kz}, -(k+1)\frac{z}{1+kz}\right) \\
&\qquad = \left(\frac{1}{1+kz}\frac{1}{1-\dfrac{k(k+1)z}{1+kz}}, \frac{(k+1)^2\dfrac{z}{1+kz}}{1-(k+1)\dfrac{kz}{1+kz}}\right) \\
&\qquad = \left(\frac{1}{1-k^2z}, (k+1)^2\frac{z}{1-k^2z}\right) \\
&\qquad = (1, z).
\end{aligned}$$

that gives the solution $k = 0$, which is the identity element. □

The intersection of the Associated and the Hitting-time subgroups and the intersection of the Derivative and the Bell subgroups, lead us to the differential equation

$$f'(z) - \frac{f(z)}{z} = 0$$

and its solution $f(z) = cz$, where c is a constant. This forms the subsets

$$1_c = \{(1, cz) | c \in \mathbb{Z}\}\,; 1_{c,c} = \{(c, cz) | c \in \mathbb{Z}\}$$

respectively, which are Riordan subgroups.

Corollary 4.3.3. *The Riordan subgroups 1_c and $1_{c,c}$ are abelian.*

Corollary 4.3.4. *The only involutions in 1_c and $1_{c,c}$ except for the identity are $(1, -z)$ and $(-1, -z)$, respectively.*

Now, let us focus on the products of the pairs of subgroups, from which 1_c and $1_{c,c}$ originated. The Associated · Hitting-time subgroups and the Derivative · Bell subgroups, respectively.

Proposition 4.3.7. *Let*

$$\begin{aligned}
A &= \big(1, f(z)\big) \in Y_f[0,0,0] \text{ (Associated subgroup), and} \\
T &= \left(\frac{zf'(z)}{f(z)}, f(z)\right) \in Y_f[-1,1,0] \text{ (Hitting-time subgroup) , and} \\
D &= \big(f'(z), f(z)\big) \in Y_f[0,1,0] \text{ (Derivative subgroup), and} \\
B &= \left(\frac{f(z)}{z}, f(z)\right) \in Y_f[1,0,0] \text{ (Bell subgroup).}
\end{aligned}$$

For $A, T, D, B \in I\big(RC_6(f)\big)$, for a fixed $f(z)$, we have that

$$A \cdot T = B \cdot D = \left(\frac{f(z)}{zf'(z)}, z \right) \text{ and} \tag{4.14}$$

$$T \cdot A = D \cdot B = \left(\frac{zf'(z)}{f(z)}, z \right). \tag{4.15}$$

Proof. Easily shown using Proposition 4.1.3. □

Investigating when eqs 4.14 and 4.15 are equal, we get the function

$$f(z) = \frac{z}{1 - cz},$$

which gives us the Riordan array

$$C_0 = (1 - cz, z),$$

which is neither an involution, nor a pseudo-involution.
Usually, matrices do not satisfy commutativity and Riordan arrays are not an exception, in general. However, all those new found Riordan subgroups are abelian. As a result, this discovery partially answers the question about the existence of other commutative subgroups [49]. Additionally, subgroups such as 1_c and $1_{c,c}$, c-extensions of the unit subgroup $(1, z)$, are abelian. Nevertheless, commutativity cannot be inherited to any c-extension form of a commutative Riordan subgroup. A counterexample is the Appell subgroup which is abelian, while its c-Appell extension is not.

4.3.1 The Power-Bell subgroups

The Power-Bell subgroups $\left(\left(\frac{f(z)}{z} \right)^n, f(z) \right)$, represent a whole family of Riordan subgroups, for any n, where $n \in \mathbb{Z}$ [49] . We denote as Power-Bell (n), each of these Riordan subgroups which are produced by n. Hence, we have that

$$\text{Power-Bell}\,(n) = \left\{ \left(\left(\frac{f(z)}{z} \right)^n, f(z) \right) \middle| n \in \mathbb{Z} \right\}.$$

These Riordan subgroups are also known as Reciprocal subgroups [12] , while for the trivial cases of $n = 0$ and $n = 1$, it collapses to the Associated and the Bell subgroups, respectively. Common Riordan elements with other subgroups of RC_6 can be found in some of the following cases.
The intersections of a subgroup of the form Power-Bell(n) with the Hitting-time subgroup, lead us to the differential equations

$$\left(\frac{f(z)}{z}\right)^n = \frac{zf'(z)}{f(z)} \tag{4.16}$$

and to the Riordan subgroup of the general form [22]

$$\mathcal{H}_{n,c} = \left\{ \left(\frac{1}{1-cz^n}, \frac{z}{\sqrt[n]{1-cz^n}} \right) \middle| n \in \mathbb{Z}, c \in \mathbb{C} \right\} \tag{4.17}$$

which contains the cyclic Riordan subgroup [22] generated by the element

$$\left(\frac{1}{1-z^n}, \frac{z}{\sqrt[n]{1-z^n}} \right), \forall n \in \mathbb{Z}.$$

We observe that the term $\frac{f(z)}{z}$ of the LHS of eq 4.16, is also contained in the RHS. Solving for $f'(z)$, we get

$$\left(\frac{f(z)}{z}\right)^{n+1} = f'(z).$$

Substitute $n+1 = N$, we have

$$\left(\frac{f(z)}{z}\right)^{N} = f'(z), \tag{4.18}$$

which is the differential equation for the intersection of Power-Bell$(n+1)$ and the Derivative subgroups. Hence, this intersection gives us the Riordan subgroups

$$\mathcal{D}_{n+1,c} = \left\{ \left(\frac{1}{(1-cz^n)^{\frac{n+1}{n}}}, \frac{z}{\sqrt[n]{1-cz^n}} \right) \middle| n \in \mathbb{Z}, c \in \mathbb{C} \right\} \tag{4.19}$$

This observation leads us to the following result.

Corollary 4.3.5. *For $n \in \mathbb{Z}$, we have that*

$$\mathcal{D}_{n+1,c} = \left(\frac{f(z)}{z}, z\right) \cdot \mathcal{H}_{n,c}, \text{ where } f(z) = \frac{z}{\sqrt[n]{1-cz^n}}, \tag{4.20}$$

Corollary 4.3.5 can also be extended, by using "Power-Derivative" and "Power-Hitting-time" subgroups, as follows.

Corollary 4.3.6. *For $m, k \in \mathbb{Z}$, we have the family of Riordan subgroups*

$$\left(\frac{1}{(1-cz^{\frac{m}{k}})^{\frac{k}{m}+1}}, \frac{z}{(1-cz^{\frac{m}{k}})^{\frac{k}{m}}}\right) = \left(\frac{1}{(1-cz^{\frac{m}{k}})^{\frac{k}{m}}}, z\right) \cdot \left(\frac{1}{1-cz^{\frac{m}{k}}}, \frac{z}{(1-cz^{\frac{m}{k}})^{\frac{k}{m}}}\right)$$

We present the new Riordan subgroups of the class in the following diagram. Dotted lines in the diagram are used for subgroups that do not belong to the class and their intersections have not been presented yet.

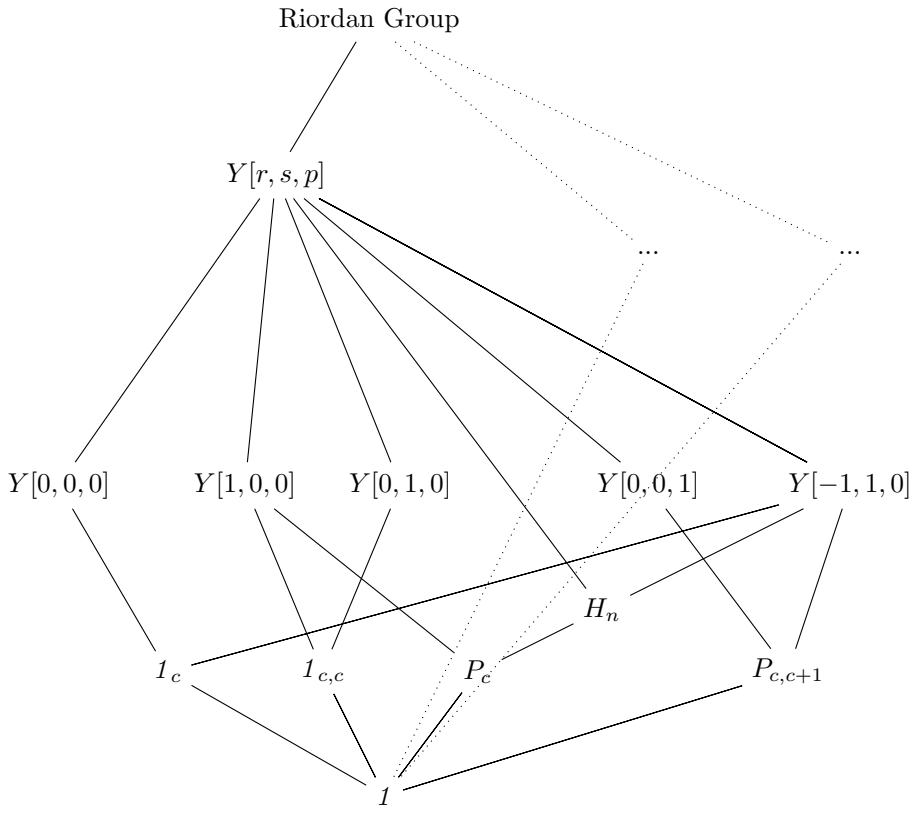


FIGURE 4.1: Diagram of subgroups of the class.

4.4 Relationships of RC_6 and other Riordan subgroups

4.4.1 Relationships of RC_6 with the Appell subgroup

The Riordan group can be expressed as semi-direct products of the Appell and the Associated subgroups, as shown in eq 3.5, and the Appell and the Bell subgroups, as shown in eq 3.6. However, there are some alternative ways to

write the Riordan group as a semi-direct product, using the appropriate form of the Appell subgroup. Firstly, we observe that the second gf of each element of the Appell subgroup, z, is the vital factor to do that, as it allows us to be more flexible on the compositional part of the $\cdot$ operation.
We observe by eq 3.6, that the first gf of this form of Appell arrays is the function $g(z)$ multiplied by the multiplicative inverse of the first gf of the Bell arrays.
Now, using the Riordan multiplication for an arbitrary Appell element $\big(G(z),z\big)$ and an arbitrary Riordan element $\big(k(z),f(z)\big)$, we have that

$$\big(G(z),z\big)\cdot\big(k(z),f(z)\big) \;=\; \big(G(z)k(z),f(z)\big).$$

In order to have a semi-direct product of the Riordan group, we get

$$G(z) \;=\; \frac{g(z)}{k(z)}.$$

According to Criterion 2 of Definition A.1.16 in Subsection A.1 of the Appendix, the subgroup $\{(G(z),z)|G\in\mathbb{F}_0\}$ of the Appell subgroup has to be normal. However, normality is not an inherited property in subgroups, in general.
Let $K=\left\{\left(\frac{g(z)}{k(z)},z\right)\,\middle|\,\frac{g(z)}{k(z)}\in\mathbb{F}_0\right\}$ be a Riordan subgroup of the Appell subgroup and $\big(h(z),z\big)$, an arbitrary Riordan element of the Appell subgroup, where $h(z)\in\mathbb{F}_0$. Then, we have that

$$\big(h(z),z\big)^{-1} \;=\; \left(\frac{1}{h(z)},z\right).$$

So,

$$\begin{aligned}\big(h(z),z\big)^{-1}\cdot\big(G(z),z\big)\cdot\big(h(z),z\big) &= \left(\frac{1}{h(z)},z\right)\cdot\left(\frac{g(z)}{k(z)},z\right)\cdot\big(h(z),z\big)\\ &= \left(\frac{g(z)}{h(z)k(z)},z\right)\cdot\big(h(z),z\big)\\ &= \left(\frac{g(z)}{k(z)},z\right),\end{aligned}$$

which means that

$$\big(h(z),z\big)^{-1}\cdot\big(G(z),z\big)\cdot\big(h(z),z\big)\in K.$$

Hence, by Definition A.1.14 in Subsection A.1 of the appendix, every subgroup of the form $\{(G(z),z)|G \in \mathbb{F}_0\}$ is normal in the Appell subgroup. In the following table, we present alternative ways to express the Riordan group as a semi-direct product.

Form of the Appell subgroup	$\ltimes$	Subgroup
$\left(\frac{(z-1)g(z)}{f(z)-1}, z\right)$		Stochastic
$\left(\frac{g(z)f(z)}{zf'(z)}, z\right)$		Hitting-time
$\left(\frac{g(z)}{f'(z)}, z\right)$		Derivative
$\left(g(z)\left(\frac{z}{f(z)}\right)^n, z\right)$		Power-Bell

TABLE 4.3: Semi-direct products

We note that we are also able to express the Riordan group as a semi-direct product of the Appell and the Stabilizer subgroups as

$$(g(z),f(z)) = \left(\frac{h(f(z))g(z)}{h(z)}, z\right) \ltimes \left(\frac{h(z)}{h(f(z))}, f(z)\right).$$

The Stabilizer subgroup is not included in the collection of isomorphic subgroups, because of the existence of the arbitrary function $h(z)$. Nevertheless as it can be used in a semi-direct product, by Lemma A.1.12, we get the following corollary.

Corollary 4.4.1. *The Stabilizer subgroup is isomorphic to any Riordan subgroup which are contained in RC_6.*

The intersections of the Riordan subgroups of each of these semi-direct products are equal to $I = \{(1,z)\}$. Moreover, elements of the Appell subgroup have already appeared earlier in this section. Returning to Proposition 4.1.3 we have the following corollary.

Corollary 4.4.2. *Elements of the Appell subgroup can be expressed as products of involutions of RC_6.*

4.4.2 Relationships of RC_6 with the Checkerboard subgroup

As we have already mentioned the Checkerboard subgroup is contained in Cheon subgroup [49]. The set of elements $\big(g_e(z), f_o(z)\big)$, where g_e is an even and f_o is an odd function, is quite broad, so there are elements of other Riordan subgroups that can be expressed in this form. Riordan matrices of RC_6 are

also elements of the Checkerboard subgroup, if and only if f is an odd function. Nevertheless, this condition is not satisfied in the case of the Stochastic subgroup, as for $\big(g(z), f(z)\big)$ where $g(z) = \frac{f(z)-1}{z-1}$ and f is odd, for $g(z) \neq 1$ we have that

$$\begin{aligned} g(-z) &= \frac{f(-z)-1}{-z-1} \\ &= \frac{-f(z)-1}{-z-1} \\ &= \frac{-\big(f(z)+1\big)}{-(z+1)} \\ &= \frac{f(z)+1}{z+1} \\ &\neq g(z). \end{aligned}$$

So, g is not even.

Example 4.4.3. *Let $f(z) = \frac{1-\sqrt{1-4z^2}}{2z}$ be an odd function, then*

$$\begin{aligned} g(z) &= \frac{f(z)-1}{z-1} \\ &= \frac{1-2z-\sqrt{1-4z^2}}{2z(z-1)} \end{aligned}$$

is not even.

Elements of the Appell subgroup, where g is an even function also belong to the Checkerboard subgroup. While similarly, elements of the Associated subgroup, where f is an odd function, also belong to the Checkerboard subgroup. Hence, we have the following proposition.

Proposition 4.4.1. *The Checkerboard subgroup can be written as a semi-direct product of the subgroups*

$$\begin{aligned} A_{e,z} &= \big\{(g_e(z), z)\big| g_e : \textit{even}, z \in \mathbb{C}^*\big\}, \\ A_{1,o} &= \{(1, f_o(z))\big| f_o : \textit{odd}, z \in \mathbb{C}^*\}, \end{aligned}$$

which are Riordan subgroups of the Appell and the Associated subgroups, respectively.

Proof. It is clear that we can decompose the Checkerboard subgroups as

$$\begin{aligned} \big(g_e(z), f_o(z)\big) &= \big(g_e(z), z\big) \cdot \big(1, f_o(z)\big) \\ &= A_{e,z} \cdot A_{1,o}\ , \end{aligned}$$

and also

$$A_{e,z} \cap A_{1,o} = I.$$

According to Definition A.1.16 of the Appendix, it suffices to show that $A_{e,z}$ is a normal subgroup of the Checkerboard subgroup. Now, let $\big(e(z), o(z)\big)$ be an arbitrary element of the Checkerboard subgroup, where $e(z)$, and $o(z)$ are even and odd functions, respectively. Then, the inverse of this element will be

$$\big(e(z), o(z)\big)^{-1} = \left(\frac{1}{e\big(\bar{o}(z)\big)}, \bar{o}(z)\right)$$

So, we have that

$$\begin{aligned}
\big(e(z), o(z)\big)^{-1} \cdot A_{e,z} \cdot \big(e(z), o(z)\big) &= \left(\frac{1}{e\big(\bar{o}(z)\big)}, \bar{o}(z)\right) \cdot \big(g_e(z), z\big) \cdot \big(e(z), o(z)\big) \\
&= \left(\frac{g_e\big(\bar{o}(z)\big)}{e\big(\bar{o}(z)\big)}, \bar{o}(z)\right) \cdot \big(e(z), o(z)\big) \\
&= \left(\frac{g_e\big(\bar{o}(z)\big)}{e\big(\bar{o}(z)\big)} e\big(\bar{o}(z)\big), o\big(\bar{o}(z)\big)\right) \\
&= \Big(g_e\big(\bar{o}(z)\big), z\Big),
\end{aligned}$$

and $\bar{o}(z)$ has to be an odd function, so $g_e\big(\bar{o}(z)\big)$ is an even function. Which means that

$$\big(e(z), o(z)\big)^{-1} \cdot A_{e,z} \cdot \big(e(z), o(z)\big) \in A_{e,z}.$$

□

4.4.3 Relationships of RC_6 with the Stabilizer subgroup

The first gf of the general form of any element of the Stabilizer subgroup, $\frac{h(z)}{h\big(f(z)\big)}$ contains the arbitrary function $h(z)$. C. Jean-Louis and A. Nkwanta wrote that the Stochastic Riordan subgroup stabilizes the column vector associated with the coefficients of the gf $h(z) = \frac{1}{z-1}$, as according to Theorem 3.1.1, we have that

$$\begin{aligned}(g(z), f(z)) \cdot \frac{1}{z-1} &= \frac{1}{z-1} \\ \Rightarrow g(z)\frac{1}{f(z)-1} &= \frac{1}{z-1} \\ \Rightarrow g(z) &= \frac{f(z)-1}{z-1}.\end{aligned}$$

C. Jean-Louis and A. Nkwanta also questioned the existence of other column vectors which are stabilized by any other Riordan subgroup [49], while T.-X. He added that not all subgroups are stabilizers [43]. We have found that five out of six of the subgroups of RC_6 are stabilizers, except for the derivative. In Table 4.4 we present the gf of the column vector which is stabilized by each subgroup, that is the subgroup that satisfy

$$S_h = \{(g(z), f(z)) \,|\, (g(z), f(z)) \cdot h(z) = h(z)\}.$$

S_h subgroup	$h(z)$ Stabilizer transformation
Associated	c, for $c \in \mathbb{C}^*$
Bell	$\pm\frac{1}{z}$
Derivative	**?**
Stochastic	$\pm\frac{1}{z-1}$
Hitting-time	$\pm z$
Power-Bell	$\pm\frac{1}{z^r}$ $\quad r \neq 0$

TABLE 4.4: Riordan stabilizers of the RC_6 class

Returning to the family of Riordan subgroups $Y[r, s, p]$, introduced in eq 4.2. For the case of $s = 0$, and by collapsing the s factor which corresponds to the derivative term, we get the Riordan subfamily

$$Y[r, 0, p] = \left(\left(\frac{f(z)}{z}\right)^r \left(\frac{f(z)-1}{z-1}\right)^p, f(z) \right).$$

Proposition 4.4.2. *The Riordan subfamily $Y[r, 0, p]$ is the stabilizer of columns of the form $\frac{1}{z^r(1-z)^p}$, where p is even.*

Proof. We have

$$\begin{aligned} Y[r,0,p]\cdot\frac{1}{z^r(1-z)^p} &= \left(\left(\frac{f(z)}{z}\right)^r\left(\frac{f(z)-1}{z-1}\right)^p, f(z)\right)\cdot\frac{1}{z^r(1-z)^p} \\ &= \left(\frac{f(z)}{z}\right)^r\left(\frac{f(z)-1}{z-1}\right)^p\frac{1}{f^r(z)\big(1-f(z)\big)^p} \\ &= \frac{1}{z^r}\left(\frac{f(z)-1}{z-1}\right)^p\frac{1}{\big(1-f(z)\big)^p}. \end{aligned}$$

For $p=2k+1$, we have

$$\begin{aligned} \frac{1}{z^r}\left(\frac{f(z)-1}{z-1}\right)^{2k+1}\frac{1}{\big(1-f(z)\big)^{2k+1}} &= -\frac{1}{z^r(z-1)^{2k+1}} \\ &\neq \frac{1}{z^r(z-1)^{2k+1}}. \end{aligned}$$

While, for $p=2k$ we get

$$\frac{1}{z^r}\left(\frac{f(z)-1}{z-1}\right)^{2k}\frac{1}{\big(1-f(z)\big)^{2k}} = \frac{1}{z^r(z-1)^{2k}}.$$

Hence, p has to be even. □

A further condition allowing $Y[r,s,p]$ to be a stabilizer of columns of a specific form, is by constructing a "Power-Hitting-time" subfamily of it.

Proposition 4.4.3. *Let $Y[r,s,p]$ be a family of Riordan subgroups. If $r=-s$, then the subfamily of Riordan subgroups $Y[-s,s,p]$ is the stabilizer of columns of the form $\frac{z^s}{(z-1)^p}$, where p is even.*

Proof. We write $\frac{1}{(\ln z)'}$ instead of z for technical reasons. So, we have that

$$
\begin{aligned}
Y[-s,s,p]\cdot\frac{1}{\left((\ln z)'\right)^s(1-z)^p} &= \left(\left(\frac{zf'(z)}{f(z)}\right)^s\left(\frac{f(z)-1}{z-1}\right)^p, f(z)\right) \\
&\qquad\cdot\frac{1}{\left((\ln z)'\right)^s(1-z)^p} \\
&= \left(\frac{zf'(z)}{f(z)}\right)^s\left(\frac{f(z)-1}{z-1}\right)^p\left(\frac{1}{\left(\ln(f(z))\right)'}\right)^s \\
&\qquad\cdot\frac{1}{(1-f(z))^p} \\
&= \left(\frac{zf'(z)}{f(z)}\right)^s\left(\frac{f(z)-1}{z-1}\right)^p\left(\frac{f(z)}{f'(z)}\right)^s \\
&\qquad\cdot\frac{1}{(1-f(z))^p} \\
&= \frac{z^s(f(z)-1)^p}{(z-1)^p(1-f(z))^p}.
\end{aligned}
$$

For $p=2k+1$, eq 4.21 will become

$$
\begin{aligned}
\frac{z^s(f(z)-1)^{2k+1}}{(z-1)^{2k+1}(1-f(z))^{2k+1}} &= -\frac{z^s}{(z-1)^{2k+1}} \\
&\neq \frac{z^s}{(z-1)^{2k+1}}.
\end{aligned}
$$

For $p=2k$, we get

$$
\frac{z^s(f(z)-1)^{2k}}{(z-1)^{2k}(1-f(z))^{2k}} = \frac{z^s}{(z-1)^{2k}}.
$$

Hence, we proved that

$$
Y[-s,s,2k]\cdot\frac{1}{\left((\ln z)'\right)^s(1-z)^{2k}} = \frac{z^s}{(z-1)^{2k}}.
$$

□

Example 4.4.4. *Let $Y[r,0,p]$, and $Y[-s,s,p]$ be subfamilies of $Y[r,s,p]$, where $(r,s,p)\in\mathbb{Z}^3$.*

A. *For the case of* $r = 3, s = 0$, *and* $p = 4$, *we have the Riordan subgroup* $Y[3,0,4] \in Y[r,0,p]$, *and we get*

$$Y[3,0,4] \cdot \frac{1}{z^3(1-z)^4} = \frac{1}{z^3(1-z)^4}$$

B. *For the case of* $r = -s$, $s = 1$, *and* $p = 2$, *we have the Riordan subgroup* $Y[-1,1,2] \in Y[-s,s,p]$, *and we get*

$$Y[-1,1,2] \cdot \frac{z}{(z-1)^2} = \frac{z}{(z-1)^2}$$

For elements of the Checkerboard subgroup which can be expressed in a stabilizer form, we have the following proposition.

Proposition 4.4.4. *A Stabilizer element* $\left(\frac{h(z)}{h\big(f(z)\big)}, f(z)\right)$ *is contained in the Checkerboard subgroup, if and only if* f *is an odd function and* h *is odd or even.*

Proof. Suppose that f is an odd function, i.e. $f(z) = -f(-z)$, then we have two cases.
If h is also odd, we have $h(z) = -h(-z)$. Then for $h \circ f$, we have that

$$h\big(f(z)\big) = h\big(-f(-z)\big) = -h\big(f(-z)\big).$$

Hence, $h \circ f$ is odd and $\frac{h(z)}{h\big(f(z)\big)}$ is then even as the quotient of two odd functions.
Similarly, if h is even, then the composition $h \circ f$ is even and the quotient $\frac{h(z)}{h\big(f(z)\big)}$ is also even. In both cases, $\left(\frac{h(z)}{h\big(f(z)\big)}, f(z)\right)$ is contained in the Checkerboard subgroup.
Now, suppose that $\left(\frac{h(z)}{h\big(f(z)\big)}, f(z)\right)$ is contained in the Checkerboard subgroup, it can be easily proven that f has to be an odd function and h has to be an odd or even function. □

Applying Proposition 4.4.4, we present the following example.

Example 4.4.5. *Let* $f(z) = -\frac{z}{1-z^2}$, *be a function in* $\mathbb{F}_1$, *and* $h_1(z) = 1 - z^2$, $h_2 = 1 - z$ *functions in* $\mathbb{F}_0$, *and* $h_3(z) = \frac{1-\sqrt{1-4z^2}}{2z}$ *is a function in* $\mathbb{F}_1$. *It can be easily shown that* $f(z), h_3(z)$ *are odd functions,* $h_1(z)$ *is an even function, while* $h_2(z)$ *is neither an odd, nor an even. Using these functions, we generate the Riordan element* $\left(\frac{h(z)}{h\big(f(z)\big)}, f(z)\right)$. *We have three cases:*

- $\left(\frac{h_1(z)}{h_1\left(f(z)\right)}, f(z)\right)$ *will be equal to* $\left(\frac{(1-z^2)^3}{(1-z^2)-z^2}, -\frac{z}{1-z^2}\right)$ *that gives us the Riordan matrix*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & -3 & 0 & -3 & 0 & -1 & 0 & \cdots \\ 5 & 0 & 5 & 0 & 4 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.21}$$

- $\left(\frac{h_2(z)}{h_2\left(f(z)\right)}, f(z)\right) = \left(\frac{(1-z)(1-z^2)}{1-z-z^2}, -\frac{z}{1-z^2}\right)$ *that gives us the Riordan matrix*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 2 & -2 & 1 & -3 & 0 & -1 & 0 & \cdots \\ 3 & -3 & 4 & -1 & 4 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.22}$$

- $\left(\frac{h_3(z)}{h_3\left(f(z)\right)}, f(z)\right) = \left(\frac{\sqrt{1-4z^2}-1}{1-z^2-\sqrt{1-6z^2+z^4}}, -\frac{z}{1-z^2}\right)$ *that gives us the Riordan matrix*

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & -1 & 0 & -1 & 0 & 0 & \cdots \\ 0 & -2 & 0 & 2 & 0 & 1 & 0 & \cdots \\ 7 & 0 & 1 & 0 & -3 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.23}$$

We observe that the Riordan matrix 4.22 is the only one that does not have a Checkerboard structure, which confirms Proposition 4.4.4.

