Kramers' law for a bistable system with time-delayed noise

D. Goulding,[1] S. Melnik,[1,2] D. Curtin,[1,2] T. Piwonski,[1,2] J. Houlihan,[3] J. P. Gleeson,[4] and G. Huyet[1,2]
[1]*Tyndall National Institute, Lee Maltings, Cork, Ireland*
[2]*Department of Applied Physics and Instrumentation, Cork Institute of Technology, Cork, Ireland*
[3]*Department of Computing, Maths and Physics, Waterford Institute of Technology, Ireland*
[4]*Department of Applied Mathematics, National University of Ireland, University College Cork, Ireland*
(Received 5 May 2006; revised manuscript received 11 July 2007; published 25 September 2007)

We demonstrate that the classical Kramers' escape problem can be extended to describe a bistable system under the influence of noise consisting of the superposition of a white Gaussian noise with the same noise delayed by time τ. The distribution of times between two consecutive switches decays piecewise exponentially, and the switching rates for $0<t<\tau$ and $\tau<t<2\tau$ are calculated analytically using the Langevin equation. These rates are different since, for the particles remaining in one well for longer than τ, the delayed noise acquires a nonzero mean value and becomes negatively autocorrelated. To account for these effects we define an effective potential and an effective diffusion coefficient of the delayed noise.

DOI: 10.1103/PhysRevE.76.031128 PACS number(s): 05.40.Ca, 42.65.Pc, 42.65.Sf

The erratic motion of pollen particles floating in water, observed by Brown in 1827 [1], has motivated many theoretical studies starting from the seminal work of Einstein [2] and led to the development of stochastic methods [3] that have applications in a variety of fields including biology, chemistry, quantum optics, and laser science. Many of the developments have been achieved within the Markov approximation which assumes an evolution without memory, i.e., the evolution of a particle depends only on its current state. For example, Kramers [4] studied the thermal escape over a potential barrier $U(x)$ which can be described by a corresponding Langevin type equation,

$$\dot{x} = -U'(x) + \sqrt{2D}\xi(t), \tag{1}$$

where $\xi(t)$ is a white Gaussian noise term with $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$. Kramers showed that the probability density function of the residence times distribution (distribution of times between two consecutive switches, RTD) is given by

$$P_{RTD}(t) = r_k \exp(-r_k t),$$

where r_k is the well-known Kramers' escape rate which in the low-noise limit $D \ll \Delta U$ is given by

$$r_k = \frac{\sqrt{U''(x_{min})|U''(x_{max})|}}{2\pi}\exp\left(-\frac{\Delta U(x)}{D}\right). \tag{2}$$

Here x_{min} and x_{max} are the positions of the potential minima and maxima, respectively, and ΔU is the potential barrier height.

However, most dynamical systems encountered in nature are non-Markovian and as a result the thermal escape over a potential barrier has been extended to take into account noise correlations, time-varying external forces, and other memory effects in bistable potentials [5,6]. In particular, the presence of exponential correlations as described by the Ornstein-Uhlenbeck process can be taken into account by adding an extra dimension to the dynamical system and can be used to derive an escape rate equivalent to Kramers' law [7,8]. The introduction of a periodic driving force can also strongly alter the qualitative features of the RTD when the Kramers' time is similar to the period of the time varying force and leads to stochastic resonance [9–11]. More recently, researchers have considered the stochastic dynamics of time-delayed dynamical systems. For example, the prototype equation [12]

$$\dot{x} = x - x^3 + \gamma x(t-\tau) + \sqrt{2D}\xi(t) \tag{3}$$

describes the stochastic evolution of the position $x(t)$ of a particle trapped in a double well potential $U(x)=x^4/4-x^2/2$ in the presence of a time delayed force $\gamma x(t-\tau)$ and of white Gaussian noise $\xi(t)$. By developing an analogy with a two-state dynamical system, where the switching rate depends on the location of the particle at $t-\tau$, it is possible to calculate the approximate power spectrum [5] and the probability distribution of the residence time in one well [13,14].

In this paper we extend the analysis of the dynamics of a particle trapped in a bistable potential to include time-delayed white Gaussian noise. This is modeled via

$$\dot{x} = -U'(x) + \sqrt{\frac{2D}{1+\varepsilon^2}}[\xi(t) + \varepsilon\xi(t-\tau)], \tag{4}$$

where $\xi(t)$, τ, and $U(x)$ have the same meaning as in Eq. (3) and ε gives the relative strength of the delayed noise term. In order to calculate the RTD, we consider particles jumping from one well to the other and find the probability of the particle jumping back at a time t. For $t<\tau$, the RTD can be calculated under the assumption that the noise terms at t and $t-\tau$ are uncorrelated. Since the noise caused a particle to jump at $t=0$, at $t=\tau$ the delayed noise will tend to hinder (assist) a switch in the opposite direction for positive (negative) feedback, and therefore the RTD exhibits a negative (positive) peak. In this paper it is shown that for trajectories which do not switch in a time τ, the noise has two distinct features: (i) a nonzero mean value and (ii) a negative autocorrelation function (ACF). Both of these factors change the escape rate for $\tau<t<2\tau$. The nonzero mean value leads to an effective potential, $V(x)$, and thus increases (decreases) the rate for negative (positive) values of ε. The negative ACF leads to a lower effective diffusion coefficient of the delayed noise and therefore reduces the switching rate.

Noise can propagate along several paths before reaching the system under study and such noise sources are commonly encountered in nature and have been recently studied numerically, for example, in [6]. The transport of nanoparticles in biological [15,16] and artificial channels [17], as well as application in gravitational-wave interferometers such as the VIRGO detector [18], are just a few typical examples.

A system with time-delayed noise can be realized experimentally using a vertical cavity surface emitting laser (VCSEL) that exhibits polarization switching as the injection current is varied. Around this switching point, a small amount of electrical noise is sufficient to induce random polarization switching which follows Kramers' law [19]. Such an experiment has been used by several authors to study the dynamics of bistable dynamical systems under the influence of noise and period forcing where stochastic resonance can be observed [20,21], or noise and delay to describe a Kramers' law with memory [14]. In this latter experiment, one polarization component of the laser output is added to the device injection current after some delay. This situation has also been shown to exhibit excitability [22]. In the experiment considered here, a similar VCSEL was operated in a polarization unstable region and electrical noise was superimposed onto the low noise dc bias. The output from a noise generator was split and propagated along electrical lines of differing length before being added to the injection current of the VCSEL, thereby introducing a time delay into the system. The noise level of the function generator was fixed and the delayed noise level controlled using an electrical attenuator in the longer electrical line. The polarization resolved output was analyzed on a digital oscilloscope and the residence time distribution was calculated.

Figure 1(a) shows the experimentally obtained RTD in one polarization state of the VCSEL at different levels of electrical noise injection with a delay time of $\tau=0.2$ μs. In the absence of delayed noise, the RTD displays an exponential decay as predicted by Kramers' theory. When the delayed noise is introduced into the system, the RTD begins to deviate from Kramers' law while still remaining piecewise exponentially decaying. Such behavior can also be observed by integrating Eq. (4) with D being increased with ε in order to keep the term $\sqrt{2D/(1+\varepsilon^2)}$ constant, which is the case in the experiment. Figure 1(b) shows a typical RTD from the simulation of Eq. (4) exhibiting similar features to those observed experimentally for the case of negative feedback.

For $\varepsilon \neq 0$ and τ sufficiently long, the RTD displays a piecewise exponential decay with rates r_1 for $0<t<\tau$ and r_2 for $\tau<t<2\tau$. The escape rate r_1 can be calculated assuming that the noise terms at t and $t-\tau$ in Eq. (4) are uncorrelated. Since the variance of uncorrelated terms is given by the sum of the separate variances $2D/(1+\varepsilon^2)$ and $2D\varepsilon^2/(1+\varepsilon^2)$, the resulting diffusion coefficient is equal to D [3] and the problem reduces to Kramers' with the escape rate

$$r_1 = \frac{1}{\pi\sqrt{2}} \exp\left(-\frac{1}{4D}\right). \tag{5}$$

For smaller τ the corresponding rate was calculated numerically in [6] and reaches the limit of r_1 as τ increases.