Corollary 4.4.6. *The intersection of the Stabilizer and Checkerboard subgroups gives rise to the Riordan subgroup*

$$\begin{aligned} Stab \cap Checkb = &\left\{ \left(\frac{h_e(z)}{h_e(f_o(z))}, f_o(z) \right) \middle| f_o : odd, h_e : even \right\} \\ &\cup \left\{ \left(\frac{h_o(z)}{h_o(f_o(z))}, f_o(z) \right) \middle| f_o : odd, h_o : odd \right\} \\ &= Stab(h_e, f_o) \cup Stab(h_o, f_o). \end{aligned} \tag{4.24}$$

We observe that this new-formed subgroup, contrary to other Riordan subgroups which came as intersections of already known subgroups, is not abelian. Nevertheless, eq 4.24 allows us to characterise Riordan subgroups according to the form of their h function. Hence, we have two main categories of subgroups of $Stab \cap Checkb$, let us denote them as (h_e, f_o) and (h_o, f_o).

4.5 Summary

The diagram in Fig 4.2 contains the known Riordan subgroups that we investigated, and the new Riordan subgroups that we found in this investigation.

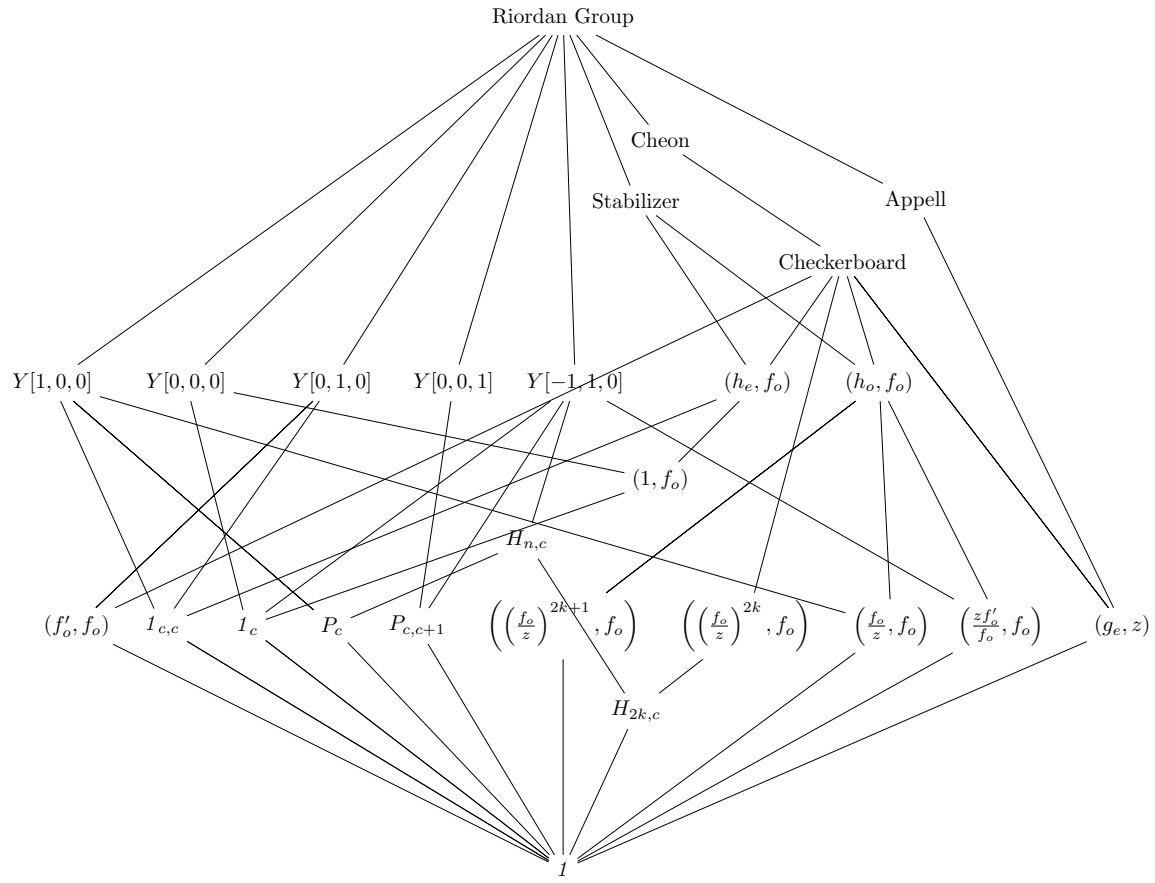


FIGURE 4.2: Diagram of Riordan subgroups.

Chapter 5

Quasi-involutions

A special kind of Riordan arrays are quasi-involutions, which were first presented in 'Structural properties of Riordan matrices and extending the matrices' [21], and were also described in 'The elements of finite order in the Riordan group over the complex field' [22] by Cheon et al. The property that makes them special is the structure of these matrices that differs from the rest of Riordan arrays, because of the similarity to their inverses. The classification of these elements was first made by Cheon et al., where quasi-involutions are described as essential self-inversing matrices after inserting minus signs [21]. In the current chapter, we discuss known Riordan quasi-involutions, we analyse the structure of these matrices, and we present some applications of quasi-involutions to Riordan arrays generated by Bessel polynomials.
This chapter is dedicated to the memory of Dr Hana Kim, one of the greatest researchers in the field of Riordan arrays that we had the pleasure to meet in person. Part of her work on quasi-involutions is presented in this chapter.

5.1 A Riordan quasi-involution

Suppose that we are given the recursive formula

$$q_{n+1,k} = q_{n,k-1} + q_{n-1,k} + q_{n,k+1}. \tag{5.1}$$

Using the same method as Shapiro [86] to find the generating function that we can get by (5.1), we have that

$$\begin{aligned} g(z)f(z)^k &= zg(z)f(z)^{k-1} + z^2g(z)f(z)^k + zg(z)f(z)^{k+1} \\ &= z\big(g(z)f(z)^{k-1} + g(z)f(z)^{k+1}\big) + z^2g(z)f(z)^k \end{aligned}$$

Dividing both of the sides by $g(z)f(z)^{k-1}$, we get that

$$\begin{aligned} f(z) &= z\big(1 + f(z)^2\big) + z^2f(z) \\ &= z + zf(z)^2 + z^2f(z), \end{aligned}$$

which leads us to the quadratic equation[21]

$$zf(z)^2 + (z^2 - 1)f(z) + z = 0, \tag{5.2}$$

and the solutions

$$f(z) = \frac{1 - z^2 \pm \sqrt{1 - 6z^2 + z^4}}{2z}.$$

Continuing with the solution which yields positive coefficients for the gf, we have that

$$f(z) = \frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z}, \tag{5.3}$$

which is the gf for the aerated large Schröder numbers [OEIS, A006318]. This gf is used to generate the Riordan matrix [21]

$$Q = (q_{n,k})_{n,k\in\mathbb{N}} = \left(\frac{f(z)}{z}, f(z)\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 0 & 6 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 16 & 0 & 8 & 0 & 1 & 0 & \cdots \\ 22 & 0 & 30 & 0 & 10 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{5.4}$$

which is a quasi-involution as the inverse of Q is

$$\begin{aligned} Q^{-1} &= \left(\frac{\bar{f}(z)}{z}, \bar{f}(z)\right) \\ &= \left(\frac{-1 - z^2 + \sqrt{1 + 6z^2 + z^4}}{2z^2}, \frac{-1 - z^2 + \sqrt{1 + 6z^2 + z^4}}{2z}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 0 & -6 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 16 & 0 & -8 & 0 & 1 & 0 & \cdots \\ -22 & 0 & 30 & 0 & -10 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned} \tag{5.5}$$

Following a similar description to the one presented as "recursions or dot diagrams" in section 4.1 in [86], we see that the recursive formula 5.1 is satisfied for the entries of the matrix 5.4. For instance, for the number at the $q_{6,2}$ entry, we have the following diagram:

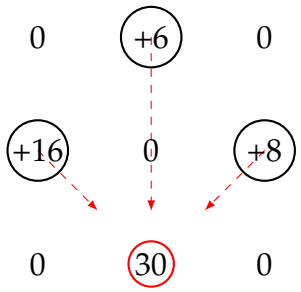


FIGURE 5.1: Diagram of recursive formula 5.1

We set

$$\frac{f(z)}{z} = \frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z^2} = g(z^2),$$

and the Riordan matrix in eq 5.4 is written as

$$\begin{aligned} Q &= (g(z^2), zg(z^2)) \\ &= \left(\frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z^2}, \frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z} \right), \end{aligned} \tag{5.6}$$

and the inverse of the matrix Q is of the form

$$Q^{-1} = (g(-z^2), zg(-z^2)). \tag{5.7}$$

Consequently, we have that this Riordan matrix of a quasi-involution is an aerated Riordan array and its inverse contains the same entries as the initial matrix with $\pm$ signs on alternating non-zero subdiagonals, as in eq 5.5 [21]. We call this type of quasi-involution, a **quasi-involution of level** 1. Similarly, we call **quasi-involution of level** k the aerated Riordan matrices that follow the same structure, with a distance between their non zero entries of k steps.

Example 5.1.1. *Let*

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9 & 0 & 0 & 1 & 0 & \cdots \\ 18 & 0 & 0 & 12 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be a Riordan matrix. The inverse of N is

$$N^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -3 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -6 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & -9 & 0 & 0 & 1 & 0 & \cdots \\ 18 & 0 & 0 & -12 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and we say that the Riordan matrix N is a quasi-involution of level 2.

Since we consider a quasi-involution to be a self-inversing matrix after some sign changes of its entries, we can also think of matrices which are not aerated. We are going to refer to those matrices as quasi-involutions of level 0. An example of such matrix is Pascal's Triangle, as its inverse is

$$\begin{aligned} P^{-1} &= \left(\frac{1}{1+z}, \frac{z}{1+z}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & \cdots \\ -1 & 5 & -10 & 10 & -5 & 1 & 0 & \cdots \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{aligned} \tag{5.8}$$

Additionally, we observe that changing some of the signs of the recursive formula 5.1, as

$$q_{n+1,k} = -q_{n,k-1} + q_{n-1,k} - q_{n,k+1} \tag{5.9}$$

we get the gf

$$w(z) = \frac{-1 + z^2 + \sqrt{1 - 6z^2 + z^4}}{2z}$$

which although does not provide us positive coefficients as the gf 5.3 does, denoting $\frac{w(z)}{z} = u(z^2)$, we have the following:

The analogous Bell type Riordan matrix will be

$$W = \left(\frac{w(z)}{z}, w(z)\right) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & \cdots \\ -6 & 0 & -6 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 16 & 0 & 8 & 0 & 1 & 0 & \cdots \\ -22 & 0 & -30 & 0 & -10 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and its inverse is

$$W^{-1} = \left(\frac{\bar{w}(z)}{z}, \bar{w}(z)\right) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 0 & 1 & 0 & 0 & 0 & \cdots \\ -6 & 0 & 6 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 16 & 0 & -8 & 0 & 1 & 0 & \cdots \\ 22 & 0 & -30 & 0 & 10 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$\bar{w}(z) = \frac{1 + z^2 - \sqrt{1 + 6z^2 + z^4}}{2z}.$$

It is clear that the condition 5.7 is also satisfied for W, as

$$\left(\frac{\bar{w}(z)}{z}, \bar{w}(z)\right) = \left(u(-z^2), zu(-z^2)\right). \tag{5.10}$$

Both Q and W have the same structure since their inverses are the same matrices, with alternating $\pm$ signs on their subdiagonals. We also observe that because of (5.7) and (5.10), the functions $f(z)$ and $w(z)$ of these aerated matrices are odd. Hence, the Riordan matrices Q and W are also Checkerboard elements. Since these matrices are of the Bell type, by using the same functions $f(z)$ and $w(z)$, we also create quasi-involutions of other subgroups of RC_6, except for the case of the Stochastic subgroup, as we referred in subsection 4.4.2. In the following table, we present these quasi-involutions generated by the function $f(z)$. One can be lead to analogous conclusions for quasi-involutions of the function $w(z)$.

Form of Riordan element	Riordan Matrix
$Q_{Assoc} = \big(1, f(z)\big)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 6 & 0 & 6 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 16 & 0 & 10 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$
$Q_{Der} = \big(f'(z), f(z)\big)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 8 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 30 & 0 & 10 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 48 & 0 & 12 & 0 & 1 & 0 & \cdots \\ 154 & 0 & 70 & 0 & 14 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$
$Q_{H-t} = \left(\frac{zf'(z)}{f(z)}, f(z)\right)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 16 & 0 & 8 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 30 & 0 & 10 & 0 & 1 & 0 & \cdots \\ 76 & 0 & 48 & 0 & 12 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$
$Q_{Power-Bell(2)} = \left(\left(\frac{f(z)}{z}\right)^2, f(z)\right)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 16 & 0 & 8 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 30 & 0 & 10 & 0 & 1 & 0 & \cdots \\ 68 & 0 & 48 & 0 & 12 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

TABLE 5.1: Riordan quasi-involutions generated by $f(z)$

Indicatively, we explicitly present the Hitting-time matrix Q_{H-t} and its inverse. So, we have that

$$\begin{aligned}
Q_{H-t} &= \left(\frac{zf'(z)}{f(z)}, f(z)\right) \\
&= \left(\frac{1-z^4-(1+z^2)\sqrt{1-6z^2+z^4}}{(1-z^2-\sqrt{1-6z^2+z^4})\sqrt{1-6z^2+z^4}}, \frac{1-z^2-\sqrt{1-6z^2+z^4}}{2z}\right) \\
&= \left(\frac{1+z^2}{\sqrt{1-6z^2+z^4}}, \frac{1-z^2-\sqrt{1-6z^2+z^4}}{2z}\right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 6 & 0 & 1 & 0 & 0 & 0 & \cdots \\
16 & 0 & 8 & 0 & 1 & 0 & 0 & \cdots \\
0 & 30 & 0 & 10 & 0 & 1 & 0 & \cdots \\
76 & 0 & 48 & 0 & 12 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{aligned}$$

and its inverse

$$\begin{aligned}
Q_{H-t}^{-1} &= \left(\frac{z\bar{f}'(z)}{\bar{f}(z)}, \bar{f}(z)\right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & -6 & 0 & 1 & 0 & 0 & 0 & \cdots \\
16 & 0 & -8 & 0 & 1 & 0 & 0 & \cdots \\
0 & 30 & 0 & -10 & 0 & 1 & 0 & \cdots \\
-76 & 0 & 48 & 0 & -12 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\end{aligned}$$

The sequence of the first generating function of Q_{H-t} excluding 0's, is $1, 4, 16, 76, 384, 2004, \ldots$ [OEIS, A241023], which express the central terms of the triangle of the OEIS sequence, A102413, as seen in figure below.

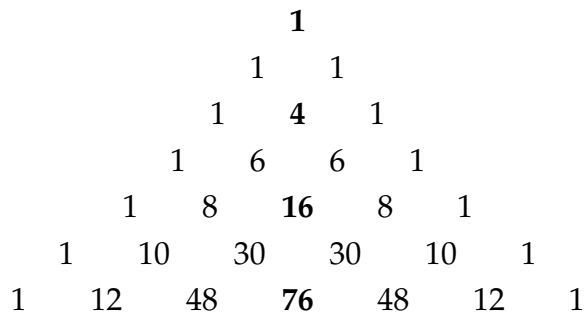


FIGURE 5.2: Triangle pattern of A102413

In Table 5.1, we have examined the case of the quasi-involution of a power-Bell matrix for $n = 2$, where again the sequence of the generating function of its first column, excluding 0's, counts royal paths in a lattice [OEIS, A006319], as we defined them in Section 1.6. For the cases of $n = 3$, and $n = 4$, we have the OEIS sequences A006320, and A006321, respectively.

5.2 Riordan subgroups of quasi-involutions

Before we proceed further to the structure of a Riordan quasi-involution, we consider some already known Riordan subgroups that contain such matrices. The elements of a family of Riordan subgroups that we have presented earlier in subsection 4.3.1, that came as intersections of the power-Bell of power $n+1$ and the Derivative subgroups, $D_{n+1,c}$, and intersections of the power-Bell of power n and the Hitting-time subgroups, $H_{n,c}$ are quasi-involutions [22]. Nevertheless, instead of referring to these intersections by two gfs, our approach focuses on the fact that only one gf is needed for these matrices to be defined. More precisely, both of the Riordan subgroups of the forms (4.17), (4.19) can be expressed as

$$\left\{ \left(\left(\frac{f(z)}{z} \right)^b, f(z) \right) \middle| b \in \mathbb{Z} \right\} \tag{5.11}$$

where

$$f(z) = \frac{z}{\sqrt[N]{1 - cz^N}}, \tag{5.12}$$

for $N \in \mathbb{Z}^*$, and $c \in \mathbb{R}$. From (5.12), we have

$$\begin{aligned} f(z)^N(1 - cz^N) &= z^N \\ \Rightarrow f(z)^N &= cz^N f(z)^N + z^N \end{aligned}$$

Multiplying this equation by $g(z)f(z)^{k-N}$, we get

$$g(z)f(z)^k = z^N g(z)f(z)^{k-N} + cz^N g(z)f(z)^k$$

and the recursive formula

$$a_{n+1,k} = a_{n+1-N,k-N} + ca_{n+1-N,k}. \tag{5.13}$$

Hence, from (5.13), and the function 5.12, we get the following useful corollary.

Corollary 5.2.1. *Let $A = (a_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array of the form*

$$\left(\left(\frac{f(z)}{z}\right)^b, f(z)\right), \text{ where } f(z) = \frac{z}{(1-cz^N)^{\frac{1}{N}}} \tag{5.14}$$

for fixed b, c, N, where $b \in \mathbb{Z}$,$c \in \mathbb{R}$, and $N \in \mathbb{Z}^*$. Then A is a quasi-involution of level $N-1$ and the entries of this Riordan matrix satisfy the recursive formula

$$a_{n+1,k} = a_{n+1-N,k-N} + ca_{n+1-N,k}. \tag{5.15}$$

From (5.11), and for the case of $b = 1$, we get a family of Riordan subgroups of the Bell type,

$$\left\{\left(\frac{1}{\sqrt[N]{1-cz^N}}, \frac{z}{\sqrt[N]{1-cz^N}}\right) \middle| c \in \mathbb{Z}, N \in \mathbb{N}^*\right\}.$$

We present some examples of Bell quasi-involutions of different levels.

Example 5.2.2. *For $N = 2$, and $c = 5$ we get the gf $f(z) = \frac{z}{(1-5z^2)^{\frac{1}{2}}}$ which gives rise to the Riordan array T, where*

$$\begin{aligned} T &= \left(\frac{1}{(1-5z^2)^{\frac{1}{2}}}, \frac{z}{(1-5z^2)^{\frac{1}{2}}}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{5}{2} & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \frac{75}{8} & 0 & \frac{15}{2} & 0 & 1 & 0 & 0 & \cdots \\ 0 & 25 & 0 & 10 & 0 & 1 & 0 & \cdots \\ \frac{625}{16} & 0 & \frac{375}{8} & 0 & \frac{25}{2} & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned} \tag{5.16}$$

This matrix is self-inversing by alternating signs on its subdiagonals, as its inverse is $T^{-1} = \left(\frac{1}{(1+5z^2)^{\frac{1}{2}}}, \frac{z}{(1+5z^2)^{\frac{1}{2}}}\right)$*, and its entries (except for the first column) satisfy the recursive formula*

$$a_{n+1,k} = a_{n-1,k-2} + 5a_{n-1,k} \tag{5.17}$$

Example 5.2.3. *Following the same procedure for the case of* $N = 3$ *(quasi-involutions of level* 2*),* $c = 9$*, and the gf*

$$f(z) = \frac{z}{\sqrt[3]{1-9z^3}},$$

we get the recursive formula

$$a_{n+1,k} = a_{n-2,k-3} + 9a_{n-2,k}. \tag{5.18}$$

which satisfies the Riordan matrix of Example 5.1.1. Hence,

$$S = \left(\frac{1}{\sqrt[3]{1-9z^3}}, \frac{z}{\sqrt[3]{1-9z^3}}\right).$$

The diagram of the linear combination 5.18 of the entries of the matrix S is shown in Fig 5.3.

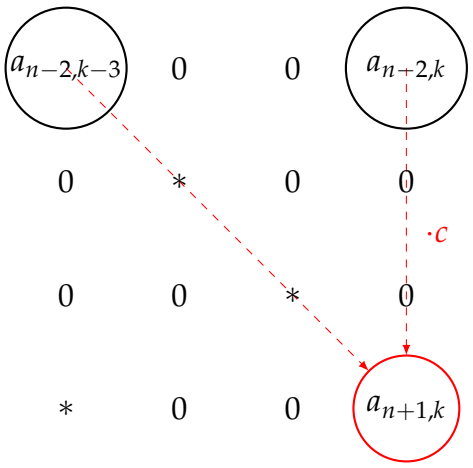


FIGURE 5.3: Diagram of recursive formula 5.18.

Since the function $f(z) = \frac{z}{\sqrt[N]{1-cz^N}}$ can generate quasi-involutions of different levels for every $N \in \mathbb{N}$, by setting $h(z^N) = \frac{f(z)}{z}$, we generalise eqs 5.6, 5.7 to

$$\left(h(z^N), zh(z^N)\right)^{-1} = \left(h(-z^N), zh(-z^N)\right) \tag{5.19}$$

for its Bell form quasi-involution of level $N-1$.

Additionally, as the gf $f(z)=\frac{z}{\sqrt[N]{1-cz^N}}$ resulted from the intersection of more than two Riordan subgroups, the same gf gives rise to quasi-involutions of other Riordan subgroups of the class RC_6, which are generated by one gf, as we saw earlier in Section 4.1. We also observe that the power-Bell forms of the Derivative and the Hitting-time subgroups depend on the parameter $N\in\mathbb{N}$, since

$$\begin{aligned} f'(z)=\left(\frac{z}{\sqrt[N]{1-cz^N}}\right)' &= \frac{1}{\left(\sqrt[N]{1-cz^N}\right)^{N+1}} \\ &= \left(\frac{f(z)}{z}\right)^{N+1} \end{aligned} \tag{5.20}$$

So, for the gf $f(z)=\frac{z}{\sqrt[N]{1-cz^N}}$ we have the following table:

Name	Form	power $b\in\mathbb{Z}$
Associated	$\left(\left(\frac{f(z)}{z}\right)^0, f(z)\right)$	0
Bell	$\left(\left(\frac{f(z)}{z}\right)^1, f(z)\right)$	1
Hitting-time	$\left(\left(\frac{f(z)}{z}\right)^N, f(z)\right)$	N
Derivative	$\left(\left(\frac{f(z)}{z}\right)^{N+1}, f(z)\right)$	N+1
Power-Bell(n)	$\left(\left(\frac{f(z)}{z}\right)^n, f(z)\right)$	n

TABLE 5.2: Riordan elements of subgroups of RC_6 as powers of the Power-Bell subgroup

For the case of the Stochastic subgroup, we have that

$$\frac{f(z)-1}{z-1}=\frac{z-\sqrt[N]{1-cz^N}}{(z-1)\sqrt[N]{1-cz^N}}, \tag{5.21}$$

which cannot be written in a Power-Bell form, and consequently it cannot generate a quasi-involution for $N\in\mathbb{N}$.

5.3 Compressions of quasi-involutions

T.-X. He introduced the compression of a double Riordan array [44], according to which an aerated matrix can be transformed into a non-aerated matrix, by minimizing the powers of the variable that appears in the gfs of a Riordan array. Nevertheless, the compression of an aerated Riordan array is not necessarily a Riordan array. We present two examples of compressions of Riordan matrices.

Example 5.3.1. *[44] The compression of the double Riordan array*

$$F = \left(\frac{1}{1-z^2}; z, \frac{z}{1-z^2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 3 & 0 & 3 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is the matrix

$$F^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 3 & 2 & 1 & 0 & \cdots \\ 1 & 1 & 4 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This matrix cannot be expressed as a Riordan array, as from the gf of the its first column $\frac{1}{1-z}$, and the multiplier function z that gives the second column, we cannot generate the remainder of the matrix.

Example 5.3.2. *The compression of the aerated Riordan matrix,*

$$K = \left(\frac{1}{\sqrt{1-4z^2}}, \frac{1-\sqrt{1-4z^2}}{2z}\right)$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
6 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 10 & 0 & 5 & 0 & 1 & 0 & 0 & 0 & \cdots \\
20 & 0 & 15 & 0 & 6 & 0 & 1 & 0 & 0 & \cdots \\
0 & 35 & 0 & 21 & 0 & 7 & 0 & 1 & 0 & \cdots \\
70 & 0 & 56 & 0 & 28 & 0 & 8 & 0 & 1 & \cdots \\
0 & 126 & 0 & 84 & 0 & 36 & 0 & 9 & 0 & \cdots \\
252 & 0 & 210 & 0 & 120 & 0 & 45 & 0 & 10 & \cdots \\
0 & 462 & 0 & 330 & 0 & 165 & 0 & 55 & 0 & \cdots \\
924 & 0 & 792 & 0 & 495 & 0 & 220 & 0 & 66 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

is the Riordan matrix

$$K^* = \left(\frac{1}{\sqrt{1-4z}}, \frac{1-\sqrt{1-4z}}{2}\right)$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
6 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\
20 & 10 & 4 & 1 & 0 & 0 & 0 & \cdots \\
70 & 35 & 15 & 5 & 1 & 0 & 0 & \cdots \\
252 & 126 & 56 & 21 & 6 & 1 & 0 & \cdots \\
924 & 462 & 210 & 84 & 28 & 7 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

where the sequence of the first column $1, 2, 6, 20, 70, 252, 924, \ldots$ *represents the central binomial coefficients [OEIS, A000984].*

We note that neither F, or K of the previous examples are quasi-involutions. Similarly, the compression of the Riordan quasi-involution 5.4, will be

$$Q^* = \left(\frac{1-z-\sqrt{1-6z+z^2}}{2z}, \frac{1-z-\sqrt{1-6z+z^2}}{2}\right) \tag{5.22}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 22 & 16 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 90 & 68 & 30 & 8 & 1 & 0 & 0 & \cdots \\ 394 & 304 & 146 & 48 & 10 & 1 & 0 & \cdots \\ 1806 & 1412 & 714 & 264 & 70 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now, for its inverse $(Q^*)^{-1}$, we observe that although the sign of the entries is alternated on the subdiagonals, this matrix does not follow the same structure, as

$$(Q^*)^{-1} = \left(\frac{1-z}{1+z}, \frac{z(1-z)}{1+z}\right)$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & -4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 8 & -6 & 1 & 0 & 0 & 0 & \cdots \\ 2 & -12 & 18 & -8 & 1 & 0 & 0 & \cdots \\ -2 & 16 & -38 & 32 & -10 & 1 & 0 & \cdots \\ 2 & -20 & 66 & -88 & 50 & -12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

Hence, Q^* is not a quasi-involution.
Compressions of Riordan matrices which are generated by functions of the form $f(z) = \frac{z}{\sqrt[N]{1-cz^N}}$ are defined similarly, but they are not quasi-involutions in general. A compression of such a matrix remains a quasi-involution if $N = 2k$, where $k \in \mathbb{N}^*$. In this case, we minimize the power of the variable and the order of the root, from $2k$ to k, simultaneously. We note that following this method, although we preserve the quasi-involution property of the initial matrix, the entries of the compressed matrix will not remain the same. Therefore, we name this matrix as **quasi-compression** of Q. We have the following example.