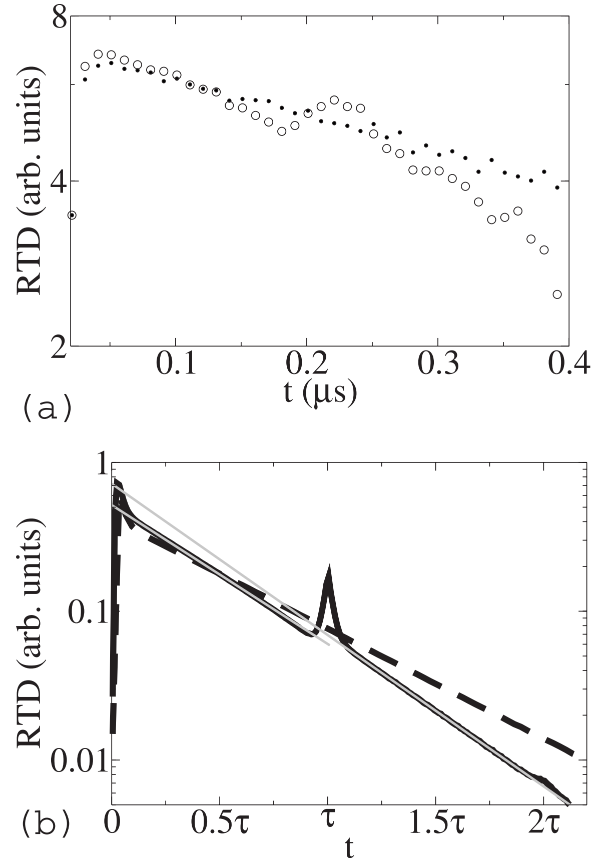

FIG. 1. (a) Experimental RTD with (circles) and without (dots) the delayed noise. (b) Numerical RTD calculated using Eq. (4) with $\varepsilon=0$ and $D=0.15$ (dashed curve) and $\varepsilon=-0.34$ and $D=0.167$ (solid curve). The thin gray lines are included to show the difference in the escape rates before and after τ in the presence of delayed noise.

We now calculate the escape rate r_2, not considered in [6], by reducing to a Kramers' type problem, defining an effective potential of the system and an effective diffusion coefficient of the delayed noise for $\tau<t<2\tau$.

In order to calculate the effective potential, consider the ensemble of all trajectories, $x(t)$, and remove trajectories once they reach some threshold position, x_{th}. For definiteness it is assumed that the initial position, $x(0)$, is to the left of the threshold, i.e., $x(0)<x_{th}$. Those trajectories which remain in the well up to time τ along with the associated noise terms are collectively referred to as the *-ensemble. The term $\xi(t)$ in the *-ensemble has a negative mean value for $0<t<\tau$, which is constant apart from the small regions near $t=0$ and τ, where it differs due to the fact that there has been a switch at $t=0$. This negative mean value appears because the trajectories whose noise has positive mean value are more likely to reach x_{th} (switch) in a time τ, thus being excluded from the *-ensemble. The nonzero mean value affects the switching rate r_2 by changing the effective potential of the system when the noise term $\xi(t)$ comes back in a time τ as $\xi(t-\tau)$. It is possible to calculate the mean value of $\xi(t)$ over the *-ensemble, using the discrete time approximation to Eq. (4).