Example 5.3.3. *For $N = 4$ and $c = 16$, let $f(z) = \frac{z}{\sqrt[4]{1-16z^4}}$ be the function that gives rise to the Riordan matrix D of the Bell form,*

$$
\begin{aligned}
D = \left(\frac{f(z)}{z}, f(z)\right) &= \left(\frac{1}{\sqrt[4]{1-16z^4}}, \frac{z}{\sqrt[4]{1-16z^4}}\right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 12 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 16 & 0 & 0 & 0 & 1 & 0 & \cdots \\
40 & 0 & 0 & 0 & 20 & 0 & 0 & 0 & 1 & \cdots \\
0 & 96 & 0 & 0 & 0 & 24 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{aligned}
$$

which is a quasi-involution. Its quasi-compression is

$$
\begin{aligned}
D^* &= \left(\frac{1}{\sqrt{1-16z^2}}, \frac{z}{\sqrt{1-16z^2}}\right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
8 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 16 & 0 & 1 & 0 & 0 & 0 & \cdots \\
96 & 0 & 24 & 0 & 1 & 0 & 0 & \cdots \\
0 & 256 & 0 & 32 & 0 & 1 & 0 & \cdots \\
1,280 & 0 & 480 & 0 & 40 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\end{aligned}
$$

and the inverse of D^*,

$$
\begin{aligned}
(D^*)^{-1} &= \left(\frac{1}{\sqrt{1+16z^2}}, \frac{z}{\sqrt{1+16z^2}}\right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
-8 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & -16 & 0 & 1 & 0 & 0 & 0 & \cdots \\
96 & 0 & -24 & 0 & 1 & 0 & 0 & \cdots \\
0 & 256 & 0 & -32 & 0 & 1 & 0 & \cdots \\
-1,280 & 0 & 480 & 0 & -40 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\end{aligned}
$$

Summarising the above, we present the next proposition that relates quasi-involutions and their quasi-compressions.

Proposition 5.3.1. *Let Q_{2k}, $k \in \mathbb{N}$ be a Riordan quasi-involution of the form*

$$Q_{2k} = \left(\frac{1}{\sqrt[2k]{1-cz^{2k}}}, \frac{z}{\sqrt[2k]{1-cz^{2k}}} \right),$$

and

$$Q_k = \left(\frac{1}{\sqrt[k]{1-cz^{k}}}, \frac{z}{\sqrt[k]{1-cz^{k}}} \right)$$

be its quasi-compression. For these two quasi-involutions, there exists a Riordan matrix W, such that

$$Q_{2k} = W \cdot Q_k \tag{5.23}$$

where $W = \left(\frac{1}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}}, \frac{z}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}} \right)$.

Proof. Let $W = \big(g(z), f(z)\big)$, so eq 5.23 becomes

$$\left(\frac{1}{\sqrt[2k]{1-cz^{2k}}}, \frac{z}{\sqrt[2k]{1-cz^{2k}}} \right) = \big(g(z), f(z)\big) \cdot \left(\frac{1}{\sqrt[k]{1-cz^{k}}}, \frac{z}{\sqrt[k]{1-cz^{k}}} \right)$$

and,

$$\left(\frac{1}{\sqrt[2k]{1-cz^{2k}}}, \frac{z}{\sqrt[2k]{1-cz^{2k}}} \right) = \left(\frac{g(z)}{\sqrt[k]{1-cf(z)^{k}}}, \frac{f(z)}{\sqrt[k]{1-cf(z)^{k}}} \right)$$

Solving for $f(z)$, we get

$$f(z) = \frac{z}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}}, \tag{5.24}$$

and

$$g(z) = \frac{1}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}}, \tag{5.25}$$

so W is written as a Bell type Riordan matrix,

$$W = \big(g(z), f(z)\big) = \left(\frac{f(z)}{z}, f(z) \right)$$

□

We note that the matrix W of Proposition 5.3.1 is not a quasi-involution. Since W allows us to transition between quasi-involutions, we name it as a **quasi-transitional matrix**. Eq 5.23 is written as

$$\left(\frac{1}{\sqrt[2k]{1-cz^{2k}}}, \frac{z}{\sqrt[2k]{1-cz^{2k}}}\right) = \left(\frac{1}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}}, \frac{z}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}}\right) \cdot \left(\frac{1}{\sqrt[k]{1-cz^k}}, \frac{z}{\sqrt[k]{1-cz^k}}\right) \tag{5.26}$$

Example 5.3.4. *For the Riordan quasi-involution of level* 3 *in Example 5.3.3, we have that*

$$D = W_1 \cdot D^*$$

which becomes

$$\left(\frac{1}{\sqrt[4]{1-16z^4}}, \frac{z}{\sqrt[4]{1-16z^4}}\right) = \left(\frac{1}{\sqrt{\sqrt{1-16z^4}+16z^2}}, \frac{z}{\sqrt{\sqrt{1-16z^4}+16z^2}}\right) \cdot \left(\frac{1}{\sqrt{1-16z^2}}, \frac{z}{\sqrt{1-16z^2}}\right). \tag{5.27}$$

The Riordan array W_1 generates the aerated matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -8 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -16 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 100 & 0 & -24 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 264 & 0 & -32 & 0 & 1 & 0 & \cdots \\ -1,376 & 0 & 492 & 0 & -40 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Now, in case k is also an even number, the same process of "quasi-compressing" of Q_k is repeated. While, Q_k cannot be factorised further, if k is odd. Hence, we have the next proposition.

Proposition 5.3.2. *Let $Q_{2^n\lambda} = \left(\frac{1}{\sqrt[2^n\lambda]{1-cz^{2^n\lambda}}}, \frac{z}{\sqrt[2^n\lambda]{1-cz^{2^n\lambda}}}\right)$, be a Riordan quasi-involution of level $2^n\lambda - 1$ where $n \in \mathbb{N}^*$, and λ an odd integer. Then $Q_{2^n\lambda}$ is equal to*

$$\prod_{i=1}^{n} \left(\frac{1}{\sqrt[2^{n-i}\lambda]{\sqrt{1-cz^{2^{n+1-i}\lambda}}+cz^{2^{n-i}\lambda}}}, \frac{z}{\sqrt[2^{n-i}\lambda]{\sqrt{1-cz^{2^{n+1-i}\lambda}}+cz^{2^{n-i}\lambda}}}\right) \cdot Q_\lambda \tag{5.28}$$

Proof. For $Q_{2^n\lambda}$, we have

$$Q_{2^n\lambda} = \left(\frac{1}{\sqrt[2^{n-1}\lambda]{\sqrt{1-cz^{2^n\lambda}} + cz^{2^{n-1}\lambda}}}, \frac{z}{\sqrt[2^{n-1}\lambda]{\sqrt{1-cz^{2^n\lambda}} + cz^{2^{n-1}\lambda}}} \right) \cdot Q_{2^{n-1}\lambda} \tag{5.29}$$

Similarly, $Q_{2^{n-1}\lambda}$ can be analysed as

$$\left(\frac{1}{\sqrt[2^{n-2}\lambda]{\sqrt{1-cz^{2^{n-1}\lambda}} + cz^{2^{n-2}\lambda}}}, \frac{z}{\sqrt[2^{n-2}\lambda]{\sqrt{1-cz^{2^{n-1}\lambda}} + cz^{2^{n-2}\lambda}}} \right) \cdot Q_{2^{n-2}\lambda}, \tag{5.30}$$

and so on, until we get

$$Q_{2\lambda} = \left(\frac{1}{\sqrt[\lambda]{\sqrt{1-cz^{2\lambda}} + cz^{\lambda}}}, \frac{z}{\sqrt[\lambda]{\sqrt{1-cz^{2\lambda}} + cz^{\lambda}}} \right) \cdot Q_{\lambda} \tag{5.31}$$

Hence, substituting each of (5.30),(5.31) to the previous quasi-involution, we get eq 5.28. □

Example 5.3.5. *Continuing the process described in Proposition 5.3.2, for Example 5.3.4, we factorise the matrix D^* as*

$$D^* = W_2 \cdot D^{*(2)} \tag{5.32}$$

where

$$D^{*(2)} = \left(\frac{1}{1-16z}, \frac{z}{1-16z} \right),$$

is the quasi-compression of the quasi-compression D^*, and

$$W_2 = \left(\frac{1}{\sqrt{1-16z^2} + 16z}, \frac{z}{\sqrt{1-16z^2} + 16z} \right).$$

From Example 5.3.3, we also have that $D = W_1 \cdot D^*$, hence

$$D = W_1 \cdot W_2 \cdot D^{*(2)}. \tag{5.33}$$

Example 5.3.6. *For the Riordan quasi-involution Q_{176}, we have that $2^4 \cdot 11 = 176$, so*

$$Q_{176} = \prod_{i=1}^{4} W_i \cdot Q_{11}.$$

For other quasi-involutions which are elements of other Riordan subgroups, since they are expressed as powers of the Bell subgroup, we get the following corollary.

Corollary 5.3.7. *Riordan quasi-involution of the forms*

$$Q_{2k} = \left(\left(\frac{1}{\sqrt[2k]{1-cz^{2k}}} \right)^{b_0}, \frac{z}{\sqrt[2k]{1-cz^{2k}}} \right),$$

and

$$Q_k = \left(\left(\frac{1}{\sqrt[k]{1-cz^{k}}} \right)^{b_0}, \frac{z}{\sqrt[k]{1-cz^{k}}} \right)$$

where $b_0 \in \mathbb{Z}$, satisfy the equation

$$Q_{2k} = W \cdot Q_k$$

if $W = \left(\left(\frac{1}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}} \right)^{b_0}, \frac{z}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}} \right)$.

In the case where the matrix is based on the quasi-compression of Q_{2k}, with different power-Bell form, we have the next proposition.

Proposition 5.3.3. *Let*

$$Q_{2k} = \left(\left(\frac{1}{\sqrt[2k]{1-cz^{2k}}} \right)^{b_0}, \frac{z}{\sqrt[2k]{1-cz^{2k}}} \right),$$

and

$$Q_k = \left(\left(\frac{1}{\sqrt[k]{1-cz^{k}}} \right)^{b_1}, \frac{z}{\sqrt[k]{1-cz^{k}}} \right)$$

be two Riordan quasi-involutions, where $b_0 \neq b_1$, then their quasi-transitional matrix will be

$$W = \left(\frac{\left(\sqrt[2k]{1-cz^{2k}} \right)^{b_1-b_0}}{\left(\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k} \right)^{b_1}}, \frac{z}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}} \right)$$

Proof. Let $W = \big(g(z), f(z)\big)$ be the quasi-transitional matrix of Q_{2k} and Q_k. The function $f(z)$ can be easily found as before, while the function $g(z)$ needs to satisfy

$$g(z) \left(\frac{1}{\sqrt[k]{1-c\dfrac{z^k}{\sqrt{1-cz^{2k}}+cz^k}}} \right)^{b_1} = \left(\frac{1}{\sqrt[2k]{1-cz^{2k}}} \right)^{b_0}, \tag{5.34}$$

which gives us

$$g(z) = \frac{\left(\sqrt[2k]{1-cz^{2k}}\right)^{b_1-b_0}}{\left(\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}\right)^{b_1}}.$$

□

Example 5.3.8. *Let S be a Riordan quasi-involution, such that*

$$\begin{aligned} S &= \left(\frac{1}{\sqrt[6]{1-12z^6}}, \frac{z}{\sqrt[6]{1-12z^6}}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & \cdots \\ 14 & 0 & 0 & 0 & 0 & 0 & 14 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{aligned}$$

The quasi-compression of S is

$$\begin{aligned} S^* &= \left(\frac{1}{\sqrt[3]{1-12z^3}}, \frac{z}{\sqrt[3]{1-12z^3}}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 8 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 12 & 0 & 0 & 1 & 0 & \cdots \\ 32 & 0 & 0 & 16 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \end{aligned}$$

and we have the Power-Bell(2) type of S^*,

$$
\begin{aligned}
S^{*(2)} &= \left(\left(\frac{1}{\sqrt[3]{1-12z^3}} \right)^2, \frac{z}{\sqrt[3]{1-12z^3}} \right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
8 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 12 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 16 & 0 & 0 & 1 & 0 & \cdots \\
80 & 0 & 0 & 20 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\end{aligned}
$$

According to Proposition 5.3.3, the quasi-transitional matrix W will be

$$
\begin{aligned}
W &= \left(\frac{(\sqrt[6]{1-12z^6})}{(\sqrt[3]{\sqrt{1-12z^6}+12z^3})^2}, \frac{z}{\sqrt[3]{\sqrt{12z^6}+12z^3}} \right) \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
-8 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & -12 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & -16 & 0 & 0 & 1 & 0 & \cdots \\
82 & 0 & 0 & -20 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{aligned}
$$

we can easily check that $S = W \cdot S^{*(2)}$.

5.3.1 Hankel transforms of the generating function $\frac{1}{\sqrt{1-4pz^2}}$

The gf of the central binomial coefficients $\binom{2n}{n}$ [OEIS - A000984], $\frac{1}{\sqrt{1-4z}}$, is expressed as [8]

$$
\frac{1}{\sqrt{1-4z}} = \cfrac{1}{1-2z-\cfrac{2z^2}{1-2z-\cfrac{z^2}{1-2z-\cfrac{z^2}{1-2z-\cfrac{z^2}{1-\ddots}}}}}. \tag{5.35}
$$

By Theorem 1.7.2, we have that the Hankel transform of the corresponding integer sequence of this function $1, 2, 6, 20, 70, 252, 924, 3,432, \ldots$ [OEIS - A000984], is expressed as powers of 2. This is one of the seven sequences that have been found with the same Hankel transform [50]. In Example 5.3.2, $g(z) = \frac{1}{\sqrt{1-4z^2}}$ is the gf of the aeration of the central binomial numbers $\binom{2n}{n}$, since $\frac{1}{\sqrt{1-4z^2}}$ may be represented as an S-fraction [7]

$$\frac{1}{\sqrt{1-4z^2}} = \cfrac{1}{1-\cfrac{2z^2}{1-\cfrac{z^2}{1-\cfrac{z^2}{1-\cfrac{z^2}{1-\ddots}}}}} \tag{5.36}$$

We observe that the Hankel transform of (5.36) is the same as its compression 5.35.

Generating functions of the form $\frac{1}{\sqrt{1-4pz^2}}$ can also be expressed as

$$\frac{1}{\sqrt{1-4pz^2}} = \cfrac{1}{1-\cfrac{2pz^2}{1-\cfrac{pz^2}{1-\cfrac{pz^2}{1-\cfrac{pz^2}{1-\ddots}}}}}$$

Again, for the Hankel determinant h_n as it is defined in Theorem 1.7.2, we have that

$$h_n = 1^{n+1}\,(2p)^n\,p^{n-1}\cdots p^2 p = 2^n \prod_{k=1}^{n} p^{n+1-k}, \tag{5.37}$$

which corresponds to the sequence $1, 2p, 4p^3, 8p^6, 16p^{10}, \ldots$.

Example 5.3.9. *For $p = 2$ we have the function $\frac{1}{\sqrt{1-8z^2}}$, which is written as*

$$\frac{1}{\sqrt{1-8z^2}} = \cfrac{1}{1-\cfrac{4z^2}{1-\cfrac{2z^2}{1-\cfrac{2z^2}{1-\cfrac{2z^2}{1-\ddots}}}}}$$

and its Hankel transform is given by the formula

$$h_n = 2^n \prod_{k=1}^{n} 2^{n+1-k} = \{1, 2^2, 2^5, 2^9, \dots\}$$

5.4 Exponential quasi-involutions and Bessel polynomials

Returning to Section 5.2, we see that an ordinary quasi-involution does not always contain integer entries, in general. Now, instead of taking an ordinary Riordan array, we use the same generating functions as Example 5.2.2 to an Exponential Riordan matrix, T_ϵ. This matrix will be

$$T_\epsilon = \left[\frac{1}{(1-5z^2)^{\frac{1}{2}}}, \frac{z}{(1-5z^2)^{\frac{1}{2}}}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 30 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 225 & 0 & 90 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 3,000 & 0 & 200 & 0 & 1 & 0 & 0 & \cdots \\ 28,125 & 0 & 16,875 & 0 & 375 & 0 & 1 & 0 & \cdots \\ 0 & 630,000 & 0 & 63,000 & 0 & 630 & 0 & 1 & \cdots \\ 6,890,625 & 0 & 5,512,500 & 0 & 183,750 & 0 & 980 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Subsequently, the structure of a quasi-involution is preserved by its exponential manipulation of the ordinary Riordan array, but the difference is that T_ϵ contains solely integer entries.
Especially, for the exponential matrices which are produced by the quasi-involutions of Riordan subgroups that we presented in Section 5.2, we have the following proposition.

Proposition 5.4.1. *Let* $Q = \left(\left(\frac{f(z)}{z}\right)^b, f(z)\right)$ *be an ordinary Riordan quasi-involution, where* $f(z) = \frac{z}{(1-cz^N)^{\frac{1}{N}}}$, $b \in \mathbb{Z}$, $c \in \mathbb{R}$, $N \in \mathbb{Z}^*$, *and the exponential Riordan quasi-involution* $eQ = \left[\left(\frac{f_\epsilon(z)}{z}\right)^b, f_\epsilon(z)\right]$, *using the equivalent of the ordinary function* f *in an exponential form. For the* A *and* A_ϵ *sequences of* Q *and* eQ, *respectively, we have*

$$A_\epsilon(z) = A(z)^{N+1} \tag{5.38}$$

Proof. From eqs 2.7 and 5.20, we get

$$\begin{aligned} A_\epsilon(z) &= \left(\frac{f_\epsilon(\bar{f}_\epsilon(z))}{\bar{f}_\epsilon(z)}\right)^{N+1} \\ &= \left(\frac{z}{\bar{f}_\epsilon(z)}\right)^{N+1} \end{aligned}$$

Hence, (5.38) is proven. □

Example 5.4.1. *For the case of the generalised Pascal's Triangle* $P_c = \left(\frac{1}{1-cz}, \frac{z}{1-cz}\right)$, *where* $c \in \mathbb{R}$, *we have that* $A(z) = 1 + cz$, *while the A-sequence of the exponential array* $eQ_c = \left[\frac{1}{1-cz}, \frac{z}{1-cz}\right]$, *is* $A_\epsilon(z) = A(z)^2 = (1 + cz)^2$.

5.4.1 Bessel polynomials

By exploring properties of quasi-involutions, we found a connection between a certain family of quasi-involutions and the generalized Bessel polynomials. We present Riordan arrays generated by Bessel polynomials and compressions of quasi-involutions which are related to them.
The **Bessel polynomials**, named after the German mathematician, astronomer and physicist Friedrich Wilhelm Bessel (1784-1846), are a class of orthogonal polynomials $y_n(x)$ that although they have been mentioned and studied before, they were firstly presented in 1948 [51]. These polynomials come as the solutions of the second-order differential equation

$$x^2 y_n''(x) + 2(x+1)y_n'(x) = n(n+1)y_n(x) \tag{5.39}$$

where $n \in \mathbb{N}$, and satisfy the initial condition $y_n(0) = 1$. The general solution of (5.39) is given by the formula

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} x^k \tag{5.40}$$

The coefficients of the terms of these polynomials are known as Bessel coefficients. We present the two kinds of Bessel numbers.
The coefficient of the term x^{n-k} in the $(n-1)$-th Bessel polynomials $y_{n-1}(x)$ denoted by $a(n,k)$, is called the signless Bessel number of the first kind. We set

$$b(n,k) = (-1)^{n-k} a(n,k) \tag{5.41}$$

and we call $b(n,k)$, the **Bessel number of the first kind**. So, for the number $b(n,k)$ we have

$$b(n,k) = \begin{cases} (-1)^{n-k}\frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}, & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } 1 \leq n < k. \end{cases} \tag{5.42}$$

By convention, we put $a(0,k) = b(0,k) = \delta_{0,k}$. [48] Now, since the number $b_{n,k}$ depends on the parameters n and k, we define the Bessel matrix of the first kind to be the infinite lower-triangular matrix b [104], such that

$$b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & -3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -15 & 15 & -6 & 1 & 0 & 0 & \cdots \\ 0 & 105 & -105 & 45 & -10 & 1 & 0 & \cdots \\ 0 & -945 & 945 & -420 & 105 & -15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.43}$$

whose (n,k)-th entry is equal to $b(n,k)$.

The **Bessel number of the second kind** $B(n,k)$, is defined to be the number of partitions of $[n] = \{1,2,3,\ldots,n\}$ into k non-empty blocks of size 1 or 2 [48], and it is given by

$$B(n,k) = \begin{cases} \frac{n!}{2^{n-k}(2k-n)!(n-k)!}, & \text{if } \lceil \frac{n}{2} \rceil \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{5.44}$$

We define the Bessel matrix of the second kind to be the infinite lower-triangular matrix B, such that

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 15 & 10 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 15 & 45 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.45}$$

The matrices 5.43, and 5.45 can be expressed using exponential Riordan arrays. More specifically, we have that

$$b = [1, \sqrt{1+2z} - 1] \text{ , and } B = \left[1, z + \frac{z^2}{2}\right] \tag{5.46}$$

Now, if $f(z) = \sqrt{1+2z} - 1$, its compositional inverse will be $\bar{f}(z) = z + \frac{z^2}{2}$, which means that the matrix b is the inverse of matrix B [104].
Except for the Bessel polynomials, we also define the **Reverse Bessel polynomials** [104] as the class of orthogonal polynomials $g_n(x)$ that satisfy the second-order differential equation

$$xg_n''(x) - 2(x+n)g_n'(x) + 2ng_n(x) = 0 \tag{5.47}$$

where $n \in \mathbb{N}$, and the general solution of (5.47) is given by the formula

$$g_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} x^{n-k}. \tag{5.48}$$

The reason for the name of these polynomials, comes from the fact that their coefficients are the same as the Bessel polynomials, but in reverse order.

Example 5.4.2. *The fourth-degree Bessel polynomial is*

$$y_4(x) = 105x^4 + 105x^3 + 45x^2 + 10x + 1,$$

while the fourth-degree reverse Bessel is

$$g_4(x) = x^4 + 10x^3 + 45x^2 + 105x + 105.$$

This form of Bessel polynomials has applications in an area of electronics, known as Filter Design. In signal processing, operators named as filters are used in order to remove unwanted components and improve the signal of transmission. These operators can be designed by known polynomial sequences. Especially, the operators which are constructed by using reverse Bessel polynomials, are known as **Bessel Filters** [62, 91].
The Reverse Bessel polynomials have coefficient array given by the exponential Riordan array

$$\left[\frac{1}{\sqrt{1-2z}}, 1-\sqrt{1-2z}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 15 & 15 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 105 & 105 & 45 & 10 & 1 & 0 & 0 & \cdots \\ 945 & 945 & 420 & 105 & 15 & 1 & 0 & \cdots \\ 10,395 & 10,395 & 4,725 & 1,260 & 210 & 21 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{5.49}$$

which is an element of the exponential Derivative subgroup, as for $f(z) = 1-\sqrt{1-2z}$, it is expressed as $[f'(z), f(z)]$.
Generalising the concept of the matrix of eq 5.49, we define the *r*-**Bessel polynomials** to be the polynomials with coefficient array given by the exponential Riordan array

$$B_r = \left[\frac{1}{(1-rz)^{\frac{r-1}{r}}}, 1-(1-rz)^{\frac{1}{r}}\right]. \tag{5.50}$$

Example 5.4.3. *For $r = 3$, we have the exponential Riordan matrix*

$$\begin{aligned} B_3 &= \left[\frac{1}{(1-3z)^{\frac{2}{3}}}, 1-(1-3z)^{\frac{1}{3}}\right] \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 10 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 80 & 52 & 12 & 1 & 0 & 0 & \cdots \\ 880 & 600 & 160 & 20 & 1 & 0 & \cdots \\ 12,320 & 8,680 & 2,520 & 380 & 30 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{aligned}$$

The sequence of the first gf of the matrix B_3, $1, 2, 10, 80, 880, 12,320, \ldots$ corresponds to the triple factorial numbers (OEIS,A008544), which are given by the formula $\prod_{k=0}^{n-1}(3k+2)$.

Now, let us have a look on a general form of the exponential Riordan quasi-involution, constructed by the gfs of the ordinary Riordan quasi-involution 5.14, parameterised by r, for $c = r$ and $b = r-1$. So, we have the exponential Riordan array

$$\begin{aligned} eQ_r &= \left[\left(\frac{f_\epsilon(z)}{z}\right)^{r-1}, f_\epsilon(z)\right] \\ &= \left[\frac{1}{(1-rz^r)^{\frac{r-1}{r}}}, \frac{z}{(1-rz^r)^{\frac{1}{r}}}\right], \end{aligned} \tag{5.51}$$

We take the compression of $f_\epsilon(z)$,

$$f^*_\epsilon(z) = \frac{z}{(1-rz)^{\frac{1}{r}}}$$

and we now form the function

$$F_r(z) = 1 - \frac{z}{f^*(z)} = 1 - (1-rz)^{\frac{1}{r}}. \tag{5.52}$$

Then we have the exponential Riordan array

$$\left[F'_r(z), F_r(z)\right] = B_r. \tag{5.53}$$

Proposition 5.4.2. *A Riordan matrix B_r can be analysed as an expression of $f^*_\epsilon(z)$, as*

$$B_r = \left[\left(\frac{f^*_\epsilon(z)}{z}\right)^r, z\right] \cdot \left[\frac{z}{f^*_\epsilon(z)}, 1 - \frac{z}{f^*_\epsilon(z)}\right]. \tag{5.54}$$

Proof. The derivative of $F_r(z)$ is written as

$$F'_r(z) = \left(\frac{1}{1-rz}\right)^{\frac{r-1}{r}}, \tag{5.55}$$

which is equal to $\left(\frac{f^*_\epsilon(z)}{z}\right)^{r-1}$, so (5.53) will be

$$B_r = \left[\left(\frac{f^*_\epsilon(z)}{z}\right)^{r-1}, 1 - \left(\frac{f^*_\epsilon(z)}{z}\right)^{-1}\right] \tag{5.56}$$

By eq 5.52, we get that $F_r(z) \in \mathbb{F}_1$, and $(1-rz)^{\frac{1}{r}} \in \mathbb{F}_0$, for every possible value of r. That means that these functions can be used to generate a Riordan array. So, we factorise the RHS of (5.56) to the RHS of (5.54). □

5.4.2 The production matrix of the r-Bessel polynomial matrix

To find the A_ϵ and Z_ϵ sequences of the r-Bessel polynomial matrix

$$B_r = \left[F_r'(z), F_r(z)\right],$$

we are going to need $\bar{F}_z(z)$. From eq 5.52, we have

$$\begin{aligned} z &= 1 - \left(1 - r\bar{F}_r(z)\right)^{\frac{1}{r}} \\ \Rightarrow 1 - r\bar{F}_r(z) &= (1-z)^r \\ \Rightarrow \bar{F}_r(z) &= \frac{1}{r}\left(1 - (1-z)^r\right). \end{aligned} \tag{5.57}$$

Hence, from eqs 2.7, the A_ϵ-sequence is

$$A_\epsilon(z) = F_r'\left(\bar{F}_r(z)\right) = (1-z)^{1-r}, \tag{5.58}$$

and the Z_ϵ-sequence is given by the formula

$$Z_\epsilon(z) = \frac{\left(F_r'\left(\bar{F}_r(z)\right)\right)'}{F_r'\left(\bar{F}_r(z)\right)}$$

which can also written as

$$\begin{aligned} Z_\epsilon(z) &= \frac{\left(A_\epsilon(z)\right)'}{A_\epsilon(z)} \\ &= \frac{(r-1)(1-z)^{-r}}{(1-z)^{1-r}} \\ &= \frac{r-1}{1-z}. \end{aligned} \tag{5.59}$$

Proposition 5.4.3. *The production matrix of B_r is given by the exponential Riordan array*

$$\left[\frac{1}{(1-z)^{r-1}}, z\right]$$

with its first row removed.

Proof. We recall that the production matrix of the matrix B_r is defined as

$$P_{B_r} = B_r^{-1} \cdot \overline{B_r},$$

where $\overline{B_r}$ denotes the matrix B_r with its top row removed. We also recall from (2.8), the bivariate generating function of the production matrix of an exponential Riordan array,

$$e^{zy}\left(Z_\epsilon(z) + yA_\epsilon(z)\right).$$

So, for B_r, and eqs 5.58 and 5.59, the production matrix $P_{B_r} = B_r^{-1} \cdot \overline{B_r}$ of the r-Bessel polynomial matrix is generated by

$$e^{zy}\left(\frac{r-1}{(1-z)^r} + \frac{y}{(1-z)^{r-1}}\right).$$

The generating function of $\left[\frac{1}{(1-z)^{r-1}}, z\right]$ is given by $\frac{e^{zy}}{(1-z)^{r-1}}$, and hence

$$P_{B_r} = \overline{\left[\frac{1}{(1-z)^{r-1}}, z\right]}.$$

□

Example 5.4.4. *For $r = 4$, we have the Riordan matrix B_4,*

$$\begin{aligned} B_4 &= \left[\frac{1}{(1-4z)^{3/4}}, 1-(1-4z)^{1/4}\right] \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 21 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 231 & 111 & 18 & 1 & 0 & 0 & \cdots \\ 3,465 & 1,785 & 345 & 30 & 1 & 0 & \cdots \\ 65,835 & 35,595 & 7,650 & 825 & 45 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \end{aligned}$$

where its production matrix will be

$$P_{B_4} = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 60 & 36 & 9 & 1 & 0 & 0 & \cdots \\ 360 & 240 & 72 & 12 & 1 & 0 & \cdots \\ 2,520 & 1,800 & 600 & 120 & 15 & 1 & \cdots \\ 20,160 & 15,120 & 5,400 & 1,200 & 180 & 18 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The exponential Riordan array $\left[\frac{1}{(1-z)^3}, z\right]$ generates the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 60 & 36 & 9 & 1 & 0 & 0 & \cdots \\ 360 & 240 & 72 & 12 & 1 & 0 & \cdots \\ 2,520 & 1,800 & 600 & 120 & 15 & 1 & \cdots \\ 20,160 & 15,120 & 5,400 & 1,200 & 180 & 18 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which gives us the production matrix of P_{B_4}, by removing its first row.

Chapter 6

Almost-Riordan arrays

An almost-Riordan array expands the concept of a Riordan array, by adding one or more extra columns on the left of a Riordan matrix. This addition has some effects on the algebraic structure of the new matrices, as we will see in this chapter. We first present the definition, the properties and past work on almost-Riordan arrays [11], and then we expand the existing theory to almost-Riordan matrices that contain k extra columns, for $k \in \mathbb{N}^*$. In the final part of the chapter, we present algebraic properties of the almost-Riordan groups and subgroups, and we discuss elements of the almost-Riordan group of significant meaning such as involutions, pseudo-involutions and quasi-involutions.

6.1 Introduction to the almost-Riordan group

Definition 6.1.1. *[11] An **almost-Riordan array** is an ordered triple*

$$\big(a(z)|g(z), f(z)\big),$$

where

$$\begin{aligned} a(z) &= \sum_{n=0}^{\infty} a_n z^n, a_0 \neq 0, \\ g(z) &= \sum_{n=0}^{\infty} g_n z^n, g_0 \neq 0, \\ f(z) &= \sum_{n=0}^{\infty} f_n z^n, f_0 = 0, f_1 \neq 0 \end{aligned}$$

in such a way that the matrix which is generated by the array produced by $a(z), g(z)$ and $f(z)$, has column vectors

$$\Big(a(z), zg(z), zg(z)f(z), zg(z)f(z)^2, zg(z)f(z)^3, \ldots\Big). \tag{6.1}$$

The set of almost-Riordan arrays is denoted as $\alpha\mathcal{R}$.

We notice that $a(z), g(z) \in \mathbb{F}_0$, therefore to achieve this lower triangular arrangement, the gf $g(z)$ needs to be multiplied by z to shift all the entries of the second column of the almost-Riordan matrix. Similarly, the gf of the next column will be $zg(z)f(z)$, and so on. Hence, for the gfs of (6.1), we have that

$$a(z) \in \mathbb{F}_0, \text{ and } zg(z)f(z)^{k-1} \in \mathbb{F}_k, \text{ for } k \in \mathbb{N}^* \tag{6.2}$$

A typical almost-Riordan matrix is of the form

$$\begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & g_0 & 0 & 0 & 0 & \cdots \\ a_2 & g_1 & g_0f_1 & 0 & 0 & \cdots \\ a_3 & g_2 & g_0f_2+g_1f_1 & g_0f_1^2 & 0 & \cdots \\ a_4 & g_3 & g_0f_3+g_1f_2+g_2f_1 & 2g_0f_1f_2+g_1f_1^2 & g_0f_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We use the notation $\big(a(z)\big|g(z), f(z)\big)$ to discriminate the almost-Riordan arrays from other Riordan arrays that demand more than two gfs to be defined, such as the Double Riordan arrays, where we use the notation $\big(G(z); f_1(z), f_2(z)\big)$, as presented in Chapters 1 and 2. The main difference between these two types of Riordan array is their matrix generation.

Example 6.1.1. *Let*

$$a(z) = \frac{1}{1-2z^2}, \quad g(z) = \frac{1}{1-z}, \quad f(z) = \frac{z}{1-z}.$$

The matrix

$$\left(\frac{1}{1-2z^2}\middle|\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & 0 & \cdots \\ 4 & 1 & 3 & 3 & 1 & 0 & \cdots \\ 0 & 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is an almost-Riordan matrix.

The analogous **Fundamental Theorem of almost-Riordan arrays**, (FTa-RA) is given in the following proposition.

Proposition 6.1.1. *(FTa-RA) [11] Let $\big(a(z)\big|g(z), f(z)\big)$ be an almost-Riordan array, and $h(z) \in \mathbb{F}_0$ be a power series. Then*

$$\big(a(z)\big|g(z), f(z)\big) \cdot h(z) = h_0a(z) + zg(z)\tilde{h}\big(f(z)\big), \tag{6.3}$$

where

$$\tilde{h}(z) = \frac{h(z) - h_0}{z}. \tag{6.4}$$

The product $\cdot$ of two almost-Riordan matrices is defined as follows.

Proposition 6.1.2. *[11] Let* $\big(a(z)\big|g(z), f(z)\big)$ *and* $\big(b(z)\big|u(z), v(z)\big)$ *be two elements of* $\alpha\mathcal{R}$*. Then*

$$\big(a(z)\big|g(z), f(z)\big) \cdot \big(b(z)\big|u(z), v(z)\big) = \Big(\big(a(z)\big|g(z), f(z)\big) \cdot b(z)\Big|g(z)u\big(f(z)\big), v\big(f(z)\big)\Big). \tag{6.5}$$

Eq 6.5 can be analysed further, using eq 6.3 for the first column. So we have that

$$\big(a(z)\big|g(z), f(z)\big) \cdot \big(b(z)\big|u(z), v(z)\big) = \left(b_0 a(z) + zg(z)\frac{b\big(f(z)\big) - b_0}{f(z)}\middle|g(z)u\big(f(z)\big), v\big(f(z)\big)\right).$$

Now, for the almost-Riordan arrays $\big(a(z)\big|g(z), f(z)\big)$, $\big(b(z)\big|u(z), v(z)\big)$, and $\big(c(z)\big|h(z), t(z)\big)$, we have that

$$\Big(\big(a(z)\big|g(z), f(z)\big) \cdot \big(b(z)\big|u(z), v(z)\big)\Big) \cdot \big(c(z)\big|h(z), t(z)\big)$$

is equal to

$$\Big(\big(a(z)\big|g(z), f(z)\big) \cdot b(z)\Big|g(z)u\big(f(z)\big), v\big(f(z)\big)\Big) \cdot \big(c(z)\big|h(z), t(z)\big),$$

according to eq 6.5, and then for the second almost-Riordan product, we have

$$\Big(\Big(\big(a(z)\big|g(z), f(z)\big) \cdot b(z)\Big|g(z)u\big(f(z)\big), v\big(f(z)\big)\Big) \cdot c(z)\Big| g(z)u\big(f(z)\big)h\Big(v\big(f(z)\big)\Big), t\Big(v\big(f(z)\big)\Big)\Big) \tag{6.6}$$

Similarly, for

$$\big(a(z)\big|g(z), f(z)\big) \cdot \Big(\big(b(z)\big|u(z), v(z)\big) \cdot \big(c(z)\big|h(z), t(z)\big)\Big),$$

we get

$$\big(a(z)\big|g(z),f(z)\big)\cdot\Big(\big(b(z)\big|u(z),v(z)\big)\cdot c(z)\Big|u(z)h\big(v(z)\big),t\big(v(z)\big)\Big),$$

which becomes

$$\Big(\big(a(z)\big|g(z),f(z)\big)\cdot\big(b(z)\big|u(z),v(z)\big)\cdot c(z)\Big|g(z)u\big(f(z)\big)h\Big(v\big(f(z)\big)\Big),\\ t\Big(v\big(f(z)\big)\Big)\Big).$$

This is equal to eq 6.6, so the operation $\cdot$ is associative. Additionally, the element $(1|1,z)$ is the almost-Riordan identity matrix, and the inverse of an almost-Riordan array $\big(a(z)\big|g(z),f(z)\big)$ [11] is an almost-Riordan array of the form

$$\big(a(z)\big|g(z),f(z)\big)^{-1}=\left(a^{\star}(z)\left|\frac{1}{g\big(\bar{f}(z)\big)},\bar{f}(z)\right.\right),$$

where

$$a^{\star}(z)=\left(1\left|-\frac{1}{g\big(\bar{f}(z)\big)},\bar{f}(z)\right.\right)a(z).$$

Hence, we have the following definition.

Definition 6.1.2. *[11] The set $\alpha\mathcal{R}$ together with the operation $\cdot$ define the **almost-Riordan group**, denoted as $\langle\alpha\mathcal{R},\cdot\rangle$.*

The Riordan group $\mathcal{R}$ is in fact a subgroup of the almost-Riordan group [11]. While, for appropriate forms of the gfs of other subgroups of $\alpha\mathcal{R}$, we have that subgroups of $\alpha\mathcal{R}$ are isomorphic to $\mathcal{R}$ [11]. More specifically, we have that

$$\mathcal{R}\simeq\left(g(z)\left|g(z)\frac{f(z)}{z},f(z)\right.\right)\simeq\Big(1\Big|g(z),f(z)\Big).$$

A subgroup of $\alpha\mathcal{R}$ which is analogous to the Appell subgroup of $\mathcal{R}$ can be defined as $\alpha\mathcal{N}=\{(a(z)|1,z);a(z)\in\mathbb{F}_0\}$ and it contains almost Riordan elements of the form

$$\begin{bmatrix}a_0&0&0&0&0&0&\cdots\\a_1&1&0&0&0&0&\cdots\\a_3&0&1&0&0&0&\cdots\\a_4&0&0&1&0&0&\cdots\\a_5&0&0&0&1&0&\cdots\\a_6&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix},$$

where $a_0, a_1, a_2, a_3, ...$ are coefficients of the polynomial $a(z) \in \mathbb{F}_0$ [11].

Proposition 6.1.3. *[11] Let $\alpha\mathcal{R}$ be the group of almost-Riordan arrays, then*

$$\alpha\mathcal{N} = \{(a(z)|1, z); a(z) \in \mathbb{F}_0\}$$

is a normal subgroup of $\alpha\mathcal{R}$.

Now, by the First Isomorphism Theorem A.1.14, as defined in subsection A.1.2 of Appendix A, we have the following proposition.

Proposition 6.1.4. *[11] Since $\alpha\mathcal{N} = \{(a(z)|1, z); a(z) \in \mathbb{F}_0\}$ is a normal subgroup of the group of almost-Riordan arrays $\alpha\mathcal{R}$, and $\mathcal{R}$ be the group of Ordinary Riordan arrays. Then*

$$\alpha\mathcal{R}/\alpha\mathcal{N} \simeq \mathcal{R}.$$

The production matrix of an almost-Riordan array is given by the following proposition.

Proposition 6.1.5. *[11] The production matrix of the almost-Riordan array $(a(z)|g(z), f(z))$ is generated by three gfs:*

$$\begin{aligned} \omega(z) &= (a(z)|g(z), f(z))^{-1}\tilde{a}(z) \text{ , for the first column,} \\ Z(z) &= (a(z)|g(z), f(z))^{-1}g(z) \text{ , for the second column,} \\ A(z) &= \frac{z}{\bar{f}(z)} \text{ , for the subsequent columns.} \end{aligned}$$

Example 6.1.2. *The production matrix of the almost-Riordan array*

$$\left(\frac{1}{1-2z^2}\middle|\frac{1}{1-z}, \frac{z}{1-z}\right)$$

in Example 6.1.1, is the matrix where the gf of its first column is

$$\begin{aligned} \omega(z) &= \left(1 - \frac{2(z+1)^2z^2}{1+2z}\middle|\frac{1}{1+z}, \frac{z}{1+z}\right) \cdot \frac{2z}{1-2z^2} \\ &= \frac{2z}{1+2z-z^2}, \end{aligned}$$

which gives the sequence $0, 2, -4, 10, -24, 58, -140, 338, -816, 1970, ...$ $(OEIS, A163271)$,

$$\begin{aligned} Z(z) &= \left(1 - \frac{2(z+1)^2z^2}{1+2z}\middle|\frac{1}{1+z}, \frac{z}{1+z}\right) \cdot \frac{1}{1-z} \\ &= \frac{(1+z)(1+2z-3z^2-2z^3)}{1+2z-z^2}, \end{aligned}$$

the gf of the second column, which corresponds to the sequence $1, 1, -2, 0, -4, 8, -20, 48, -116, 280, -676, ...$, *and the gf for the rest of the columns*

$$\begin{aligned} A(z) &= \frac{z}{\left(\frac{z}{1-z}\right)} \\ &= 1+z. \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 1 & 0 & 0 & \cdots \\ -4 & -2 & 0 & 1 & 1 & 0 & \cdots \\ 10 & 0 & 0 & 0 & 1 & 1 & \cdots \\ -24 & -4 & 0 & 0 & 0 & 1 & \cdots \\ 58 & 8 & 0 & 0 & 0 & 0 & \cdots \\ -140 & -20 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Adding two columns instead of one on the left of a Riordan matrix, we define the set of almost-almost-Riordan arrays or almost-Riordan arrays of level 2 or $\alpha\mathcal{R}(2)$, in the sense that

$$\mathcal{R} = \alpha\mathcal{R}(0) \text{ and } \alpha\mathcal{R} = \alpha\mathcal{R}(1),$$

which means that $\alpha\mathcal{R}(1)$ corresponds to almost-Riordan matrices that are expressed by a triad of gfs $\big(a(z)\big|g(z), f(z)\big)$, and $\alpha\mathcal{R}(2)$ corresponds to almost-Riordan matrices that are expressed by a quartet of gfs $\big(a(z), b(z)\big|g(z), f(z)\big)$, where $a(z), b(z), g(z) \in \mathbb{F}_0$, and $f(z) \in \mathbb{F}_1$. Hence, for the gfs of the almost-Riordan array of order 2, $\big(a(z), b(z)\big|g(z), f(z)\big)$ we have that

$$a(z) \in \mathbb{F}_0, \quad zb(z) \in \mathbb{F}_1, \quad z^2 g(z) f(z)^{k-2} \in \mathbb{F}_k, \text{ for } k \in \mathbb{N}^* \setminus \{1\}, \tag{6.7}$$

which generates the matrix

$$\begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & b_0 & 0 & 0 & 0 & 0 & \cdots \\ a_2 & b_1 & g_0 & 0 & 0 & 0 & \cdots \\ a_3 & b_2 & g_1 & g_0 f_1 & 0 & 0 & \cdots \\ a_4 & b_3 & g_2 & g_0 f_2 + g_1 f_1 & g_0 f_1^2 & 0 & \cdots \\ a_5 & b_4 & g_3 & g_0 f_3 + g_1 f_2 + g_2 f_1 & 2 g_0 f_1 f_2 + g_1 f_1^2 & g_0 f_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Example 6.1.3. *Let*

$$a(z) = \frac{1}{1-3z-z^2}, \quad b(z) = 1, \quad g(z) = \frac{1}{1-z}, \quad f(z) = \frac{(1-2z)z}{1-z-z^2},$$

where $a(z)$ *generates the sequence* $1, 3, 10, 33, 109, 360, \ldots$ *[OEIS, A006190], and* $f(z)$ *generates the sequence of Lucas numbers (beginning with* 1*)* $1, 3, 4, 7, 11, 18, 29, 47, \ldots$ *[OEIS, A000204].*
Then the matrix

$$\left(\frac{1}{1-3z-z^2}, 1 \middle| \frac{1}{1-z}, \frac{(1-2z)z}{1-z-z^2}\right) = \left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 33 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 109 & 0 & 1 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 360 & 0 & 1 & 23 & 9 & 1 & 0 & 0 & \cdots \\ 1,189 & 0 & 1 & 82 & 45 & 12 & 1 & 0 & \cdots \\ 3,927 & 0 & 1 & 280 & 182 & 76 & 15 & 1 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right],$$

is an almost-Riordan matrix of level 2*.*

Now, for the analogous binary operation for $\alpha\mathcal{R}(2)$, we first define the **Fundamental Theorem for almost-Riordan arrays of level** 2, (FTa-RA(2)).

Proposition 6.1.6. *(FTa-RA(2)) [11] Let* $\big(a(z), b(z)\big|g(z), f(z)\big)$ *be an almost-Riordan array of level* 2*, and the power series* $h(z) = \sum_{n=0}^{\infty} h_n z^n$*. Then*

$$\big(a(z), b(z)\big|g(z), f(z)\big) \cdot h(z) = h_0 a(z) + h_1 z b(z) + z^2 g(z) \tilde{\tilde{h}}\big(f(z)\big), \tag{6.8}$$

where

$$\tilde{\tilde{h}}(z) = \frac{h(z) - h_0 - h_1 z}{z^2}. \tag{6.9}$$

This leads us to the operation $\cdot$ of $\alpha\mathcal{R}(2)$, which is defined as follows:

Proposition 6.1.7. *[11] Let*

$$\mathcal{A} = \big(a(z), b(z)\big|g(z), f(z)\big), \text{ and } \mathcal{B} = \big(h(z), k(z)\big|u(z), v(z)\big),$$

be two elements of $\alpha\mathcal{R}(2)$*, then the product* $\mathcal{A} \cdot \mathcal{B}$ *is equal to*

$$\Big(\big(a(z), b(z)\big|g(z), f(z)\big) \cdot h(z), \big(b(z)\big|g(z), f(z)\big) \cdot k(z) \Big| g(z)u\big(f(z)\big), v\big(f(z)\big)\Big). \tag{6.10}$$

Applying FTa-RA(2), FTa-RA and FTRA to the first three gfs in (6.10) respectively, we get that

$$\mathcal{A}\cdot\mathcal{B}=\Big(h_0a(z)+h_1zb(z)+z^2g(z)\tilde{\tilde{h}}\big(f(z)\big),k_0b(z)+zg(z)\tilde{k}\big(f(z)\big)\Big|\big(g(z),u(z)\big)f(z),v\big(f(z)\big)\Big). \tag{6.11}$$

The identity element of the set $\alpha\mathcal{R}(2)$ is $(1,1|1,z)$, while the inverse of an element $\big(a(z),b(z)\big|g(z),f(z)\big)$ of $\alpha\mathcal{R}(2)$ [11], is defined as

$$\big(a(z),b(z)\big|g(z),f(z)\big)^{-1}=\left(a^{\star\star}(z),b^{\star}(z)\Big|\frac{1}{g\big(\bar{f}(z)\big)},\bar{f}(z)\right), \tag{6.12}$$

where

$$\begin{aligned} b^{\star}(z) &= \big(1\big|-g(z),f(z)\big)^{-1}b(z)\\ &= \left(1\Big|-\frac{1}{g\big(\bar{f}(z)\big)},\bar{f}(z)\right)b(z), \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} a^{\star\star}(z) &= \big(1,-b(z)\big|-g(z),f(z)\big)^{-1}a(z)\\ &= \left(1,-b^{\star}(z)\Big|-\frac{1}{g\big(\bar{f}(z)\big)},\bar{f}(z)\right)a(z). \end{aligned} \tag{6.14}$$

Hence, we have the following definition.

Definition 6.1.3. *[11] The set $\alpha\mathcal{R}(2)$ together with the operation $\cdot$ define the* ***almost-Riordan group of level 2****, denoted as $\langle\alpha\mathcal{R}(2),\cdot\rangle$.*

Additionally, a normal subgroup of $\alpha\mathcal{R}(2)$ is

$$\alpha\mathcal{N}(2)=\big\{\big(a(z),b(z)\big|1,z\big);a(z),b(z)\in\mathbb{F}_0\big\},$$

which can also be used for the isomorphism [11]

$$\alpha\mathcal{R}(2)/\alpha\mathcal{N}(2)\simeq\alpha\mathcal{R}(0)=\mathcal{R}.$$

In next sections, we present our work in almost-Riordan arrays, starting with the general case of almost-Riordan arrays with k extra columns, for $k\in\mathbb{N}$.

6.2 Almost-Riordan arrays of level k

For $\alpha\mathcal{R}(k)$, we have almost-Riordan arrays of level k, of the form

$$\Big(\underbrace{a(z), b(z), c(z), \ldots, w(z)}_{k \text{ gfs}} \Big| g(z), f(z)\Big), \tag{6.15}$$

where $k \in \mathbb{N}$, $a(z), b(z), ..., w(z), g(z) \in \mathbb{F}_0$, and $f(z) \in \mathbb{F}_1$.

If $a(z) = b(z) = \cdots = w(z) = 1$, $g(z) \in \mathbb{F}_0$ and $f(z) \in \mathbb{F}_1$, the almost-Riordan array is called **trivial**.

Example 6.2.1. *Let*

$$a(z) = \frac{1}{1-3z^2}, \quad b(z) = 1 + 2z^2, \quad c(z) = 1,$$

$$g(z) = \frac{1 - z + z^2 - \sqrt{1 - 2z - z^2 - 2z^3 + z^4}}{2z^2}, \quad f(z) = z,$$

where $g(z)$ is the gf of Generalized Catalan numbers, which is also known as the RNA generating function, and it gives rise to the sequence $1, 1, 1, 2, 4, 8, 17, 37, ...$ [OEIS, A004148].
Then the array

$$\left(\frac{1}{1-3z^2}, 1 + 2z^2, 1 \middle| \frac{1 - z + z^2 - \sqrt{1 - 2z - z^2 - 2z^3 + z^4}}{2z^2}, z\right)$$

gives rise to the matrix

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
9 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
27 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 4 & 2 & 1 & 1 & 1 & 0 & \cdots \\
81 & 0 & 0 & 8 & 4 & 2 & 1 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

which is an almost-Riordan matrix of level 3.

We first present the **Fundamental Theorem of almost-Riordan arrays of level k** (FTa-RA(k)), for $k \in \mathbb{N}$.

Proposition 6.2.1. *(FTa-RA(k)) Let*

$$\alpha R(k) = \Big(a_0(z), a_1(z), a_2(z), \dots, a_{k-1}(z) \Big| g(z), f(z)\Big)$$

be an almost-Riordan array of level k, where

$$a_k = \sum_{i=0}^{\infty} a_{ik} z^i \in \mathbb{F}_0, \text{ for } k \in \mathbb{N}$$

and $h(z) = \sum_{n=0}^{\infty} h_n z^n \in \mathbb{F}_0$ *be a power series, then*

$$\begin{aligned} \alpha R(k) \cdot h(z) = \sum_{n=1}^{k} h_{n-1} a_{n-1}(z) z^{n-1} + g(z) \left(\frac{z}{f(z)}\right)^k \\ \left(h\big(f(z)\big) - \sum_{n=1}^{k} h_{n-1} f(z)^{n-1}\right). \end{aligned} \tag{6.16}$$

Proof. First, we have that

$$\begin{aligned} \alpha R(k) \cdot h(z) &= \Big(a_0(z), a_1(z), a_2(z), \dots, a_{k-1}(z) \Big| g(z), f(z)\Big) \cdot h(z) \\ &= \big(a_0(z), a_1(z)z, \dots, a_{k-1}(z)z^{k-1}, g(z)z^k, \dots\big) \cdot \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{k-1} \\ h_k \\ \vdots \\ \vdots \end{bmatrix} \\ &= h_0 a_0(z) + h_1 a_1(z) z + \cdots + h_{k-1} a_{k-1}(z) z^{k-1} + h_k g(z) z^k \\ &\qquad + h_{k+1} g(z) z^k f(z) + \cdots \\ &= h_0 a_0(z) + h_1 a_1(z) z + \cdots + h_{k-1} a_{k-1}(z) z^{k-1} + g(z) z^k (h_k + \\ &\qquad + h_{k+1} f(z) + \cdots) \end{aligned} \tag{6.17}$$

Now, as

$$h(f(z)) = \sum_{k=0}^{\infty} h_k f(z)^k,$$

we have that

$$h\big(f(z)\big) - h_0 - h_1 f(z) - \cdots - h_{k-1} f(z)^{k-1} = h_k f(z)^k + h_{k+1} f(z)^{k+1} + \cdots$$

which becomes

$$\frac{h\big(f(z)\big) - h_0 - h_1 f(z) - \cdots - h_{k-1} f(z)^{k-1}}{f(z)^k} = h_k + h_{k+1} f(z) + \cdots \tag{6.18}$$

Substituting (6.18) to (6.17), we get (6.16).