Numerically Eq. (4) is integrated using a discrete time step Δt as

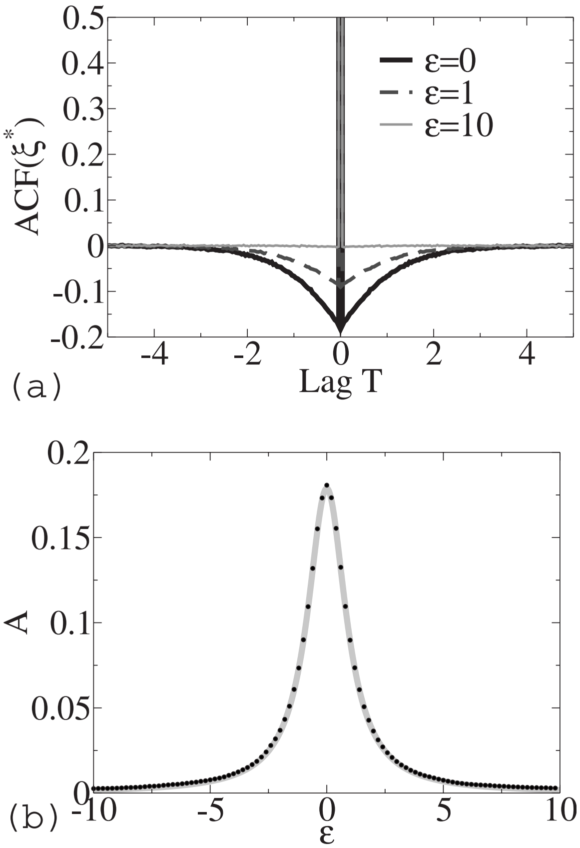

FIG. 2. (a) The ACF approaches the white noise limit as ε increases. The parameters are $D=0.15$, $\tau=100$, and $x_{th}=0.5$. The ACF does not depend on τ. (b) Values of A found from simulation together with the expression $A=A_0/1+\varepsilon^2$ with $A_0 \approx 0.1788$ and $\gamma \approx 1$ for parameters $D=0.15$, $\tau=100$, and $x_{th}=0.5$.

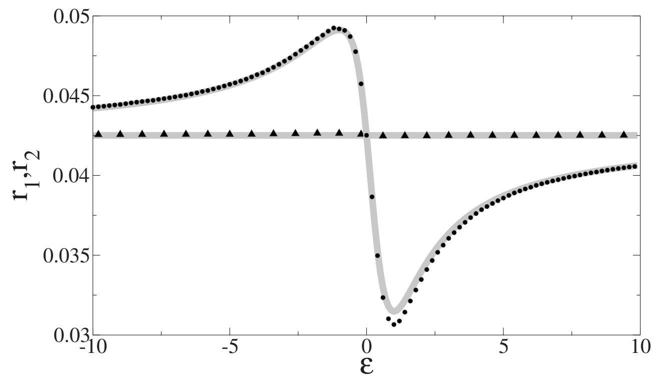

FIG. 3. Comparison of theoretical (curves) and numerically calculated (symbols) escape rates r_1 (triangles) and r_2 (circles). The numerical data are for $D=0.15$, $\tau=100$, and $x_{th}=0.5$. For the numerical simulations the value of $x_{th}=0.5$ is used to match r_1 from simulation with the theoretical r_1 for $D=0.15$.

$$x_{n+1} = x_n - U'(x_n)\Delta t + \sqrt{\frac{2D\Delta t}{1+\varepsilon^2}}(\xi_n + \varepsilon \xi_{n-N}), \tag{6}$$

where ξ_n is a sequence of normally distributed random numbers of mean zero and variance one and $N=\tau/\Delta t$ is the number of discrete time steps which corresponds to the delay time τ. In the numerical approximation, the *-ensemble is made up of sequences x_n^* (with all $x_n^* < x_{th}$), ξ_n^*, and ξ_{n-N}^* which are related via Eq. (6).

Assuming that the time τ is sufficiently long ($\tau \gg 1/r_1$), the system enters a quasistationary state, as for the Kramers' escape problem. The probability density which solves the associated Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}[U'P] + D\frac{\partial^2 P}{\partial x^2}, \tag{7}$$

with an absorbing boundary condition, $P(x_{th},t)=0$, is approximately given by

$$P(x,t) \approx e^{-r_1 t}\Phi(x), \tag{8}$$

where $\Phi(x)$ is the eigenfunction to the eigenvalue r_1, for sufficiently small D [3]. This can be interpreted as the slow escape of particles from the well at a rate r_1, but with the remaining particles in the well distributed according to the probability density $\Phi(x)$. In this quasistationary state, $\langle x_n^* \rangle = \langle x_{n+1}^* \rangle$ and therefore averaging the *-ensemble across Eq. (6) and given that $\langle \xi_{n-N}^* \rangle = \varepsilon \langle \xi_n^* \rangle$ for $0 < t < \tau$ [23], we obtain

$$\langle \xi_n^* \rangle = \langle U'(x_n^*) \rangle \sqrt{\frac{\Delta t}{2D(1+\varepsilon^2)}}.$$