□

In line with eqs 6.4 and 6.9 , we rewrite (6.16) as

$$\alpha R(k) \cdot h(z) = \sum_{n=1}^{k} h_{n-1} a_{n-1}(z) z^{n-1} + z^k g(z) \widetilde{h}^{(k)}\big(f(z)\big) \tag{6.19}$$

where

$$\widetilde{h}^{(k)}(z) = \frac{h\big(f(z)\big) - \sum_{n=1}^{k} h_{n-1} f(z)^{n-1}}{f(z)^k} \tag{6.20}$$

To find the row sum of an almost-Riordan matrix of level k, we need to split the matrix in two parts, the "almost" and the "Riordan", as in the matrix below.

$$\left[\begin{array}{cccc|cccc}
a_{00} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
a_{10} & a_{11} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
a_{20} & a_{21} & \ddots & 0 & 0 & 0 & 0 & 0 & \cdots \\
a_{30} & a_{31} & \ddots & a_{kk} & 0 & 0 & 0 & 0 & \cdots \\
a_{40} & a_{41} & \ddots & a_{(k+1)k} & g_0 & 0 & 0 & 0 & \cdots \\
a_{50} & a_{51} & \ddots & a_{(k+2)k} & g_1 & g_0 f_1 & 0 & 0 & \cdots \\
a_{60} & a_{61} & \ddots & a_{(k+3)k} & g_2 & g_0 f_2 + g_1 f_1 & g_0 f_1^2 & 0 & \cdots \\
a_{70} & a_{71} & \cdots & a_{(k+4)k} & g_3 & g_0 f_3 + g_1 f_2 + g_2 f_1 & 2 g_0 f_1 f_2 + g_1 f_1^2 & g_0 f_1^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right].$$

For the Riordan submatrix, we use the same formula as for the case of the Ordinary Riordan arrays (3.1), where $g(z)$ is shifted k positions down. Whereas, for the first $k+1$ rows, we have

$$\sum_{i=0}^{k} a_{ji} z^i = a_{j0} + a_{j1} z + a_{j2} z^2 + \cdots + a_{jk} z^k$$

Hence the row sum formula of almost Riordan arrays of level k is

$$\sum_{i=0}^{k} a_{ji} z^i + \frac{g(z) z^k}{1 - f(z)} \tag{6.21}$$

The product of two almost-Riordan arrays of level k, where $k \in \mathbb{N}$

$$\alpha A = \big(a_0(z), a_1(z), \cdots, a_{k-1}(z) \big| g(z), f(z)\big),$$

and

$$\alpha B = \big(b_0(z), b_1(z), \cdots, b_{k-1}(z) \big| u(z), v(z)\big)$$

is

$$\alpha A \cdot \alpha B = \big(a_0(z), \cdots, a_{k-1}(z) \big| g(z), f(z)\big) \cdot \big(b_0(z), b_1(z), \cdots, b_{k-1}(z) \big| u(z), v(z)\big)$$

which is equal to

$$\Big(\big(a_0(z), \ldots, a_{k-1}(z) \big| g(z), f(z)\big) \cdot b_0(z), \ldots, \big(a_{k-2}(z), a_{k-1}(z) \big| g(z), f(z)\big) \cdot b_{k-2}(z), \big(a_{k-1}(z) \big| g(z), f(z)\big) \cdot b_{k-1}(z) \Big| g(z) u\big(f(z)\big), v\big(f(z)\big) \Big)$$

Now, for the power series of the k-th row

$$a_k(z) = a_{0k} + a_{1k} z + a_{2k} z^2 + \cdots = \sum_{j=0}^{\infty} a_{jk} z^j,$$

and

$$b_k(z) = b_{0k} + b_{1k} z + b_{2k} z^2 + \cdots = \sum_{j=0}^{\infty} b_{jk} z^j,$$

where $k \in \mathbb{N}$, we have that

$$\alpha A \cdot \alpha B = \Big(\sum_{n=1}^{k} b_{0(n-1)} a_{n-1}(z) z^{n-1} + z^k g(z) \tilde{b}_0\big(f(z)\big), \ldots, b_{0(k-1)} a_{k-1}(z) + z g(z) \tilde{b}_{k-1}\big(f(z)\big) \Big| g(z) u\big(f(z)\big), v\big(f(z)\big) \Big). \tag{6.22}$$

We note that the first term of the RHS of eq 6.22 expresses the almost part of the matrix and the second part of it, expresses the part of the Riordan matrix. That leads us to the following proposition:

Proposition 6.2.2. *The set of almost-Riordan matrices with k extra columns*

$$\alpha\mathcal{R}(k) = \left\{ \big(a_0(z), a_1(z), \ldots, a_{k-1}(z)\big|g(z), f(z)\big) \Big| a_i, g \in \mathbb{F}_0, f \in \mathbb{F}_1 \right\},$$

forms a Riordan group, for every $k \in \mathbb{N}$.

Proof. Obviously, for the case of $k = 0$, we have the Riordan group and for the cases of $k = 1$ and $k = 2$, we have the almost-Riordan groups of level 1 and 2, respectively.
For the general case of the set $\alpha\mathcal{R}(k)$, we know that it is close under the almost-Riordan multiplication as we proved in eq 6.22. So, it suffices to show that for the inverse of an almost-Riordan array of level k, we have that

$$\big(aR(k)\big)^{-1} \in \alpha\mathcal{R}(k) \tag{6.23}$$

Now, let $aT = \big(a(z), b(z), c(z)\big|g(z), f(z)\big)$ be an almost-Riordan array of $\alpha\mathcal{R}(3)$, then

$$\begin{aligned} (aT)^{-1} &= \big(a(z), b(z), c(z)\big|g(z), f(z)\big)^{-1} \\ &= \left(a^{***}(z), b^{**}(z), c^{*}(z)\Big|\frac{1}{g\big(\bar{f}(z)\big)}, \bar{f}(z)\right)^{-1}, \end{aligned}$$

by adding a new column in eq (13) in [11]. Where

$$a^{***}(z) = a_0 - a_1 z b^{**}(z) - a_2 z^2 c^{*}(z) - \frac{z^3}{g\big(\bar{f}(z)\big)}\tilde{a}^{(3)}\big(\bar{f}(z)\big)$$

and

$$\tilde{a}^{(3)}(z) = \frac{a(z) - a_0 - a_1 z - a_2 z^2}{z^3} \tag{6.24}$$

Iterating the process, we have that

$$\begin{aligned} \big(aR(k)\big)^{-1} &= \big(a_0(z), a_1(z), \ldots, a_{k-1}(z)\big|g(z), f(z)\big)^{-1} \\ &= \left(a_0^{*(k)}(z), a_1^{*(k-1)}(z), \ldots, a_{k-1}^{*}(z)\Big|\frac{1}{g\big(\bar{f}(z)\big)}, \bar{f}(z)\right), \end{aligned} \tag{6.25}$$

where

$$a_p^{*(n)}(z) = a_{0_p} - \sum_{i=1}^{n-1} a_{i_p} z^i a_i^{*(n-i)}(z) - \frac{z^n}{g\big(\bar{f}(z)\big)}\tilde{a}_p^{(n)}\big(\bar{f}(z)\big),$$

and

$$\tilde{a}^{(n)}(z) = \frac{a(z) - \sum_{i=0}^{n-1} a_i z^i}{z^n}$$

for $p, n \in \mathbb{N}$. □

6.2.1 Product of almost-Riordan arrays of different level

In section 6.2, we defined the product for almost-Riordan arrays of level $k \neq 0$. A modified form of this product is also presented in the following theorem, for almost-Riordan arrays of different level.

Theorem 6.2.2. *Let $\alpha\mathcal{K}(n)$, and $\alpha\mathcal{L}(m)$ be two almost-Riordan arrays of level n, and m, respectively, where $n \neq m$. The product $\alpha\mathcal{K}(n) \cdot \alpha\mathcal{L}(m)$ is defined as*

$$\alpha\mathcal{K}(n) \cdot \alpha\mathcal{L}(m) = \alpha\mathcal{R}(p),$$

where $p = max\{n, m\}$.

Proof. Suppose that $m < n$, and let

$$\alpha\mathcal{K}(n) = \big(a_0(z), a_1(z), \ldots\ldots, a_{n-1}(z) \big| g(z), f(z)\big),$$

and

$$\alpha\mathcal{L}(m) = \big(b_0(z), b_1(z), \ldots, b_{m-1}(z) \big| u(z), v(z)\big),$$

where

$$a_k(z) = \sum_{j=0}^{\infty} a_{jk} z^j, \text{ and } b_k(z) = \sum_{j=0}^{\infty} b_{jk} z^j,$$

for $k \in \mathbb{N}$. Now, we expand $\alpha\mathcal{L}(m)$ as

$$\left(b_0, b_1, \ldots, b_{m-1}, \underbrace{u, \frac{uv}{z}, \frac{uv^2}{z^2}, \ldots, \frac{uv^{n-m-1}}{z^{n-m-1}}, \frac{uv^{n-m}}{z^{n-m}}}_{(n-m)\text{terms}} \middle| \frac{uv^{n-m+1}}{z^{n-m+1}}, v\right) \tag{6.26}$$

and let $U(z) = \frac{uv^{n-m+1}}{z^{n-m+1}}$. Hence, we get that $\alpha\mathcal{K}(n) \cdot \alpha\mathcal{L}(m)$ is equal to

$$\big(a_0(z), a_1(z), \ldots, a_{n-1}(z) \big| g(z), f(z)\big) \cdot \left(b_0(z), b_1(z), \ldots, \frac{u(z)v(z)^{n-m}}{z^{n-m}} \middle| U(z), v(z)\right),$$

and

$$\Bigg(\Big(a_0(z), a_1(z), \ldots, a_{n-1}(z) \Big| g(z), f(z)\Big) \cdot b_0(z), \ldots, \\ \Big(a_{n-2}(z), a_{n-1}(z) \Big| g(z), f(z)\Big) \cdot u(z) \frac{v(z)^{n-m-1}}{z^{n-m-1}}, \\ \Big(a_{n-1}(z) \Big| g(z), f(z)\Big) \cdot u(z) \frac{v(z)^{n-m}}{z^{n-m}} \\ \Big| g(z)U\big(f(z)\big), v\big(f(z)\big) \Bigg), \tag{6.27}$$

which becomes an almost-Riordan array of level n, $\alpha\mathcal{R}(n)$. It can be similarly shown for the case of $m > n$. We get that

$$\alpha\mathcal{K}(n) \cdot \alpha\mathcal{L}(m) = \alpha\mathcal{R}(m). \tag{6.28}$$

Hence in both cases, we have that

$$\alpha\mathcal{K}(n) \cdot \alpha\mathcal{L}(m) = \alpha\mathcal{R}\big(max\{n, m\}\big).$$

□

6.3 Factorization of a Riordan matrix to almost-Riordan matrices

We have earlier presented factorizations of Riordan matrices such as semi-direct products of the Riordan group in Chapters 3 and 4, and decompositions of Riordan quasi-involutions to their quasi-compressions in Chapter 5. In the current section, we show another method of factorizing a Riordan matrix, using almost-Riordan matrices.
Let the Riordan matrix $R = \big(g(z), f(z)\big)$, where

$$g(z) = g_0 + g_1 z + g_2 z^2 + \cdots, \text{ and } f(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots$$

then R is analysed as

$$\big(g(z), f(z)\big) = \begin{bmatrix} g_0 & 0 & 0 & 0 & \cdots \\ g_1 & g_0 f_1 & 0 & 0 & \cdots \\ g_2 & g_0 f_2 + g_1 f_1 & g_0 f_1^2 & 0 & \cdots \\ g_3 & g_0 f_3 + g_1 f_2 + g_2 f_1 & 2g_0 f_1 f_2 + g_1 f_1^2 & g_0 f_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{6.29}$$

We observe that all columns were composed by coefficients of both gfs, except for the initial one which solely contains coefficients of the first gf, in an ascending order. If we exclude the first row and the first column of the matrix, we get

$$\begin{aligned} \big(g(z), f(z)\big) &= \left[\begin{array}{c|ccc} g_0 & 0 & 0 & 0 & \cdots \\ \hline g_1 & g_0 f_1 & 0 & 0 & \cdots \\ g_2 & g_0 f_2 + g_1 f_1 & g_0 f_1^2 & 0 & \cdots \\ g_3 & g_0 f_3 + g_1 f_2 + g_2 f_1 & 2g_0 f_1 f_2 + g_1 f_1^2 & g_0 f_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] \\ &= \left[\begin{array}{c|c} g_0 & \vec{0} \\ \hline G(z) & \left(\frac{g(z)f(z)}{z}, f(z)\right) \end{array}\right] \end{aligned} \tag{6.30}$$

where $G(z) = g(z) - g_0 = g_1 z + g_2 z^2 + \cdots$, and $\vec{0} = (0, 0, 0, 0, \ldots)$. Now, the internal submatrix of (6.30), is analysed further as

$$\left(\frac{g(z)f(z)}{z}, f(z)\right) = \left(\frac{f(z)}{z}, z\right) \cdot \big(g(z), f(z)\big), \tag{6.31}$$

and it follows that every Riordan matrix 6.29 can be factorised by

$$\big(g(z), f(z)\big) = \left[\begin{array}{c|c} g_0 & \vec{0} \\ \hline G(z) & \left(\frac{f(z)}{z}, z\right) \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & \vec{0} \\ \hline \vec{0} & \big(g(z), f(z)\big) \end{array}\right]. \tag{6.32}$$

By induction, we have that

$$\big(g(z), f(z)\big) = \prod_{k \geq 0} \left(I_k \oplus \left[\begin{array}{c|c} g_0 & \vec{0}^T \\ \hline G(z) & \left(f(z)/z, z\right) \end{array}\right] \right),$$

which is analysed as

$$\begin{bmatrix} g_0 & 0 & 0 & 0 & \cdots \\ g_1 & f_1 & 0 & 0 & \cdots \\ g_2 & f_2 & f_1 & 0 & \cdots \\ g_3 & f_3 & f_2 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & g_0 & 0 & 0 & \cdots \\ 0 & g_1 & f_1 & 0 & \cdots \\ 0 & g_2 & f_2 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & g_0 & 0 & \cdots \\ 0 & 0 & g_1 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \cdots . \tag{6.33}$$

For the inverse of a Riordan array, we have by eq 6.32

$$\big(g(z), f(z)\big)^{-1} = \left(\left[\begin{array}{c|c} g_0 & \vec{0} \\ \hline G(z) & \left(\frac{f(z)}{z}, z\right) \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & \vec{0} \\ \hline \vec{0} & (g(z), f(z)) \end{array}\right] \right)^{-1}$$

which is written as

$$\big(g(z), f(z)\big)^{-1} = \big[I_1 \oplus \big(g(z), f(z)\big)^{-1}\big] \cdot \left[\begin{array}{c|c} g_0 & \vec{0}^T \\ \hline G(z) & \left(\frac{f(z)}{z}, z\right) \end{array}\right]^{-1}$$

where

$$\left[\begin{array}{c|c} g_0 & \vec{0}^T \\ \hline G(z) & \Big(f(z)/z, z\Big) \end{array}\right]^{-1} = \left[\begin{array}{c|c} 1/g_0 & \vec{0}^T \\ \hline \vec{\widehat{g}} & \Big(z/f(z), z\Big) \end{array}\right]$$

with the vector $\vec{\widehat{g}}$ corresponding to the generating function

$$\widehat{g}(z) = \frac{g_0 - g(z)}{g_0} f(z) \tag{6.34}$$

and $\frac{z}{f(z)} = \sum_{n\geq 0} d_n z^n$ is determined by $d_0 = \frac{1}{f_1}$ and for $n \geq 1$, and by Wronski's formula (page 18 of [46]), we have

$$d_n = \frac{(-1)^n}{f_1^{n+1}} \det \begin{bmatrix} f_2 & f_1 & 0 & \cdots & 0 \\ f_3 & f_2 & f_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_n & f_{n-1} & \cdots & f_2 & f_1 \\ f_{n+1} & f_n & \cdots & f_3 & f_2 \end{bmatrix}_{n\times n} .$$

The matrices of the RHS of eq 6.33 are not Riordan arrays. Nevertheless, they can be written as almost-Riordan arrays, where the gf of the first column of the internal matrix of each of those arrays is $\frac{f(z)}{z}$. Hence, we have that a Riordan array can be decomposed to an infinite product of almost-Riordan arrays of

different levels, as

$$(g(z), f(z)) = \left(g(z)\middle|\frac{f(z)}{z}, z\right)\left(1, g(z)\middle|\frac{f(z)}{z}, z\right)\left(1, 1, g(z)\middle|\frac{f(z)}{z}, z\right)\cdots$$

where, the product for almost-Riordan arrays of different levels is defined in Theorem 6.2.2. So, we have the following theorem:

Theorem 6.3.1. *Every Riordan matrix is factorised by infinitely many almost-Riordan matrices as*

$$(g(z), f(z)) = \prod_{k=0}^{\infty}\left(\underbrace{1, \ldots, 1}_{k \text{ terms}}, g(z)\middle|\frac{f(z)}{z}, z\right).$$

Example 6.3.2. *Let* $K = \left(\frac{1}{1-2z-z^2}, \frac{1-\sqrt{1-4z^2}}{2z}\right)$ *so we have*

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 12 & 6 & 2 & 1 & 0 & 0 & \cdots \\ 29 & 14 & 7 & 2 & 1 & 0 & \cdots \\ 70 & 36 & 16 & 8 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{6.35}$$

Writing the gfs of K as power-series, we get

$$\begin{aligned} \frac{1}{1-2z-z^2} &= 1 + 2z + 5z^2 + 12z^3 + 29z^4 + 70z^5 + \cdots \\ \frac{1-\sqrt{1-4z^2}}{2z} &= z + z^3 + 2z^5 + 5z^7 + 14z^9 + 42z^{11} + \cdots \end{aligned}$$

that correspond to the sequence of Pell numbers 1,2,5,12,29,70,... (OEIS, A000129), and the aerated Catalan numbers 1,1,2,5,14, ... (OEIS, A000108), respectively. Hence, according to Theorem 6.3.1, the matrix K is factorised as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 12 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 29 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 70 & 2 & 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 12 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 29 & 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 5 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 12 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \cdots$$

6.4 Involutions in the group of almost Riordan arrays

Searching for involutions among the almost Riordan matrices, let the almost-Riordan array $\big(a(z)\big|g(z),f(z)\big)$ be an involution, so it needs to satisfy

$$\big(a(z)\big|g(z),f(z)\big)\cdot\big(a(z)\big|g(z),f(z)\big)=(1|1,z)$$

which becomes

$$\Big(\big(a(z)\big|g(z),f(z)\big)\cdot a(z)\Big|g(z)g\big(f(z)\big),f\big(f(z)\big)\Big)=(1|1,z). \tag{6.36}$$

The same conditions as in the definition of involutions (3.1.2) for Ordinary Riordan arrays, are satisfied for the internal generating functions $g(z)$ and $f(z)$ of eq 6.36, while for the first column, we have that

$$\begin{aligned}\big(a(z)\big|g(z),f(z)\big)\cdot a(z) &= a_0a(z)+zg(z)\frac{a\big(f(z)\big)-a_0}{f(z)}\\ &= a_0a(z)+\frac{zg(z)a\big(f(z)\big)}{f(z)}-\frac{a_0zg(z)}{f(z)}\\ &= a_0\left(a(z)-\frac{zg(z)}{f(z)}\right)+\frac{zg(z)a\big(f(z)\big)}{f(z)},\end{aligned} \tag{6.37}$$

which for an involution is equal to 1. According to the definition of an almost-Riordan array, the gf of the initial column $a(z)\in\mathbb{F}_0$, while $g(z)\in\mathbb{F}_0$, and $f(z)\in\mathbb{F}_1$.

Proposition 6.4.1. *If $\big(g(z),f(z)\big)$ is an involution in $\mathcal{R}$, then $\big(1\big|g(z),f(z)\big)$ is an involution in $\alpha\mathcal{R}(1)$.*

Proof. We have $a(z)=1$ and thus

$$a_0\left(a(z)-\frac{zg(z)}{f(z)}\right)+\frac{zg(z)a\big(f(z)\big)}{f(z)}=1-\frac{zg(z)}{f(z)}+\frac{zg(z)}{f(z)}=1,$$

as required. □

Proposition 6.4.2. *Let $\big(g(z),f(z)\big)$ be an ordinary Riordan involution, then the almost-Riordan array $\big(a(z)\big|g(z),f(z)\big)$ is also an involution if $a(z)=\frac{zg(z)}{f(z)}$.*

Proof. By choosing $a(z) = \frac{zg(z)}{f(z)}$, we get a power series in $\mathbb{F}_0$, and eq 6.37 becomes

$$\begin{aligned} a_0\left(a(z) - \frac{zg(z)}{f(z)}\right) + \frac{zg(z)a\big(f(z)\big)}{f(z)} &= \frac{zg(z)a\big(f(z)\big)}{f(z)} \\ &= \frac{zg(z)\dfrac{f(z)g\big(f(z)\big)}{f\big(f(z)\big)}}{f(z)} \\ &= zg(z)\frac{g\big(f(z)\big)}{f\big(f(z)\big)} \end{aligned} \tag{6.38}$$

Now, since $\big(g(z), f(z)\big)$ is an involution, we have that

$$\big(g(z), f(z)\big) \cdot \big(g(z), f(z)\big) = (1, z)$$

and

$$\big(g(z), f(z)\big) \cdot \big(g(z), f(z)\big) = \Big(g(z)g\big(f(z)\big), f\big(f(z)\big)\Big),$$

which leads us to the conditions

$$g(z) = \frac{1}{g\big(f(z)\big)}, \text{ and } f\big(f(z)\big) = z \tag{6.39}$$

Applying the conditions 6.39 to eq 6.38, we have that this (6.38) is equal to 1 as required. □

We note that in the general case, the almost Riordan array $\left(\frac{zg(z)}{f(z)}\middle| g(z), f(z)\right)$ is in fact a Riordan array. It coincides with the Riordan array $\left(\frac{zg(z)}{f(z)}, f(z)\right)$. Generalizing, we get the following proposition for $\left(\frac{z^n}{f(z)^n}g(z), f(z)\right)$.

Proposition 6.4.3. *If $\big(g(z), f(z)\big)$ is an involution in $\mathcal{R}$, then so is $\left(\frac{z^n}{f(z)^n}g(z), f(z)\right)$, for $n \in \mathbb{N}$.*

Proof. Let $\big(g(z), f(z)\big)$ be an involution. For the Riordan array $\big(G(z), f(z)\big)$, where $G(z) = \frac{z^n}{f(z)^n}g(z)$, it can be easily shown that

$$G\big(f(z)\big) = \frac{1}{G(z)}.$$

□

We also note that the Riordan array $\left(\frac{z^n}{f(z)^n}g(z), f(z)\right)$, for a fixed $n \in \mathbb{N}$ can be written as an almost-Riordan array as $\left(\frac{z^n}{f(z)^n}g(z)\middle|\frac{z^n}{f(z)^{n-1}}g(z), f(z)\right)$.

For the Riordan family of subgroups which are solely defined by their second gf $f(z)$, $\Upsilon[r,s,p]$, we only need the condition $f(z) = \bar{f}(z)$, as the $g(z)$ function depends on $f(z)$. Hence, we notice that if

$$v_f[\rho,\sigma,\pi] = \left(\left(\frac{f(z)}{z}\right)^{\rho}(f'(z))^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z)\right)$$

is a Riordan involution, then the almost-Riordan array

$$\left(\left(\frac{f(z)}{z}\right)^{\rho-1}(f'(z))^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}\middle|\left(\frac{f(z)}{z}\right)^{\rho}(f'(z))^{\sigma}\left(\frac{f(z)-1}{z-1}\right)^{\pi}, f(z)\right)$$

is also an involution.

Example 6.4.1. *Let $f(z) = -\frac{z}{1+z}$, so $v_f[1,1,1]$ has the form $\left(\frac{f(z)}{z}f'(z)\frac{f(z)-1}{z-1}, f(z)\right)$, which corresponds to the matrix*

$$\left(\frac{1+2z}{(1+z)^4(1-z)}, -\frac{z}{1+z}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & -3 & -1 & 0 & 0 & 0 & 0 \\ -4 & 2 & 6 & 4 & 1 & 0 & 0 & 0 \\ 10 & 2 & -8 & -10 & -5 & -1 & 0 & 0 \\ -18 & -12 & 6 & 18 & 15 & 6 & 1 & 0 \\ 30 & 30 & 6 & -24 & -33 & -21 & -7 & -1 \end{pmatrix},$$

which is an involution, as $f(z) = \bar{f}(z)$. Now, the almost-Riordan matrix which contains $v_f[1,1,1]$ is also an involution and has the form

$$\left(f'(z)\frac{f(z)-1}{z-1}\middle|\frac{f(z)}{z}f'(z)\frac{f(z)-1}{z-1}, f(z)\right) = \left(\frac{1+2z}{(1+z)^3(1-z)}, -\frac{z}{1+z}\right)$$

which is equal to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & -3 & -3 & -1 & 0 & 0 & 0 \\ 6 & -4 & 2 & 6 & 4 & 1 & 0 & 0 \\ -8 & 10 & 2 & -8 & -10 & -5 & -1 & 0 \\ 12 & -18 & -12 & 6 & 18 & 15 & 6 & 1 \end{pmatrix}$$

6.5 Pseudo-involutions in the almost-Riordan group

6.5.1 Pseudo-involutions in the group of almost-Riordan arrays of first order

Almost-Riordan arrays can also form pseudo-involutions under specific conditions. We recall the definition of a pseudo-involution for a Riordan array $R = \big(g(z), f(z)\big)$, is $(R \cdot M)^2 = I$, where $M = (1, -z)$. As we presented in Chapter 2, Pascal's Triangle is a known pseudo-involution. Now, if the initial column $(1,1,1,1,1,1,...)^T$ of this Riordan matrix is replaced by $(1,r,r,r,r,r,...)^T$, we get

$$A_r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ r & 1 & 0 & 0 & 0 & 0 & \cdots \\ r & 2 & 1 & 0 & 0 & 0 & \cdots \\ r & 3 & 3 & 1 & 0 & 0 & \cdots \\ r & 4 & 6 & 4 & 1 & 0 & \cdots \\ r & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

which is not a Riordan matrix, for $r \neq 1$. However, this matrix can be expressed as an almost-Riordan matrix.

Proposition 6.5.1. *The almost-Riordan array*

$$A_r = \left(\frac{1 + z(r-1)}{1-z} \middle| \frac{1}{(1-z)^2}, \frac{z}{1-z} \right)$$

for $r \in \mathbb{Z} - \{1\}$, is a pseudo-involution in the almost-Riordan group.