It is worthwhile to note that $\langle \xi_n^* \rangle$ neither depends on n nor on τ. This nonzero mean noise in the *-ensemble implies that particles remaining trapped in the well for a time $t > \tau$ experience an effective potential $V(x)$ given by

$$V(x) = U(x) - x\frac{\varepsilon}{1+\varepsilon^2}\langle U'(x^*) \rangle, \tag{9}$$

where the constant tilt of the potential can be calculated for the quasistationary distribution by

$$\langle U'(x^*) \rangle = \frac{\int_{-\infty}^{x_{th}} U'(x)\Phi(x)dx}{\int_{-\infty}^{x_{th}} \Phi(x)dx}. \tag{10}$$

The value of $\langle U'(x_n^*) \rangle$ can be calculated from the Fokker-Planck equation and the boundary condition $\Phi(x_{th})=0$. To see this, insert the quasistationary probability distribution function Eq. (8) into Eq. (7) to obtain

$$-r_1\Phi(x) = [U'(x)\Phi(x)]' + D\Phi''(x).$$

Integrating over the range $(-\infty, x)$ and rearranging yields

$$U'(x)\Phi(x) = -r_1 \int_{-\infty}^{x} \Phi(y)dy - D\Phi'(x).$$

Further integrating over the range $(-\infty, x_{th})$ and using the boundary condition $\Phi(x_{th})=0$, we obtain

$$\int_{-\infty}^{x_{th}} U'(x)\Phi(x)dx = -r_1 \int_{-\infty}^{x_{th}} (x_{th}-x)\Phi(x)dx. \tag{11}$$

For low levels of the noise, it is expected that most trajectories will be found close to the minimum of the potential and so the approximation

$$\langle x^* \rangle = \frac{\int_{-\infty}^{x_{th}} x\Phi(x)dx}{\int_{-\infty}^{x_{th}} \Phi(x)dx} \approx x_{min} \tag{12}$$

is valid up to terms of order D. Combining Eqs. (10)–(12) gives

$$\langle U'(x^*) \rangle \approx -r_1(x_{th} - x_{min}). \tag{13}$$

Using Eqs. (9) and (13) the potential barrier height of $V(x)$ for the case of a quartic bistable potential $U(x)$ is

$$\Delta V \approx \frac{1}{4} + r_1(1 + x_{th})\frac{\varepsilon}{1+\varepsilon^2}. \tag{14}$$

Next, in order to calculate the effective diffusion coefficient, consider the effect of the negative ACF of the noise term $\xi(t)$ in the *-ensemble. This negative ACF appears because the trajectories whose noise exhibits positive ACF are more likely to cause a switch and are therefore filtered from the *-ensemble.

A typical ACF as a function of lag T is shown in Fig. 2(a). This function can be approximated as

$$\langle \xi^*(T)\xi^*(0) \rangle = \delta(T) - Ae^{-\gamma T}.$$

The value of A is found to depend on ε as

$$A(\varepsilon) = A_0/(1+\varepsilon^2)$$

as shown in Fig. 2(b). The value of γ remains constant as the value of ε varies.

Taking half the integral of the ACF over all lags we can define the diffusion correction coefficient k for the colored noise $\xi^*(t)$ as

$$k = \frac{1}{2}\int_{-\infty}^{\infty} \langle \xi^*(T)\xi^*(0) \rangle dT = \frac{1}{2} - \frac{A_0}{2\gamma(1+\varepsilon^2)}.$$

Finally, consider Eq. (4) for $\tau < t < 2\tau$ and substitute the effective noise $\sqrt{2k}\,\eta(t)$ for $\xi(t-\tau)$ together with the effective potential $V(x)$ for $U(x)$. This gives

$$\dot{x} = -V'(x) + \sqrt{\frac{2D}{1+\varepsilon^2}}[\xi(t) + \varepsilon\sqrt{2k}\,\eta(t)],$$

where $\xi(t)$ and $\eta(t)$ are two uncorrelated white Gaussian noise terms with mean zero and variance one, and $V(x)$ is defined by Eq. (9). This reduces to

$$\dot{x} = -V'(x) + \sqrt{2D\left(1 - \frac{\varepsilon^2 A_0}{\gamma(1+\varepsilon^2)^2}\right)}\xi(t). \tag{15}$$