Proof. Since the Riordan part of this matrix comes from Pascal's Triangle, the columns of the "interior" matrix satisfy the conditions for a pseudo-involution.

Now, the inverse of Pascal's triangle is the Riordan array

$$\begin{aligned} P^{-1} &= \left(\frac{1}{1+z}, \frac{z}{1+z}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 3 & -3 & 1 & 0 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots \\ -1 & 5 & -10 & 10 & -5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

Hence, it suffices to show that the first column of the inverse of A_r needs to be $(1, -r, r, -r, r, -r, ...)^T$, which has gf $1 - \frac{r}{1+z}$. After all these observations, we have

$$\left(\frac{1+z(r-1)}{1-z}\middle|\frac{1}{(1-z)^2}, \frac{z}{1-z}\right)^{-1} = \left(\left(\frac{1+z(r-1)}{1-z}\right)^*\middle|\frac{1}{(1+z)^2}, \frac{z}{1+z}\right),$$

where

$$\begin{aligned} \left(\frac{1+z(r-1)}{1-z}\right)^* &= \left(1\middle| -\frac{1}{(1-z)^2}, \frac{z}{1-z}\right)^{-1} \cdot \frac{1+z(r-1)}{1-z} \\ &= \left(1\middle| -\frac{1}{(1+z)^2}, \frac{z}{1+z}\right) \cdot \frac{1+z(r-1)}{1-z} \\ &= 1 + z\left(-\frac{1}{(1+z)^2}\right) \cdot \frac{r}{1-\frac{z}{1+z}} \\ &= 1 - \frac{z}{(1+z)^2} \cdot \frac{r(1+z)}{1+z-z} \\ &= 1 - \frac{rz}{1+z}. \end{aligned}$$

□

Working similarly for the case of the general Pascal's Triangle $\left(\frac{1}{1-cz}, \frac{z}{1-cz}\right)$, for $c \in \mathbb{Z}$, we add an extra parameter and we get the following proposition:

Proposition 6.5.2. *The almost-Riordan array*

$$A_r^p = \left(\frac{1+pz(r-1)}{1-pz}\middle|\frac{1}{(1-pz)^2}, \frac{z}{1-pz}\right)$$

is a pseudo-involution in the almost-Riordan group $\alpha\mathcal{R}(1)$.

Proof. We are going to show that the first column of the inverse of A_r^p is given by $(1, -rp, rp^2, -rp^3, ...)$, with gf $1 - \frac{prz}{1+pz}$. So, we have that

$$\left(\frac{1+pz(r-1)}{1-pz}\middle|\frac{1}{(1-pz)^2}, \frac{z}{1-pz}\right)^{-1} = \left(\left(\frac{1+pz(r-1)}{1-pz}\right)^*\middle|\frac{1}{(1+pz)^2}, \frac{z}{1+pz}\right) \tag{6.40}$$

where,

$$\begin{aligned}
\left(\frac{1+pz(r-1)}{1-pz}\right)^* &= \left(1\middle|-\frac{1}{(1-pz)^2}, \frac{z}{1-pz}\right)^{-1} \cdot \frac{1+pz(r-1)}{1-pz} \\
&= \left(1\middle|-\frac{1}{(1+pz)^2}, \frac{z}{1+pz}\right) \cdot \frac{1+pz(r-1)}{1-pz} \\
&= 1 + z\left(\frac{1}{(1+pz)^2}\right) \cdot \frac{pr}{1-p\frac{z}{1+pz}} \\
&= 1 - \frac{z}{(1+pz)^2} \cdot \frac{pr(1+pz)}{1+pz-pz} \\
&= 1 - \frac{prz}{1+pz}.
\end{aligned}$$

□

Now, we are going to prove that these pseudo-involutions form a Riordan subgroup.
For $r, s \neq 1$, and $p, q \in \mathbb{Z}$, we have that $A_r^p \cdot A_s^q$ is equal to

$$\left(\frac{1+pz(r-1)}{1-pz}\middle|\frac{1}{(1-pz)^2}, \frac{z}{1-pz}\right) \cdot \left(\frac{1+qz(s-1)}{1-qz}\middle|\frac{1}{(1-qz)^2}, \frac{z}{1-qz}\right),$$

applying almost-Riordan multiplication, we get

$$\left(\left(\frac{1+pz(r-1)}{1-pz}\middle|\frac{1}{(1-pz)^2}, \frac{z}{1-pz}\right) \cdot \frac{1+qz(s-1)}{1-qz}\middle| \frac{1}{(1-pz)^2} \cdot \frac{1}{\left(1-q\frac{z}{1-pz}\right)^2}, \frac{\frac{z}{1-pz}}{1-q\frac{z}{1-pz}}\right)$$

which simplifies to

$$\left(\left(\frac{1+pz(r-1)}{1-pz}\middle|\frac{1}{(1-pz)^2},\frac{z}{1-pz}\right)\cdot\frac{1+qz(s-1)}{1-qz}\middle|\frac{1}{(1-(p+q)z)^2},\frac{z}{1-(p+q)z}\right) \tag{6.41}$$

For the first column of this array, we have that

$$\left(\frac{1+pz(r-1)}{1-pz}\middle|\frac{1}{(1-pz)^2},\frac{z}{1-pz}\right)\cdot\frac{1+qz(s-1)}{1-qz}$$

is equal to

$$\frac{1+p(r-1)z}{1-pz}+z\frac{1}{(1-pz)^2}\frac{\dfrac{1+q\dfrac{z}{1-pz}(s-1)}{1-\dfrac{z}{1-pz}}-1}{\dfrac{z}{1-pz}},$$

which simplifies to

$$1+\frac{prz}{1-pz}+\frac{z}{(1-pz)^2}\frac{\dfrac{\dfrac{qsz}{1-pz}}{\dfrac{1-pz-qz}{1-pz}}}{\dfrac{z}{1-pz}}$$

and finally,

$$1+\frac{prz}{1-pz}+sq\frac{z}{(1-pz)(1-(p+q)z)}.$$

So, eq 6.41 becomes

$$\left(1+\frac{prz}{1-pz}+sq\frac{z}{(1-pz)(1-(p+q)z)}\middle|\frac{1}{(1-(p+q)z)^2},\frac{z}{1-(p+q)z}\right) \tag{6.42}$$

Since we proved that the set of almost-Riordan arrays A_r^p are closed under the almost-multiplication (6.5), and their inverse are of the same form (6.40), they

form an almost-Riordan subgroup. More specifically, we have the following corollary:

Corollary 6.5.1. *For $r \neq 1$, the almost-Riordan array*

$$A_r = \left(\frac{1+z(r-1)}{1-z} \middle| \frac{1}{(1-z)^2}, \frac{z}{1-z} \right)$$

generates a subgroup of pseudo-involutions in $\alpha R(1)$.

Comparing the Riordan subgroup

$$P_c = \left\{ \left(\frac{1}{1-cz}, \frac{z}{1-cz} \right), c \in \mathbb{Z} \right\}$$

which is generated by Pascal's Triangle, with the almost-Riordan subgroup, generated by A_r

$$A_r^p = \left\{ \left(\frac{1+pz(r-1)}{1-pz} \middle| \frac{1}{(1-pz)^2}, \frac{z}{1-pz} \right); r \neq 1, p \in \mathbb{Z} \right\}$$

we see that although both of them solely contain pseudo-involutions, the only subgroup which is abelian is P_c. For A_r^p and by eq 6.42, we get

$$A_r^p \cdot A_s^q \neq A_s^q \cdot A_r^p.$$

We also note that if $\big(g(z), f(z)\big)$ is a pseudo-involution, then the trivial almost-Riordan array $\big(1 \big| g(z), f(z)\big)$ is a pseudo-involution in the group of almost-Riordan arrays. Decomposing the almost-Riordan array A_r, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ r & 1 & 0 & 0 & 0 & 0 & \cdots \\ r & 2 & 1 & 0 & 0 & 0 & \cdots \\ r & 3 & 3 & 1 & 0 & 0 & \cdots \\ r & 4 & 6 & 4 & 1 & 0 & \cdots \\ r & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ r & 1 & 0 & 0 & 0 & 0 & \cdots \\ r & 0 & 1 & 0 & 0 & 0 & \cdots \\ r & 0 & 0 & 1 & 0 & 0 & \cdots \\ r & 0 & 0 & 0 & 1 & 0 & \cdots \\ r & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 0 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $\left(1 \middle| \frac{1}{(1-z)^2}, \frac{z}{1-z} \right)$ is a pseudo-involution.

6.5.2 Pseudo-involutions in the group of almost-Riordan arrays of the second order

For the case of almost-Riordan arrays of level 2, we define an almost-Riordan element of $\alpha R(2)$ by using once again Pascal's Triangle for the Riordan part of the matrices, while the same $\mathbb{F}_0$ function is used for the first two columns. So, we have

$$\left(\frac{1+z}{1-z}, \frac{1+z}{1-z}\middle| \frac{1}{(1-z)^2}, \frac{z}{1-z}\right).$$

The corresponding matrix M is then defined by

$$\begin{aligned} M_{n,k} &= [z^n] z^k \frac{1+z}{1-z}, k < 2; \\ M_{n,k} &= [z^n] \frac{z^k}{(1-z)^k}, k \geqslant 2. \end{aligned}$$

Thus the matrix M is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 2 & 2 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 2 & 2 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and its inverse is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 2 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 2 & -2 & 3 & -3 & 1 & 0 & 0 & \cdots \\ -2 & 2 & -4 & 6 & -4 & 1 & 0 & \cdots \\ 2 & -2 & 5 & -10 & 10 & -5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proposition 6.5.3. *The element* $\left(\frac{1+z}{1-z}, \frac{1+z}{1-z}\middle| \frac{1}{(1-z)^2}, \frac{z}{1-z}\right) \in \alpha R(2)$ *is a pseudo-involution.*

Proof. We have

$$\begin{aligned} b^*(z) &= \left(1\middle| -\frac{1}{(1-z)^2}, \frac{z}{1-z}\right)^{-1} \cdot \frac{1+z}{1-z} \\ &= \left(1\middle| -(1+z)^2, \frac{z}{1+z}\right) \cdot \frac{1+z}{1-z} \\ &= 1 - \frac{z}{(1+z)^2}\frac{2}{1-\frac{z}{1+z}} \\ &= 1 - \frac{2}{1+z} \\ &= \frac{1-z}{1+z}, \end{aligned}$$

which expands to give $1, -2, 2, -2, \ldots$.
We then obtain that

$$\begin{aligned} a^{**}(z) &= \left(1, -\frac{1-z}{1+z}\middle| -\frac{1}{(1+z)^2}, \frac{z}{1+z}\right) \cdot \frac{1+z}{1-z} \\ &= 1 - 2z\frac{1-z}{1+z} - \frac{z^2}{(1+z)^2}\frac{2(1+z)}{1+z-z} \\ &= \frac{1-z}{1+z}. \end{aligned}$$

□

It is possible to extend this result to higher orders. For instance, we can consider the almost Riordan array of third order defined by

$$\left(\frac{1+z}{1-z}, \frac{1+z}{1-z}, \frac{1+z}{1-z}\middle| \frac{1}{(1-z)^2}, \frac{z}{1-z}\right).$$

The corresponding matrix is defined by

$$\begin{aligned} M_{n,k} &= [z^n] z^k \frac{1+z}{1-z}, \quad k < 3; \\ M_{n,k} &= [z^n] \frac{z^k}{(1-z)^{k-1}}, \quad k \geq 3, \end{aligned}$$

and we have that

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 2 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 2 & 2 & 2 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 2 & 2 & 2 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The inverse of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 2 & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & -2 & 2 & -2 & 1 & 0 & 0 & 0 & \cdots \\ -2 & 2 & -2 & 3 & -3 & 1 & 0 & 0 & \cdots \\ 2 & -2 & 2 & -4 & 6 & -4 & 1 & 0 & \cdots \\ -2 & 2 & -2 & 5 & -10 & 10 & -5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

6.6 Quasi-involutions in the almost-Riordan group

In this section, we are going to demonstrate how to build quasi-involutions in the almost-Riordan group, by using known quasi-involutions from the Riordan group. For that purpose, we are going to use two different approaches. First, by adjoining an extra column on the left of the matrix, as we have already applied in previous sections of the chapter, and then by replacing its initial column.

6.6.1 Adding a new column

By eq 5.23, we get that

$$W = Q_{2k} \cdot Q_k^{-1} \tag{6.43}$$

which gives us

$$\left(\frac{F(z)}{z}, F(z)\right) = \left(\frac{f(z)}{z}, f(z)\right) \cdot \left(\frac{f^*(z)}{z}, f^*(z)\right)^{-1} \tag{6.44}$$

where $F(z) = \frac{z}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}}$, $f(z) = \frac{z}{\sqrt[2k]{1-cz^{2k}}}$, and $f^*(z) = \frac{z}{\sqrt[k]{1-cz^k}}$.
Now, by adding a column on each of the arrays on the RHS of eq 6.44, we have:

$$\left(A(z)\middle|\frac{1}{\sqrt[2k]{1-cz^{2k}}},\frac{z}{\sqrt[2k]{1-cz^{2k}}}\right)\cdot\left(a(z)\middle|\frac{1}{\sqrt[k]{1+cz^k}},\frac{z}{\sqrt[k]{1+cz^k}}\right) \tag{6.45}$$

where $A, a \in \mathbb{F}_0$, and their quasi-transitional matrix becomes

$$\left(A(z)+a\big(f(z)\big)+1\middle|\frac{1}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}},\frac{z}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}}\right) \tag{6.46}$$

We note that $A(z)$ needs to be a $(2k-1)$- aerated, and $a(z)$ a $(k-1)$- aerated fps, as we see on the following example.

Example 6.6.1. *Let the Riordan quasi-involution*

$$Y = \left(\frac{1}{\sqrt{1-8z^2}},\frac{z}{\sqrt{1-8z^2}}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 8 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 0 & 12 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 64 & 0 & 16 & 0 & 1 & 0 & \cdots \\ 160 & 0 & 120 & 0 & 20 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and let

$$\sigma Y = \left(K(z)\middle|\frac{1}{\sqrt{1-8z^2}},\frac{z}{\sqrt{1-8z^2}}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ B & 0 & 8 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 24 & 0 & 12 & 0 & 1 & 0 & 0 & \cdots \\ \Gamma & 0 & 64 & 0 & 16 & 0 & 1 & 0 & \cdots \\ 0 & 160 & 0 & 120 & 0 & 20 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

be the same matrix with the extra column

$$1, 0, A, 0, B, 0, \Gamma, 0, \Delta, 0, E, 0, Z, 0, H, 0, \Theta, 0, I.. \tag{6.47}$$

on the left.
We need $\sigma Y \cdot (\sigma Y)^{-1} = I$, *where*

$$(\sigma Y)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ B & 0 & -8 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 24 & 0 & -12 & 0 & 1 & 0 & 0 & \cdots \\ -\Gamma & 0 & 64 & 0 & -16 & 0 & 1 & 0 & \cdots \\ 0 & -160 & 0 & 120 & 0 & -20 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which leads us to the equations

$$B = 4A$$
$$\Delta = 4(3\Gamma - 7A)$$
$$Z = 4(5E - 320\Gamma - 3,904A)$$
$$\vdots$$

Hence, the sequence 6.47 is expressed as the fps

$$K(z) = 1 + Az^2 + 4Az^4 + \Gamma z^6 + 4(3\Gamma - 7A)z^8 + Ez^{10} + 4(5E - 320\Gamma - 3,904A)z^{12} + \cdots, \tag{6.48}$$

and we say that the almost-Riordan array

$$\sigma Y = \left(K(z) \middle| \frac{1}{\sqrt{1-8z^2}}, \frac{z}{\sqrt{1-8z^2}} \right) \tag{6.49}$$

is a quasi-involution in $\alpha\mathcal{R}(1)$.
For the quasi-compression of σY, *we take the Pascal-like array*

$$\sigma Y^* = \left(k(z) \middle| \frac{1}{1-8z}, \frac{z}{1-8z} \right), \tag{6.50}$$

where

$$k(z) = 1 + \alpha z + 4\alpha z^2 + \gamma z^3 + (12\gamma + 22\alpha)z^4 + \cdots.$$

The fps $K(z)$ *and* $k(z)$ *are used in (6.46) to link these two quasi-involutions through eq 5.23.*

6.6.2 Replacing a column

We also define quasi-involutions in $\alpha\mathcal{R}$, by replacing the first column of a given quasi-involution. Again, we have the quasi-involution of the form

$$\left(\frac{f(z)}{z}, f(z)\right) = \left(\frac{1}{\sqrt[2k]{1-cz^{2k}}}, \frac{z}{\sqrt[2k]{1-cz^{2k}}}\right). \tag{6.51}$$

Since the Riordan array structure of the matrix that we are going to construct, starts from the second column, its gf will be $\frac{f(z)^2}{z}$, while the multiplier function remains the same. So, we have

$$U_{2k} = \left(B(z)\middle|\frac{f(z)}{z}f(z), f(z)\right) = \left(B(z)\middle|\frac{z}{\sqrt[k]{1-cz^{2k}}}, \frac{z}{\sqrt[2k]{1-cz^{2k}}}\right), \tag{6.52}$$

and

$$U_k^{-1} = \left(b(z)\middle|\frac{z}{\sqrt[k]{(1+cz^k)^2}}, \frac{z}{\sqrt[k]{1+cz^k}}\right), \tag{6.53}$$

the inverse of the almost-Riordan matrix which is constructed by its quasi-compression. So, their quasi-transitional matrix $W = U_{2k} \cdot U_k^{-1}$ is

$$\left(B(z) + zf(z)^{\frac{k-1}{2}}\left(b\left(\frac{\sqrt{f(z)}}{z}\right) - 1\right)\middle|\frac{zf(z)}{\sqrt[k]{(\sqrt{1-cz^{2k}}+cz^k)^2}}, \frac{z}{\sqrt[k]{\sqrt{1-cz^{2k}}+cz^k}}\right) \tag{6.54}$$

Example 6.6.2. *Using the same Riordan quasi-involution as in Example 6.6.1, we have the almost-Riordan array*

$$\tau Y = \left(\Lambda(z)\middle|\frac{z}{1-8z^2}, \frac{z}{\sqrt{1-8z^2}}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ B & 0 & 12 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 64 & 0 & 16 & 0 & 1 & 0 & 0 & \cdots \\ C & 0 & 120 & 0 & 20 & 0 & 1 & 0 & \cdots \\ 0 & 512 & 0 & 192 & 0 & 24 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and working similarly we get that

$$\Lambda(z) = 1 + Az + 6Az^2 + Cz^3 + (280A + 14C)z^4 + \cdots$$

Additionally, the quasi-compression of τY is

$$\tau Y^* = \left(\lambda(z)\middle|\frac{z}{(1-8z)^2}, \frac{z}{\sqrt{1-8z}}\right), \tag{6.55}$$

where $\lambda(z) = 1 + az + 8az^2 + cz^3 + (512a - 16c)z^4 + \cdots$.

We note that for the appropriate values of the parameters $A, B, C, D, \ldots$ and $a, b, c, d, \ldots$ of the almost-Riordan arrays with a replaced column, these matrices are equal to their equivalent quasi-involutions of the Riordan group.

Chapter 7

Eigenvalues and Eigenvectors of Riordan matrices

In previous chapters, we mostly approach Riordan arrays via their generating functions. Nevertheless, we should not ignore the fact that these mathematical objects are also expressed as matrices. Analysing further the structure of the Stabilizer subgroup, we have developed a part of the Linear Algebra of Riordan matrices. More specifically, the study of the Stabilizer subgroup led us to a link between the eigenvalues of a Riordan array and the characteristic of the compositional function of this subgroup, information about the forms of the eigenvalues, and the existence of eigenvectors of a Riordan array.

The context of this chapter is part of our common work with G-S. Cheon and M.M. Cohen, under the title "The Linear Algebra of Proper Riordan arrays" (See Appendix C).

7.1 The Stabilizer subgroup as an Eigenvector subgroup

In [12], the Stabilizer subgroup is described as a family of Riordan subgroups, named the Eigenvector subgroups, for any function $h(z) \in \mathbb{F}_0$. Let

$$\vec{h} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \end{bmatrix}$$

be the column vector of eq 3.8, this is also the eigenvector that corresponds to the eigenvalue of the Riordan matrix that equals one, as from its first gf we have

$$\frac{h(z)}{h\big(f(z)\big)} = g(z) \Rightarrow \big(g(z), f(z)\big) \cdot h(z) = 1 \cdot h(z).$$

More generally, we have the following lemma:

Lemma 7.1.1. *Let $A = \big(g(z), f(z)\big)$ be a Riordan matrix and the gf of $\vec{h}$ is $h(z) = h_0 + h_1 z + \cdots$, then $\vec{h}$ is an eigenvector with eigenvalue λ if and only if*

$$g(z) \cdot h\big(f(z)\big) = \lambda \cdot h(z).$$

Proof. It can be easily derived from FTRA and the definition of an eigenvector, since

$$\begin{aligned} A \cdot h(z) &= \big(g(z), f(z)\big) \cdot h(z) \\ &= g(z) \cdot h\big(f(z)\big) \\ &= \lambda \cdot h(z). \end{aligned} \tag{7.1}$$

□

As we presented in subsection 3.2.7, the first gf of a Stabilizer Riordan matrix is always an expression that contains a constant term, and this result does not depend on the order of the fps $h(z)$. That means that the characteristic of an eigenvector can also be different than 0, under restrictions. Now, suppose that in Lemma 7.1.1, we have the column vector

$$h(z) = h_k z^k + h_{k+1} z^{k+1} + h_{k+2} z^{k+2} + \ldots \Rightarrow \vec{h} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ h_k \\ h_{k+1} \\ h_{k+2} \\ \vdots \end{bmatrix}$$

where $h_k \neq 0$, $k \in \mathbb{N}$. We are going to examine under which conditions, $\vec{h}$ is an eigenvector of A with eigenvalue λ. We note that an eigenvalue can be any complex number, nevertheless in the case of Riordan arrays, we limit this range of the available values, according to the next proposition.

Proposition 7.1.1. *Let $A = \big(g(z), f(z)\big)$ be a Riordan matrix, where $g(z) \in \mathbb{F}_0$, and $f(z) \in \mathbb{F}_1$. If*

$$h(z) = h_k z^k + h_{k+1} z^{k+1} + h_{k+2} z^{k+2} + \cdots,$$

with $k \in \mathbb{N}$ and $h_k \neq 0$, and if $\vec{h}$ is an eigenvector of A with eigenvalue λ, then

$$\lambda = g_0 \cdot f_1^k. \tag{7.2}$$

Proof. From (7.1), we get $A \cdot \vec{h} = \lambda \cdot \vec{h}$, so

$$\begin{aligned}\lambda \cdot h(z) &= g(z) \cdot h\big(f(z)\big) \\ &= (g_0 + g_1 z + \cdots)\left(h_k(f_1 z + f_2 z^2 + \cdots)^k + h_{k+1}(f_1 z + f_2 z^2 + \cdots)^{k+1} + \cdots\right)\end{aligned}$$

which can be written as

$$\lambda h_k z^k + \lambda h_{k+1} z^{k+1} + \cdots = (g_0 h_k f_1^k) z^k + \cdots \tag{7.3}$$

where by the first term of both sides of eq 7.3, we get

$$\lambda h_k = g_0 h_k f_1^k,$$

and since $h_k \neq 0$, we prove eq 7.2. □

Before we proceed, and since we proved that $h(z) \in \mathbb{F}_k$, where $k \in \mathbb{N}$, we discriminate the eigenvalues of a Riordan matrix as follows: An eigenvector $\vec{h} = (h_0, h_1, h_2, \dots)^T$ of the Riordan matrix $\big(g(z), f(z)\big)$ is **an eigenvector of level** k if and only if $\vec{h} = (0, \dots, 0, h_k, h_{k+1}, \dots,)^T$, with $h_k \neq 0$. While, an eigenvector of level zero ($h_0 \neq 0$) is called a **primary eigenvector**.
Additionally, we call the set that contains one eigenvector at each level $k \geq 0$, $\{\vec{h_0}, \vec{h_1}, \dots, \vec{h_k}, \dots\}$ as a **full set of eigenvectors**. We note that this is not a basis of the vector space of all infinite sequences in $\mathbb{F}$.

7.1.1 Eigenvectors - Eigenvalues of Riordan Matrix Powers

It is known from Linear Algebra that if λ is an eigenvalue of a matrix A, then λ^N is the eigenvalue of the matrix A^N. Now, suppose that we have a Riordan matrix $A = \big(g(z), f(z)\big)$, where $\vec{h} \in \mathbb{F}_k$ is an eigenvector of A, with eigenvalue λ, then for every $N \in \mathbb{N}$, we have

$$\begin{aligned} & A^N \vec{h} &&= \lambda^N \vec{h} \\ \iff & A^N \vec{h} &&= (g_0 f_1^k)^N \vec{h} \text{ (by Proposition 7.1.1)} \\ \iff & A^N \vec{h} &&= g_0^N f_1^{kN} \vec{h}\end{aligned}$$

We also have that

$$\begin{aligned} A^2 &= A \cdot A \\ &= \Big(g(z) g\big(f(z)\big), f\big(f(z)\big)\Big),\end{aligned}$$

and

$$\begin{aligned} A^3 &= A^2 \cdot A \\ &= \Big(g(z)g\big(f(z)\big)g\Big(f\big(f(z)\big)\Big), f\Big(f\big(f(z)\big)\Big)\Big), \end{aligned}$$

and by induction, we get

$$A^N = \left(\prod_{i=0}^{N-1} g\big(f^{\circ(i)}(z)\big), f^{\circ(N)}(z)\right),$$

where

$$f^{\circ(N)}(z) = \underbrace{\big(f(f(f(\cdots f(z))))}_{N-\text{ times}}.$$

We note that $f^{\circ(0)}(z)$, $f^{\circ(1)}(z)$ are z, and $f(z)$, respectively. Hence,

$$A^N h(z) = \prod_{i=0}^{N-1} g\big(f^{\circ(i)}(z)\big) h\big(f^{\circ(N)}(z)\big),$$

which means that

$$\prod_{i=0}^{N-1} g\big(f^{\circ(i)}(z)\big) h\big(f^{\circ(N)}(z)\big) = g_0^N f_1^{kN} h(z)$$

which is also written as

$$\frac{1}{g_0^N f_1^{kN}} \prod_{i=0}^{N-1} g\big(f^{\circ(i)}(z)\big) = \frac{h(z)}{h\big(f^{\circ(N)}(z)\big)} \tag{7.4}$$

For $N = 1$, the eq 7.4 provides us a different interpretation of the first generating function of Stabilizer elements $\left(\frac{h(z)}{h\big(f(z)\big)}, f(z)\right)$.

7.2 Existence and non-Existence of Eigenvectors

In this section we present the conditions under which a eigenvector of a Riordan array exists. Firstly, by the Babbage equation 3.2 and for a fps $f(z)$, we denote k as the **compositional order** of $f(z)$. Now, we present a classic theorem as a lemma about fps of finite and infinite compositional order.

Lemma 7.2.1. *[25, 80] Let $f(z) = f_1 z + f_2 z^2 + \cdots$ be a fps , with $f_1 \neq 0$.*

(a) If $f(z)$ has finite compositional order n then f_1 has multiplicative order n in $\mathbb{F}\backslash\{0\}$ and (highly nontrivial) there exists a formal series $\theta(z) \in \mathbb{F}_1$ where

$$\big(\theta \circ f \circ \overline{\theta}\big)(z) = f_1 z. \tag{7.5}$$

(b) If f_1 has infinite multiplicative order then $f(z)$ is of infinite compositional order and, (again nontrivial), there exists a formal series $\phi(z) \in \mathbb{F}_1$ where

$$\big(\phi \circ f \circ \overline{\phi}\big)(z) = f_1 z. \tag{7.6}$$

In eqs 7.5 and 7.6, we set $\ell(z) = f_1 z$, and we also consider $f(z)$ to be conjugate (A.1.13) to its linear part $\ell(z)$: $f(z) \sim \ell(z)$.

Corollary 7.2.2. *If $f(z)$ has infinite compositional order, with f_1 of finite multiplicative order, then $f(z)$ cannot be conjugate to $\ell(z)$.*

Proof. Since conjugacy is an equivalence relation and the compositional order of $\ell(z)$ equals the multiplicative order of f_1, an $f(z)$ of infinite compositional order cannot be conjugate to $\ell(z)$ of finite compositional order. □

7.2.1 The existence of a full set of eigenvectors for a Riordan matrix

The following theorem answers the question about the existence of a full set of eigenvectors of a Riordan matrix.

Theorem 7.2.3. *Let $A = \big(g(z), f(z)\big)$ be a Riordan matrix that has a primary eigenvector $\vec{h}$ with gf $h(z) \in \mathbb{F}_0$, and if there exists $\theta(z) \in \mathbb{F}_1$ such that*

$$(\theta \circ f \circ \overline{\theta})(z) = f_1 \cdot z = \ell(z),$$

then the set of columns

$$\{h(z)\theta(z)^k\}_{k=0}^{\infty} \text{ of } \big(h(z), \theta(z)\big) \in \mathcal{R}$$

form a full set of eigenvectors for $\big(g(z), f(z)\big)$.

Proof. By Lemma 7.1.1 and the Fundamental Theorem of Riordan arrays, the matrix $A = \big(g(z), f(z)\big)$ multiplies the k^{th} column $\vec{h}_k$ of the matrix $\big(h(z), \theta(z)\big)$, which has generating function $h(z)\big(\theta(z)\big)^k$, then we have that

$$\begin{aligned}
\big(g(z), f(z)\big) \cdot \Big(h(z)\big(\theta(z)\big)^k\Big) &= \underbrace{g(z)\Big(h\big(f(z)\big)}_{g_0 h(z)}\big(\theta(f(z))\big)^k\Big) \\
&= g_0 h(z)\Big(\ell\big(\theta(z)\big)\Big)^k \\
&= g_0 h(z)\Big(f_1\theta(z)\Big)^k = g_0 f_1^k \cdot h(z)\big(\theta(z)\big)^k
\end{aligned}$$

Thus $\vec{h_k}$ is an eigenvector of $\big(g(z), f(z)\big)$ with eigenvalue $g_0 f_1^k$. □

Now, if f_1 is of infinite order then there exists unique eigenvectors at each level, according to the next theorem.

Theorem 7.2.4. *Suppose that $f_1^n \neq 1$ for all $n \in \mathbb{N}$, then for every $k \geq 0$, the Riordan matrix $A = \big(g(z), f(z)\big)$ has a unique eigenvector of the form*

$$\vec{h} = \big(\underbrace{0, 0, \ldots 0}_{(k-1)\ terms}, h_k = 1,\ h_{k+1}, h_{k+2}, \ldots \big)^T.$$

Proof. Let $A \cdot \vec{h} = \lambda \cdot \vec{h}$, so

$$\begin{bmatrix} g_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ g_1 & g_0 f_1 & 0 & 0 & 0 & 0 & \cdots \\ g_2 & a_{2,1} & g_0 f_1^2 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \cdots \\ g_k & a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & g_0 f_1^k & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} h_0 \\ \vdots \\ h_{k-1} \\ h_k \\ h_{k+1} \\ h_{k+2} \\ \vdots \end{bmatrix} = \lambda \cdot \begin{bmatrix} h_0 \\ \vdots \\ h_{k-1} \\ h_k \\ h_{k+1} \\ h_{k+2} \\ \vdots \end{bmatrix}$$

If $\vec{h}$ is of level k then necessarily $h_0 = \cdots = h_{k-1} = 0$, the eigenvalue is $\lambda = g_0 f_1^k$ by Proposition 7.1.1, and we may set $h_k = 1$. If $h_1, \ldots, h_{n-1}$ have been determined for $n > k$ then, since $f_1^n \neq 1$ for all $n \in \mathbb{N}$, the product of the n^{th} row and the vector $\vec{h}$ gives

$$g_n h_0 + a_{n,1} h_1 + \cdots + a_{n,n-1} h_{n-1} + g_0 f_1^n h_n = g_0 f_1^k h_n,$$

and

$$g_n h_0 + a_{n,1} h_1 + \cdots + a_{n,n-1} h_{n-1} + g_0 f_1^k (f_1^{n-k} - 1) h_n = 0$$

and we may solve uniquely for h_n, proving the theorem. □

Corollary 7.2.5. 1. *If f_1 has infinite order then there exists an eigenvector at each level $k \geq 0$ for the matrix $\big(g(z), f(z)\big)$.*

2. *If $\big(g(z), f(z)\big)$ has finite order in $\mathcal{R}$ then there exists an eigenvector at each level $k \geq 0$ for the matrix $\big(g(z), f(z)\big)$.*

Proof. 1. It follows from the existence of a primary eigenvector as proved in Theorem 7.2.4 and from the existence of $\theta(z)$ given by Lemma 7.2.1.

2. It is proven in [26], where a general formula for all eigenvectors is also given. It is proven that the fact that $\big(g(z), f(z)\big)$ has finite order implies

the existence of a primary eigenvector and also that $f(z)$ has finite compositional order (hence that $\theta(z)$, which conjugates $f(z)$ to $f_1 z$, exists).

□

The following proposition shows that the existence of a full set of eigenvectors will follow from the existence of just a primary eigenvector and one non-primary eigenvector. To prove this, we are going to use the following theorem.

Theorem 7.2.6. *[65] Let $A(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ and $n \in \mathbb{N}$, then there exists a unique series of the form $B(z) = 1 + b_1 z + b_2 z^2 + \cdots$ such that*

$$\big(B(z)\big)^n = A(z).$$

We denote $B(z) = \big(A(z)\big)^{\frac{1}{n}}$.

Proposition 7.2.1. *Suppose that $\big(g(z), f(z)\big) \in \mathcal{R}$ has a primary eigenvector $\vec{h}$ given by $h(z)$ and another eigenvector, given by $v(z) = v_k z^k + \cdots$, with $v_k \neq 0$ and $k > 0$. Then there exists a formal power series $\theta(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, such that $\big(\theta \circ f \circ \overline{\theta}\big)(z) = f_1 z$ (and thus, by Theorem 7.2.3, there exists a full set of eigenvectors given by $h(z)\theta(z)^k$).*

Proof. As scalar multiples of eigenvectors are also eigenvectors, we may assume that $h_0 = 1$ and $v_k = 1$. Since $h(z)$ and $v(z)$ are eigenvectors of $\big(g(z), f(z)\big)$, we have

$$\begin{aligned} \big(g(z), f(z)\big) \cdot h(z) &= g(z) \cdot h\big(f(z)\big) = g_0 h(z). \\ \big(g(z), f(z)\big) \cdot v(z) &= g(z) \cdot v\big(f(z)\big) = g_0 f_1^k v(z). \end{aligned}$$

Since $h_0 \neq 0$, $h(z)$ and $h\big(f(z)\big)$ have multiplicative inverses. Thus the above equations give

$$\begin{aligned} g(z) &= g_0 \frac{h(z)}{h\big(f(z)\big)} \\ \Rightarrow g_0 \frac{h(z)}{h\big(f(z)\big)} v\big(f(z)\big) &= g_0 f_1^k v(z) \\ \Rightarrow \frac{v\big(f(z)\big)}{h\big(f(z)\big)} &= f_1^k \frac{v(z)}{h(z)}. \end{aligned}$$

We define $a(z) = \frac{v(z)}{h(z)}$, so that we have

$$a\big(f(z)\big) = f_1^k a(z).$$

We may write $\frac{1}{h(z)} = \frac{1}{1+H(z)} = \big(1 - H(z) + H(z)^2 - \cdots\big)$. Thus

$$\begin{aligned} a(z) = \frac{v(z)}{h(z)} &= \Big(z^k + v_{k+1}z^{k+1} + \cdots\Big)\Big(1 - H(z) + H(z)^2 - \cdots\Big) \\ &= z^k\Big(1 + v_{k+1}z + \cdots\Big)\Big(1 - H(z) + H(z)^2 - \cdots\Big) \\ &= z^k\Big(1 + b_1z + b_2z^2 + \cdots\Big) \\ &= \Big(z\big(1 + b_1z + b_2z^2 + \cdots\big)^{\frac{1}{k}}\Big)^k \end{aligned}$$

By Theorem 7.2.6, there exists a unique k^{th} root of $(1 + b_1z + b_2z^2 + \cdots)$ which has the form $(1 + c_1z + \cdots)$.
Hence, we have that $\Big(z\big(1 + b_1z + b_2z^2 + \cdots\big)^{\frac{1}{k}}\Big)^k = \big(\theta(z)\big)^k$, where $\theta(z)$ is of the form $\theta(z) = z + a_2z^2 + a_3z^3 + \cdots$.
Therefore

$$\begin{aligned} a\big(f(z)\big) &= f_1^k a(z) \\ \Rightarrow \Big(\theta\big(f(z)\big)\Big)^k &= f_1^k\Big(\theta(z)\Big)^k \\ \Rightarrow \theta\big(f(z)\big) &= \alpha\theta(z), \end{aligned}$$

where $\alpha^k = f_1^k$.
But, $\theta(z) = z + a_2z^2 + a_3z^3 + \cdots$ implies that

$$\begin{aligned} (\alpha z + \alpha a_2z^2 + \cdots) &= \alpha\theta(z) \\ &= \theta\big(f(z)\big) \\ &= f(z) + a_2f(z)^2 + \cdots \\ &= f_1z + \cdots \end{aligned}$$

Therefore $\alpha = f_1$. Letting $\ell(z) = f_1z$ we have

$$\begin{aligned} \theta\big(f(z)\big) &= f_1 \cdot \theta(z) \\ \Rightarrow \big(\theta \circ f\big)(z) &= \big(\ell \circ \theta\big)(z) \\ \Rightarrow \big(\theta \circ f \circ \bar{\theta}\big)(z) &= \ell(z) = f_1 \cdot z. \end{aligned}$$

□

7.2.2 Riordan arrays with primary but no higher level eigenvalues

In the next theorem, we present a case where a Riordan array does not have eigenvectors of level $k > 0$.

Theorem 7.2.7. *Let the fps $f(z), h(z) \in \mathbb{F}_1$, where f_1 has finite order, $f(z)$ has infinite compositional order, and $h_0 = 1$. Suppose that $g(z) \in \mathbb{F}_0$ such that*

$$g(z) = g_0 \cdot \frac{h(z)}{h\big(f(z)\big)}. \tag{7.7}$$

Then $\big(g(z), f(z)\big)$ has a primary eigenvector given by $h(z)$ but no eigenvectors of level greater than zero.

Proof. By definition, we have that

$$\big(g(z), f(z)\big) \cdot h(z) = g(z) \cdot h\big(f(z)\big) = g_0 \cdot h(z),$$

so $h(z)$ gives a primary eigenvector of $\big(g(z), f(z)\big)$. On the other hand, if there existed a higher level eigenvector $v(z)$ then by Proposition 7.2.1, $f(z)$ would be conjugate to the linear function $\ell(z) = f_1 z$. Thus the compositional order of $f(z)$ would equal the compositional order of $\ell(z)$, which has compositional order equal to the finite multiplicative order of f_1. This contradicts our assumption on $f(z)$. □

7.2.3 Riordan arrays with no primary eigenvectors

Eq 7.1 can be stated equivalently as

$$(A_n - \lambda I_n) \cdot \vec{h} = 0, \tag{7.8}$$

where I_n is the $n \times n$ identity matrix. Hence, we have that

$$A_n - \lambda I_n = \begin{bmatrix} g_0 - \lambda & 0 & 0 & 0 & 0 & 0 & \cdots \\ g_1 & g_0 f_1 - \lambda & 0 & 0 & 0 & 0 & \cdots \\ g_2 & a_{2,1} & g_0 f_1^2 - \lambda & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \cdots \\ g_k & a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & g_0 f_1^k - \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

So, eq (7.8) becomes

$$\begin{bmatrix} g_0-\lambda & 0 & 0 & 0 & 0 & 0 & \cdots \\ g_1 & g_0f_1-\lambda & 0 & 0 & 0 & 0 & \cdots \\ g_2 & a_{2,1} & g_0f_1^2-\lambda & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \cdots \\ g_k & a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & g_0f_1^k-\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} h_0 \\ \vdots \\ h_{k-1} \\ h_k \\ h_{k+1} \\ \vdots \\ \vdots \end{bmatrix} = 0 \tag{7.9}$$

Theorem 7.2.8. *Let $A = \big(g(z), f(z)\big)$ be a proper Riordan matrix given by*

$$g(z) = g_0 + g_r z^r + \cdots, \text{ and } f(z) = f_1 z + f_s z^s + \cdots$$

where $g_r \neq 0$ and $f_s \neq 0$. If one of the following holds then $\big(g(z), f(z)\big)$ does not have an eigenvector of the form $(h_0, h_1, h_2, \ldots)^T$, $h_0 \neq 0$ corresponding to the eigenvalue g_0.

(i) $f_1 = 1$ *and* $1 \leq r < s$.

(ii) $f_1 = -1$, $r = 1$, $g_2 \neq 0$ *and* $f_2 = g_1/g_0$.

(iii) $f_1 = -1$, $r = 2$, $r < s$.

(iv) $f_1 = -1$, $r \geq 3$ *is odd*, $r < s$ *and* $g_{r+1} \neq 0$.

(v) $f_1 = -1$, $r \geq 4$ *is even*, $r \leq s$.

Proof. • (Case 1: $f_1 = 1$ and $1 \leq r < s$) Let $f(z) = z + f_2 z^2 + f_3 z^3 + \cdots$. Since $f_1 = 1$, it is obvious that if $g_1 \neq 0$ then eq 7.9 has no solution for any $s > 1$. For $r \geq 2$, let $R_i^T = (a_{i0}, a_{i1}, \ldots)$ denote the ith row vector of $A - g_0 I$ in eq 7.9 where $i = 0, 1, \ldots$. Assume that $f_2 = \cdots = f_{r-1} = 0$. Since

$$a_{rk} = [z^r]g(z)f(z)^k = [z^r](g_0 + g_r z^r + \cdots)(z + f_r z^r + \cdots)^k, k \geq 0$$

we obtain $R_r^T = (g_r, g_0 f_r, 0, \ldots)$. From $R_r^T h = 0$ we have

$$g_r h_0 + g_0 f_r h_1 = 0.$$

Since $g_0 \neq 0$ and $g_r h_0 \neq 0$ it follows that if $f_r = 0$ then the linear system 7.9 has no solution. In conclusion, if $g(z) = g_0 + g_r z^r + \cdots$ with $g_r \neq 0$ and $f(z) = z + f_2 z + \cdots$ with $f_2 = \cdots = f_r = 0$, i.e., $1 \leq r < s$ then $\big(g(z), f(z)\big)$ has no eigenvector.

- (Case 2: $f_1 = -1, r = 1$) Let $f(z) = -z + f_2z^2 + f_3z^3 + \cdots$. Let $r = 1$, i.e. $g_1 \neq 0$. Since

$$a_{2k} = [z^2](g_0 + g_1z + g_2z^2 + \cdots)(-z + f_2z^2 + \cdots)^k, k \geq 0$$

we obtain $R_2^T = (g_2, g_0f_2 - g_1, 0, \ldots)$ from eq 7.9. It follows from $R_2^T h = 0$ that

$$g_2h_0 + (g_0f_2 - g_1)h_1 = 0.$$

Thus if $g_2 \neq 0$ and $f_2 = g_1/g_0$ then eq 7.9 has no solution.

- (Case 3: $f_1 = -1, r = 2$)If $r = 2$, i.e. $g_1 = 0$ and $g_2 \neq 0$, the above equation gives us that (7.9) has no solution for the case $f_2 = 0$.

- (Case 4: $f_1 = -1, r = 3$) Now let $r \geq 3$ be odd. Since $g_0 \neq 0, g_1 = 0$ and $f_1 = -1$ we obtain $R_1^T = (0, -2g_0, 0, \ldots)$. Hence we have $h_1 = 0$ from $R_1^T h = 0$. Assume that $f_2 = \cdots = f_r = 0$. Since

$$[z^{r+1}]g(z)f(z)^k = [z^{r+1}](g_0 + g_rz^r + g_{r+1}z^{r+1} + \cdots)(-z + f_{r+1}z^{r+1} + \cdots)^k,$$

where $k \geq 0$ and the $(r+1)^{th}$ diagonal entry of $A - g_0I$ is zero, we obtain

$$R_{r+1}^T = (g_{r+1}, -g_r + g_0f_{r+1}, 0, \ldots).$$

From $R_{r+1}^T h = 0$ with $h_1 = 0$ we obtain $h_0g_{r+1} = 0$ where $h_0 \neq 0$. Hence if $g_{r+1} \neq 0$ then eq 7.9 has no solution. In conclusion, if $g(z) = g_0 + g_rz^r + g_{r+1}z^{r+1} + \cdots$ for an odd $r \geq 3$ where g_0, g_r, g_{r+1} are nonzero, and $f_2 = \cdots = f_r = 0$ i.e. $r < s$ then $\big(g(z), f(z)\big)$ has no eigenvector.

- (Case 5: $f_1 = -1$, and $r \geq 4$ is even) Finally, let $r \geq 4$ be even. In a similar way, we obtain $h_1 = 0$. Assume that $f_2 = \cdots = f_{r-1} = 0$. Since

$$[z^r]g(z)F(z)^k = [z^r](g_0 + g_rz^r + \cdots)(-z + f_rz^r + \cdots)^k,$$

where $k \geq 0$ and the rth diagonal entry of $A - g_0I$ is zero, we obtain $R_r^T = (g_r, g_0f_r, 0, \ldots)$. From $R_r^T h = 0$ with $h_1 = 0$ we obtain $h_0g_r = 0$ where $h_0 \neq 0$. Since $g_r \neq 0$ the linear system 7.9 has no solution. In conclusion, if $g(z) = g_0 + g_rz^r + \cdots$ for an even $r \geq 4$ where g_0, g_r are nonzero, and $f_2 = \cdots = f_{r-1} = 0$ i.e. $r \leq s$ then $\big(g(z), f(z)\big)$ has no eigenvector.

□

Chapter 8

Conclusions and further directions

Riordan subgroups and their properties have been the guide of our study. In Chapter 4, working on the structure of the Riordan group, we have defined a class of Riordan subgroups which hold similar properties, we determined their relationships with other subgroups, while we also generalised a part of our results to the family of Riordan subgroups $Y[r, s, p]$. Since, most of our work is related, but not limited to the algebraic structures of the Ordinary Proper Riordan arrays a possible direction will be to generalise our findings into other kinds of Riordan arrays, explore the behaviour of Riordan sets and subgroups and eventually expand the theoretical part of Riordan Group theory. Additionally, an area that might be worth studying is the combinations of different types of Riordan arrays that can lead us to new groups, e.g. almost-exponential Riordan arrays or k-tuple $a\mathcal{R}(n)$, as a combination of k^{th} level Riordan, and n^{th} level almost-Riordan arrays.

In Chapter 5, we have studied quasi-involutions, a special type of Riordan arrays, partially analysing their structure by defining quasi-compressions and linking quasi-involutions of different levels. In addition, we expressed Riordan arrays generated by Bessel polynomials, by compressions of exponential quasi-involutions. Since, we have presented two general generating functions that give Riordan quasi-involutions, the study of different functions that are able to generate similar matrices and the characterization of quasi-involutions could be a part of our future work in this field. Our endeavours to analyse a lower triangular matrix, with all ones diagonal that it is self-inverse as a quasi-involution, lead us to the following figure.

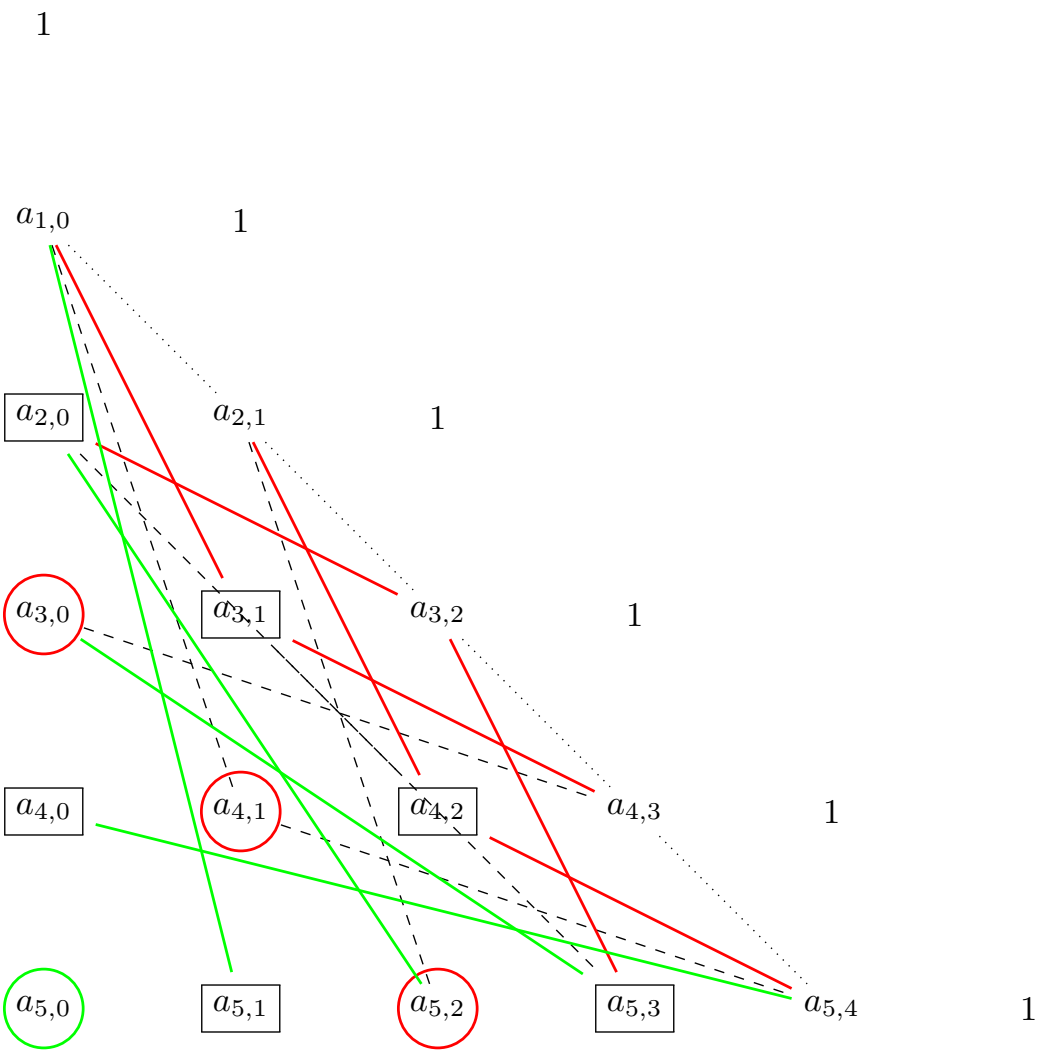


FIGURE 8.1: Structure of a self-inverse matrix

In the Fig 8.1, starting from the second subdiagonal, we represent the entries according to the subdiagonal to which they belong. Hence, we have entries of an even subdiagonal in a box, and entries of an odd subdiagonal in a colourful circle. Making pairs of entries above and on the right of any box or circle, according to the edges of the lines of the digram (dashed and dotted lines for the even subdiagonals, and green and red for the odd ones), we present the following non-linear recursive formulas for the entries of the matrix $(a_{n,k})_{n,k\in\mathbb{N}}$.

- if $n-k=2\lambda, \lambda \in \mathbb{N}^*$, then

$$a_{n,k} = \frac{1}{2} \sum_{i=k+1}^{n} (-1)^{n+i} a_{n,i} a_{i,k}$$

- if $n-k = 2\lambda+1, \lambda \in \mathbb{N}^*$, then for the entries of the same row and column of $a_{n,k}$

$$\sum_{i=k+1}^{n} (-1)^{n+i} a_{n,i} a_{i,k} = 0$$

Our work on almost-Riordan arrays in Chapter 6 has extended the already existing theory to almost-Riordan arrays with n extra columns. We have shown that every Riordan array can be factorised to trivial-like almost-Riordan arrays of different levels, while our study on involutions, pseudo-involutions, and quasi-involutions of almost-Riordan groups has led us to further links to the main Riordan group. The algebraic perspectives of almost-Riordan arrays is also a possible direction of our future research. Using the results of our findings one could go further into the theory of almost-Riordan arrays into the combinatorial behaviour of these mathematical objects.

By a simple observation on the structure of the Stabilizer subgroup, we have presented a study in the eigenvalues and eigenvectors of a Riordan matrix, in Chapter 7. There is more related work in progress which will be presented in due time.

Open problems and conjectures

During our study in the area of Riordan arrays, we found some intriguing questions. Some of them are still under investigation, while others may not have a clear answer. We list them in this subsection as open problems.

- Not all Riordan subgroups are stabilizers. (T-X. He et al, 2017).
  Is there any stabilizer transformation for the derivative subgroup?

  Equivalently, does the equation $\int h(z)dz = \int h(f(z))df$ have a non-trivial solution for $f \in \mathbb{I}\mathrm{H}_1$?

- Is there any Riordan subgroup without any non-trivial involutions and/or pseudo-involutions?

- Any almost-Riordan subgroups characterized by their extra column(s)?

- Could we factorize a Riordan array by using double (or k-tuple) Riordan arrays?

- The Riordan subgroup of the Generalised Pascal's Triangle,
  $P_c = \left(\frac{1}{1-cz}, \frac{z}{1-cz}\right)$ is cyclic. Is there any other significant cyclic subgroup in the Riordan Group?

- Could we construct a semi-direct product for the Riordan group, without using the Appell subgroup? Equivalently, is there any other normal Riordan subgroup, except for the Appell which can be used for that purpose?

- In examples 5.2.2 and 5.2.3, we observe that although both of the matrices are quasi-involutions, the entries of each matrix are not necessarily integers. Even if we limit the range of the parameter c to the set of integers, we still have $a_{n,k} \in \mathbb{Q}$.

  Conjecture 8.0.1. *The Riordan quasi-involution of level $N-1$,*

  $$G = \left(\frac{1}{(1-cz^N)^{\frac{1}{N}}}, \frac{z}{(1-cz^N)^{\frac{1}{N}}} \right)$$

  contains integer entries if and only if $c = N^2\lambda$, for $\lambda \in \mathbb{N}^*$

- By the almost-Riordan arrays, we have the following

  Conjecture 8.0.2. *Almost-Riordan involutions where the extra column is not trivial, are Riordan arrays.*

Appendix A

Group Theory

A.1 Group - Ring Theory

As our research is inseparably connected with the algebraic structure of the Riordan group, we introduce some basic definitions of Group theory, followed by some advanced theories that we will use in the sections that follow. Moreover, as the main objects of our research are matrices, we will focus on the applied group theory on infinite matrices and we will present examples of matrix groups, whenever we need to emphasise a specific part of the theory.

A.1.1 Groups

Definition A.1.1. *[38] A **group** $\langle G, * \rangle$ is a set G closed under a binary operation $*$, such that the following axioms are satisfied*
A_1 : for all $a, b, c \in G$, we have

$$(a * b) * c = a * (b * c), \text{ associativity of } *$$

A_2 : there is an element e in G, which is usually denoted by 1, such that for all $z \in G$,

$$e * z = z * e = z, \text{ identity element } e \text{ for } *$$

A_3 : corresponding to each $a \in G$, there is an element a' in G, which is usually denoted by a^{-1}, such that

$$a * a' = a' * a = e, \text{ inverse } a' \text{ of } a.$$

We usually symbolise a group, with the name of its set, G.

Example A.1.1. *The set of integers together with addition, $\langle \mathbb{Z}, + \rangle$ form a group. However, the set of all positive integers, $\mathbb{Z}^+$ is not a group, as there is no identity element for $+$ in $\mathbb{Z}^+$.*

Example A.1.2. *The general linear group of matrices of degree n over $\mathbb{R}$,*

$$GL_n(\mathbb{R}) = \{A \in \mathbb{M}_{n \times n}(\mathbb{R}) | detA \neq 0\}$$

is the set of $n \times n$ invertible matrices, together with the operation of ordinary matrix multiplication is a group.

A measure that describes the size of a group comes from the definition below.

Definition A.1.2. *[38] If G is a group, with finitely many elements, then the **order**, $|G|$, of G is the number of elements in G.*

Definition A.1.3. *[14] If g is an element of a group G, then the least positive integer m that $g^m = 1$, where $g^m = g * g * g ... * g$ (m-times), is called the **order of the element** g in G.*

Example A.1.3. *The order of the group $U_4 = \{1, -1, i, -i\}$ of fourth roots of unity under multiplication, is $|U_4| = 4$. While, the order of the element -1, is 2 as $(-1)^2 = 1$.*

Definition A.1.4. *[38] A group G is called **abelian**, if its binary operation $*$ is commutative. i.e.*

$$\forall a, b \in G : a * b = b * a.$$

Example A.1.4. *The set $\mathbb{M}_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices under matrix addition is an abelian group.*

Definition A.1.5. *[38] If a subset H of a group G is closed under the binary operation of G and if H with the induced operation from G is itself a group, then H is a **subgroup** of G, denoted as*

$$H \leq G \text{ or } H < G, \text{ if } H \neq G.$$

Example A.1.5. *Invertible matrices of $GL_n(\mathbb{R})$ have to satisfy the condition $\det A \neq 0$ for every $A \in \mathbb{M}_{n\times n}(\mathbb{R})$. Let Q be a subset of GL_n consisting of those matrices with $\det A = 1$. Then Q is a group and we have that $Q \leqslant GL_n$.*

We use the following lemma to show when a subset of a group is a subgroup.

Lemma A.1.6. *[38] Let H be a non-empty set of the group $\langle G, * \rangle$. Then H is a subgroup of G if and only if for every $a, b \in H$, we have that $a * b \in H$ and $a^{-1} \in H$.*

Definition A.1.6. *[38] If G is a group and $a \in G$, then*

$$\langle a \rangle = \{a^n | n \in \mathbb{Z}\}$$

*is a subgroup of G and it is called the **cyclic** subgroup of G which is generated by a. Also, given a group G and an element g in G, if*

$$\langle g \rangle = \{g^n | n \in \mathbb{Z}\}$$

*then g is called **generator** of G and the group $G = \langle g \rangle$ is cyclic.*

If the cyclic subgroup $\langle a \rangle$ of G is finite, then the order of the element a is the order $|\langle a \rangle|$ of this cyclic subgroup.

Example A.1.7. *Let $\mathbb{M}_{2\times 2}(n)$ be the group of 2×2 matrices such that*

$$\mathbb{M}_{2\times 2}(n) = \left\{ M(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \middle| \, n \in \mathbb{N}^* \right\}.$$

Then, $\mathbb{M}_{2\times 2}(n)$ is cyclic and the matrix $M(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is a generator of $\mathbb{M}_{2\times 2}(n)$, as

$$M^2(1) = M(1)M(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = M(2),$$

$$M^3(1) = M(1)M(1)M(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = M(3)$$

and in general

$$M^n(1) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = M(n).$$

Definition A.1.7. *[38] Let X be a set, G a group and e the identity element of G. An **action of** G **on** X is a map*

$$* : G \times X \to X$$

such that

1. *$ex = x$, for all $x \in X$,*
2. *$(g_1 g_2)(x) = g_1(g_2 x)$, for all $x \in X$ and all $g_1, g_2 \in G$.*

Under these conditions X is called a G-set.

By action of a group G on a set or on an element, we have the following definitions of subgroups of G. A particular form of a group is given by a semi-direct product. Nevertheless, before we present the definition of it, we introduce the concepts of a centralizer and the set product of two subgroups.

Definition A.1.8. *[38] Let G be a group and α be a fixed element of G, then the **centralizer of an element** α is denoted as the set of elements of G which commute with α,*

$$C_G(\alpha) = \{g \in G | \alpha g = g\alpha\}$$

and it is a subgroup of G.

Example A.1.8. *Suppose that we define a set of 2×2 matrices*

$$\mathbb{T}_{2\times 2} \;=\; \left\{ T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, det\ T \neq 0 \text{ and } a, b, c, d \in \mathbb{R} \right\},$$

and the element $N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{T}_{2\times 2}$. *The centralizer of the element* N *is the set*

$$C_{\mathbb{T}_{2\times 2}}(N) = \left\{ T \in \mathbb{T}_{2\times 2} \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\},$$

which gives us the relations $a = d$ *and* $b = c$. *Hence,*

$$C_{\mathbb{T}_{2\times 2}}(N) = \left\{ T = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a^2 - b^2 \neq 0 \text{ and } a, b \in \mathbb{R} \right\}.$$

Similarly, we define a centralizer of a subset (or a subgroup).

Definition A.1.9. *[47] Let* S *be a subset of the group* G *and let* H *be a subgroup of* G. *Then the* ***centralizer of a subset*** S *in* H, *denoted by* $C_H(S)$ *is*

$$C_H(S) = \{h \in H | hs = sh, \forall s \in S\},$$

and it is a subgroup of G.

Definition A.1.10. *[32] Let* Ω *be a set and* G *a group, then for* $\omega \in \Omega$ *and* $g \in G$, *the subset*

$$G_\Omega = \{g \in G | \omega g = \omega\},$$

where the operation is a group action, is called the ***stabilizer*** *of* ω *in* G *and it is a subgroup of* G.

Example A.1.9. *Let*

$$\mathbb{J}_{2\times 2} = \left\{ J = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| det J \neq 0 \text{ and } a, b, c, d \in \mathbb{R} \right\},$$

be a set of 2×2 *matrices and* $L = \{\begin{bmatrix} 2 & 1 \end{bmatrix}\}$, *a set with one* 1×2 *matrix. The stabilizer of* L *in* $\mathbb{J}_{2\times 2}$ *is the set*

$$\mathbb{J}_{2\times 2_L} = \left\{ J \in \mathbb{J}_{2\times 2} \middle| \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix} \right\},$$

which gives us the relations $d = 1 - 2b$ *and* $c = 2(1 - a)$. *Hence,*

$$\mathbb{J}_{2\times 2_L} = \left\{ J = \begin{bmatrix} a & b \\ 2(1-a) & 1-2b \end{bmatrix} \middle| det J \neq 0 \text{ and } a, b \in \mathbb{R} \right\},$$

and finally,

$$\mathbb{J}_{2\times 2_L} = \left\{ J = \begin{bmatrix} a & b \\ 2(1-a) & 1-2b \end{bmatrix} \middle| a \neq 2b, \text{ where } a, b \in \mathbb{R} \right\}.$$

Another proposition that will be used quite often, is the following.

Proposition A.1.1. *[38] Let H and K be subgroups of G, then*

$$H \cap K = \{z | z \in H \text{ and } z \in K\}$$

is a subgroup of G.

Definition A.1.11. *[47] Let G be a group and A, B be subgroups of G. The **set product** AB of A and B is defined as*

$$AB = \{ab | a \in A, b \in B\}.$$

However, because a set product is not always a subgroup of G, we have the following lemma.

Lemma A.1.10. *[47] Let G be a group and A, B be subgroups of G. The set product AB is a subgroup of G if and only if $AB = BA$.*

If the group G can be expressed as $G = AB$, where A and B are subgroups of G, then we say that G is the product of the subgroups A and B, while we also refer to G as a **factorised group**.
Before, we proceed further to isomorphisms between groups, we need to define the concept of a coset.

Definition A.1.12. *[13] Let G be a group and H a subgroup of G. Then a **right coset** of H in G is a subset of the form*

$$Hg = \{hg | h \in H\}$$

*for some g in G. We define a **left coset** of H in G to be a subset of the form*

$$gH = \{gh | h \in H\}.$$

These two different kinds of cosets, are equal if and only if the following proposition is satisfied. Then, we simply refer to them as cosets.

Proposition A.1.2. *[13] $\alpha H = H\alpha$ for all $\alpha \in G$ if and only if $\alpha^{-1} h \alpha \in H$ for all $h \in H$ and all $\alpha \in G$.*

Definition A.1.13. *[101] Let a and x be two elements of a group G. For an element b which is a similarity transformation of a, $b = x^{-1}ax$, we say that a and b are **conjugate** with respect to x.*

Since the conjugacy is an equivalence relation [6], for conjugate elements a, b, we write $a \sim b$, and they have the following properties:

1. Every element is conjugate with itself.
2. If a is conjugate with b with respect to x, then b is conjugate to a with respect to x.
3. If a is conjugate with b and c, then b and c are conjugate with each other.

Example A.1.11. *[6] Two complex conjugates $z = a + ib$ and $\bar{z} = a - ib$, where $a, b \in \mathbb{R}$ and, $b \neq 0$ are also conjugate according to Definition A.1.13.*

Definition A.1.14. *[13] A subgroup H of a group G is **normal** in G if $g^{-1}hg \in H$, for all $g \in G$ and all $h \in H$. We write $H \triangleleft G$.*

We have already presented examples of a normal subgroup. More specifically, in Definition A.1.9, if S is an abelian subgroup, then S is normal in $C(S)$ [13].

Proposition A.1.3. *[13] Every subgroup of an abelian group is normal.*

Definition A.1.15. *[13] Let G be a group and $N \triangleleft G$, then the set of right cosets of N in G, is called the **quotient group** (factor group) of G by N and it is denoted by G/N.*

Now, we are able to present the formal definition of a semi-direct product.

Definition A.1.16. *[47] The group G is the **semi-direct product**, or split extension, of the subgroup N by the subgroup K if the following criteria are satisfied*

1. $G = NK$.
2. $N \trianglelefteq G$ (N *is a normal subgroup of* G).
3. $N \cap K = e$.

We note that criterion 3 implies that the factorisation $g = nk$, where $n \in N$ and $k \in K$ is unique. Nevertheless, the semi-direct products are not, in general, uniquely defined up to isomorphism [1].

A.1.2 Group Homomorphisms

We define mappings between the elements of two sets. A group is also a set, as it consists of elements. Hence, we are able to define a mapping between two groups.

[1] for the definition of the isomorphism, see next subsection.

Definition A.1.17. *[38] A map ϕ of a group $\langle G, * \rangle$ into a group $\langle G', \times \rangle$ is a **homomorphism***

$$\phi : G \to G'$$

if the homomorphism property

$$\phi(a * b) = \phi(a) \times \phi(b)$$

holds for all $a, b \in G$.

Definition A.1.18. *[38] Let*

$$\phi : G \to G'$$

be a homomorphism of groups, where e is the identity of G and e' is the identity of G'. The subgroup

$$\phi^{-1}(\{e'\}) \quad = \quad \{z \in G | \phi(z) = e'\} \tag{A.1}$$

*is the **kernel** of ϕ, denoted by $Ker(\phi)$.*

Definition A.1.19. *[38] Let ϕ be a mapping of a set G to a set G', and let $A \subseteq G$ and $B \subseteq G'$. The **image** $\phi(A)$ of A in G' under ϕ, is the set $\{\phi(\alpha) | \alpha \in A\}$.*

Definition A.1.20. *[13, 38] A homomorphism $\phi : G \to G'$ is an **one-to-one** mapping, if $\phi(g_1) = \phi(g_2)$ implies $g_1 = g_2$, for $g_1, g_2 \in G$. i.e. distinct elements of G have distinct images in G' (under ϕ). i.e. $Ker\ (\phi) = \{e\}$. Then ϕ is called a **monomorphism**.*

Definition A.1.21. *[13] A homomorphism $\phi : G \to G'$ is an **onto** mapping, if for every element g' in G' there is at least one element $g \in G$ for which $\phi(g) = g'$. i.e. $Im\ (\phi) = G'$. Then ϕ is called an **epimorphism**.*

Definition A.1.22. *[38] An **isomorphism** $\phi : G \to G'$ is a one-to-one and onto homomorphism, which is denoted as $G \simeq G'$.*

The following lemma can be used in order to establish isomorphisms between the non-normal subgroups which are used in Definition A.1.16 of Subsection A.1.1.

Lemma A.1.12. *[47] Let G be a group and suppose that H, K and N are subgroups of G such that*

1. *G is the semi-direct product of N by H, and*
2. *G is the semi-direct product of N by K.*

Then, $H \simeq K$.

Theorem A.1.13. *[13] The mapping $v : G \to G/N$ defined by $v(g) = Ng$ is a homomorphism of G onto G/N, and it is called the natural homomorphism of G onto its factor group G/N.*

We now define the converse of Theorem A.1.13.

Theorem A.1.14. ***(First Isomorphism Theorem)*** *[13] If $\theta : G \to H$ is a homomorphism of a group G into a group H, then $N = \ker(\theta)$ is a normal subgroup of G, and $\eta : \theta(G) \to G/N$ defines an isomorphism of $\theta(G)$ onto G/N, such that $\eta(\theta(g)) = Ng$.*

According to Theorems A.1.13 and A.1.14, we have the following diagram.

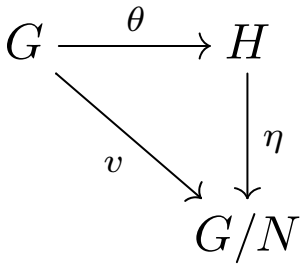


FIGURE A.1: First Isomorphism Theorem.

Theorem A.1.15. ***(Second Isomorphism Theorem)*** *[13]*
Let $N \triangleleft G$, and let H be a subgroup of G. Then $H \cap N \triangleleft H$, HN is a subgroup of G, and

$$H/(H \cap N) \simeq HN/N.$$

A.1.3 Rings, Integral Domains and Fields

In this subsection, we introduce more complex algebraic structures, which are defined by using two binary operations instead of one.

Definition A.1.23. *[38] A **ring** $\langle R, +, * \rangle$ together with two binary operations $+$ and $*$, which we usually call addition and multiplication, respectively, defined on R such that following axioms are satisfied*
R_1 : $\langle R, + \rangle$ is an abelian group.
R_2 : Multiplication is associative.
R_3 : For all $a, b, c \in R$, the left distributive law

$$a * (b + c) = (a * b) + (a * c), \text{ and}$$

the right distributive law

$$(a + b) * c = (a * c) + (b * c)$$

hold.

From now on, we simply write R to refer to a ring $\langle R, +, * \rangle$.

Definition A.1.24. *[38] A ring in which the multiplication is commutative is called a* ***commutative ring****.*

$$a * b = b * a, \forall a, b \in R.$$

Definition A.1.25. *[38] A ring with a multiplicative identity element, e, is called a* ***unitary ring****.*

$$e * a = a, \forall a \in R.$$

Definition A.1.26. *[38] An* ***integral domain*** *is a commutative ring with identity, in which there are no non-trivial zero divisors.*

$$a * b = 0 \Rightarrow a = 0 \text{ or } b = 0, \forall a, b \in R.$$

Definition A.1.27. *[38] A* ***field*** *is an integral domain such that every non-zero element has a multiplicative inverse.*

$$\exists a^{-1} \text{ such that } a * a^{-1} = 1, \forall a \in R.$$

Gathering all of the above algebraic structures, we show the relations among them by the following diagram.

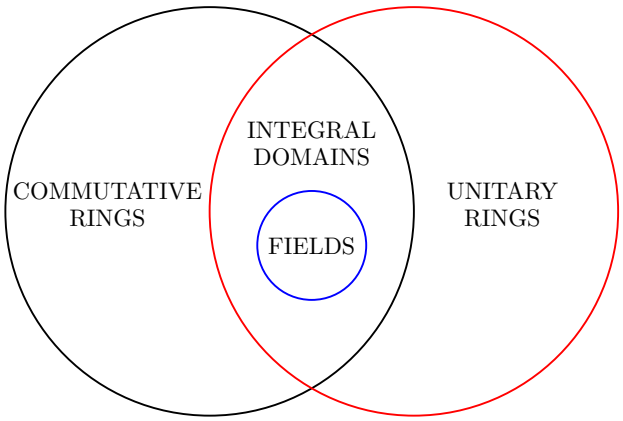


FIGURE A.2: Venn diagram of some algebraic structures [38].

Appendix B

Symbolic Code in Mathematica

B.1 Ordinary Riordan arrays

The symbolic mathematical computation program of Mathematica has been proven to be a powerful and useful tool, which extends our capability as researchers to gain time, to increase our intuition and to find results quicker. More specific, it has been often used to generate Riordan arrays, by using generating functions as inputs. For that purpose, we name the Riordan matrix as R and we create the code

$R =$`Table[SeriesCoefficient[`$(...)(...)^k, \{x, 0, n\}], \{n, 0, 10\}, \{k, 0, 10\}$`]`
`//MatrixForm`

where the desirable generating functions are inputted on the positions $(...)$ and $(...)^k$, while the numbers 0 and 10, denote the numbers of n rows and k columns that we wish to be appeared. The enumeration of both rows and columns ought to start from 0, while as we are expecting to get a square matrix, the upper limit of rows and columns needs to be the same number.
Using this matrix, we can easily found the production matrix P_R by writing the relation (2.1) as

$$\texttt{Inverse}[R[[1;;9,1;;9]]].R[[2;;10,1;;9]] \ \texttt{// MatrixForm}$$

Now, let us apply these codes on an example of a Riordan array to generate the Riordan matrix.

Example B.1.1. *Let* $R = \left(\frac{1}{1-2x-x^2}, \frac{x}{1-x}\right)$ *be a Riordan array. Hence, we have*

$R =$`Table[SeriesCoefficient[`$(\frac{1}{1-2x-x^2})(\frac{x}{1-x})^k, \{x, 0, n\}], \{n, 0, 8\}, \{k, 0, 8\}$`]`
`//MatrixForm`

which gives us the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 \\ 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 \\ 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 \\ 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 \\ 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \end{bmatrix}.$$

Nevertheless, Mathematica does not allow us to calculate the inverse of a matrix which starts from 0*, as it cannot be recognised as the number of a column or a row of R. Therefore, we have to change the enumeration of R, starting from* 1*. Additionally, the matrix* $\bar{R}$ *will be R with the top row removed, so we need the rows* $2-9$*, according to the current enumeration. Hence, we write*

$$\texttt{Inverse}[R[[1;;8,1;;8]]].R[[2;;9,1;;8]] \texttt{ // MatrixForm}$$

which outputs the production matrix of R

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

B.2 Exponential Riordan arrays

The way to represent an exponential Riordan matrix in Mathematica, is quite similar to the Ordinary Riordan array, except for the multiplication of the number $\frac{n!}{k!}$, as

$$\epsilon R = \texttt{Table}[\tfrac{n!}{k!}\texttt{SeriesCoefficient}[(...)(...)^k, \{x, 0, n\}], \{n, 0, 10\}, \{k, 0, 10\}] \texttt{ //MatrixForm}$$

The production matrix of an exponential Riordan array can be found similarly to the case of an Ordinary one.

Example B.2.1. *By using the same gfs as in Ex. B.1.1, we have the following code.*

$\epsilon R =$ `Table[`$\frac{n!}{k!}$ `SeriesCoefficient[`$(\frac{1}{1-2x-x^2})(\frac{x}{1-x})^k, \{x,0,n\}], \{n,0,6\}, \{k,0,6\}]` `//MatrixForm`,

which gives us the exponential Riordan matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 10 & 6 & 1 & 1 & 0 & 0 & 0 \\ 72 & 48 & 12 & 1 & 0 & 0 & 0 \\ 696 & 480 & 144 & 20 & 1 & 0 & 0 \\ 8,400 & 5,880 & 1,920 & 340 & 30 & 1 & 0 \\ 121,680 & 85,680 & 29,160 & 5,880 & 690 & 42 & 1 \end{bmatrix}.$$

B.3 Double Riordan arrays

In the case of double Riordan arrays, the code that we have created is slightly different as we need to find a way to describe the powers of the multiplier functions for each of the columns. For that purpose, we need the floor function $\lfloor x \rfloor$, and the ceiling function $\lceil x \rceil$.

$$\texttt{Table[SeriesCoefficient[(...) (...)}^{\text{Floor}[\frac{k+1}{2}]}\texttt{ (...)}^{\text{Ceiling}[\frac{k-1}{2}]}\texttt{, \{x,0,n\}], \{n,0,11\}, \{k,0,11\}] // MatrixForm}$$

While, the production matrix of a Double Riordan array can also be found similarly to the case of an Ordinary one. We give an example of the above code.

Example B.3.1. *For the double Riordan array* $D = \left(\frac{1}{1-x^2}, \frac{x}{1-x^2}, x\right)$, *we write*

$$D = \texttt{Table[SeriesCoefficient[}\left(\frac{1}{1-x^2}\right)\left(\frac{x}{1-x^2}\right)^{Floor\left[\frac{k+1}{2}\right]} x^{Ceiling[\frac{k-1}{2}]}\texttt{, \{x,0,n\}], \{n,0,11\}, \{k,0,11\}] // MatrixForm}$$

and we get the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & 0 & 1 & 0 & 0 \\ 1 & 0 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & 0 & 1 & 0 \\ 0 & 6 & 0 & 15 & 0 & 20 & 0 & 15 & 0 & 6 & 0 & 1 \end{bmatrix}.$$

B.4 Almost-Riordan arrays

The following codes for the case of almost-Riordan arrays were created by P. Barry. Here is an example for the almost-Riordan matrix $\left(\frac{1}{1-2z^3}\middle|\frac{1}{1-z},\frac{z}{1-z}\right)$.

$$Table[Table[If[k == 0, SeriesCoefficient[1/(1-2x^3), x, 0, n],$$
$$SeriesCoefficient[1/(1-x)(x/(1-x))^{k-1}, \{x, 0, n-1\}]], \{k, 0, 8\}], \{n, 0, 8\}]$$
$$//MatrixForm$$

which outputs the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 4 & 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 \\ 0 & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{bmatrix}.$$

The above code can be also generalised for the case of almost-Riordan arrays of level $k > 1$. Let us use the same example by adding 2 extra columns instead of 1. So, for the almost-Riordan matrix $\left(\frac{1}{1-2z^3},\frac{z-1}{z+1}\middle|\frac{1}{1-z},\frac{z}{1-z}\right)$, we write

$$Table[Table[If[k == 0, SeriesCoefficient[1/(1-2x^3), x, 0, n],$$
$$If[k == 1, SeriesCoefficient[x(x-1)/(x+1), x, 0, n],$$
$$SeriesCoefficient[1/(1-x)(x/(1-x))^{k-2}, \{x, 0, n-2\}]], \{k, 0, 8\}], \{n, 0, 8\}]$$
$$//MatrixForm$$

that gives the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 4 & -2 & 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 0 & -2 & 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 0 & -2 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix}.$$

Appendix C

Published, submitted and on progress articles

C.1 Algebraic properties of Riordan subgroups

P. Barry, A. Hennessy, N. Pantelidis (2020), Journal of Algebraic Combinatorics, DOI: 10.1007/s10801-020-00953- 4.

Abstract: We present properties of the group structure of Riordan arrays.We examine similar properties among known Riordan subgroups, and from this, we define $Y[r, s, p]$, a family of Riordan arrays. We generalise conditions for involutions, and pseudo-involutions of $Y[r, s, p]$, and we present stabilizers of this family. We find abelian subgroups as intersections of Riordan subgroups, and show some alternative semi-direct products of the Riordan group.

Keywords: Riordan subgroup, involution, pseudo-involution, semi-direct product, isomorphism, stabilizer.

C.2 On pseudo-involutions, involutions and quasi-involutions in the group of almost Riordan arrays

P. Barry and N. Pantelidis, (2019), available electronically at: arXiv:1901.03734

Abstract: The group of almost Riordan arrays contains the group of Riordan arrays as a subgroup. In this note, we exhibit examples of pseudo-involutions, involutions and quasi-involutions in the group of almost Riordan arrays.

Keywords: Almost-Riordan array, involution, pseudo-involution, quasi-involution.

C.3 The Linear Algebra of Proper Riordan Arrays

G-S. Cheon, M.M. Cohen, N. Pantelidis, , unpublished manuscript.

Abstract: Suppose that $\big(g(x), F(x)\big)$ is an element of the Riordan group $\mathcal{R}$ over a field $\mathbb{F}$ of characteristic 0, with associated matrix ("proper Riordan array"), $A = A(g(x), F(x))$. We give basic factorization theorems and diagonalization theorems for $A(g(x), F(x))$. Then, we do a complete analysis of the existence of eigenvectors of Riordan arrays of infinite order. Finally we determine, given the vector $\vec{h}$, the set $\lambda-\text{Stab}(\vec{h})$ consisting of those $A = A(g(x), F(x)) \in \mathcal{R}$ such that $A\vec{h} = \lambda \cdot \vec{h}$.

Keywords: Riordan group, Riordan array, formal series of infinite order, conjugation, eigenvectors, stabilizers.

C.4 Quasi-involutions of the Riordan group

P. Barry, A. Hennessy, N. Pantelidis, unpublished manuscript.

Abstract: A quasi-involution is a self-inverse Riordan matrix that its inverse contains the same entries with $\pm$ sign on alternating subdiagonals. We analyse the structure of these matrices and we link them to Riordan arrays which are generated by Bessel polynomials.

Keywords: Quasi-involution, Bessel polynomials, Hankel transforms, Riordan group, continuous fractions, paths.

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