Equation (15) is an effective Kramers' type model, equivalent to Eq. (4) for $\tau < t < 2\tau$. From this the escape rate r_2, using Eqs. (2) and (14), reads

$$r_2 = \frac{1}{\pi\sqrt{2}} \exp\left(- \frac{\frac{1}{4} + r_1(1+x_{th})\frac{\varepsilon}{1+\varepsilon^2}}{D\left(1 - \frac{A_0\varepsilon^2}{\gamma(1+\varepsilon^2)^2}\right)} \right). \tag{16}$$

As can be seen in Fig. 3 the theoretical values for r_1 and r_2 are in excellent agreement with the values obtained from numerical simulations. The rate r_2 has extrema at $\varepsilon = \pm 1$ and approaches r_1 in the limit of large positive or negative ε.

Also, it is worthwhile to note that the RTD may exhibit changes of escape rate at any multiples of τ, similar to those reported in [12], using Eq. (3). To describe these effects, one could extend the analysis presented here for the case $t > 2\tau$.

In conclusion, we have extended the classical Kramers' problem to account for the addition of a time delayed Gaussian white noise term. By analyzing the problem in two different regimes, we have derived analytical expressions for the two different switching rates. Such analysis could be extended to account for a superposition of any number of delayed noise terms.

The authors thank Eugene van den Hurk, Boole Centre for Research in Informatics (BCRI), University College Cork, Ireland for providing us with a computer cluster facility. This work was supported by Science Foundation Ireland under Contracts No. sfi/01/fi/co and No. 02/IN.1/IM062 and Embark under the IRCSET scheme.

[1] R. Brown, Philos. Mag. **4**, 161 (1828).
[2] A. Einstein, Ann. Phys. **17**, 549 (1905).
[3] C. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer, Berlin, 2004).
[4] H. A. Kramers, Physica (Utrecht) **7**, 284 (1940).
[5] L. S. Tsimring and A. Pikovsky, Phys. Rev. Lett. **87**, 250602 (2001).
[6] M. Borromeo, S. Giusepponi, and F. Marchesoni, Phys. Rev. E **74**, 031121 (2006).
[7] P. Hänggi, F. Marchesoni, and P. Grigolini, Z. Phys. B: Condens. Matter **56**, 333 (1984).
[8] P. Jung, A. Neiman, M. K. N. Afghan, S. Nadkarni, and G. Ullah, New J. Phys. **7**, 17 (2005).
[9] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, 453 (1981).
[10] C. Nicolis and G. Nicolis, Tellus **33**, 225 (1981).
[11] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[12] D. Curtin, S. P. Hegarty, D. Goulding, J. Houlihan, T. Busch, C. Masoller, and G. Huyet, Phys. Rev. E **70**, 031103 (2004).
[13] C. Masoller, Phys. Rev. Lett. **90**, 020601 (2003).
[14] J. Houlihan, D. Goulding, T. Busch, C. Masoller, and G. Huyet, Phys. Rev. Lett. **92**, 050601 (2004).
[15] B. Alberts, *Molecular Biology of the Cell* (Garland Science,

New York, 2002).
[16] J. Kärger and D. M. Ruthven, *Diffusion in Zeolites and Other Microporous Solids* (Wiley, New York, 1992).
[17] S. Matthias and F. Muller, Nature (London) **424**, 53 (2003).
[18] B. Caron *et al.*, Class. Quantum Grav. **14**, 1461 (1997).
[19] M. B. Willemsen, M. U. F. Khalid, M. P. van Exter, and J. P. Woerdman, Phys. Rev. Lett. **82**, 4815 (1999).
[20] G. Giacomelli, F. Marin, and I. Rabbiosi, Phys. Rev. Lett. **82**, 675 (1999).
[21] S. Barbay, G. Giacomelli, and F. Marin, Phys. Rev. E **61**, 157 (2000).
[22] T. Piwonski, J. Houlihan, T. Busch, and G. Huyet, Phys. Rev. Lett. **95**, 040601 (2005).
[23] Consider $z=x+\varepsilon y$ for zero mean unit-variance normal variables x and y, and construct a *-ensemble by keeping only values leading to $z<M$ for some fixed value of M. In this case $\langle x^*\rangle=\varepsilon\langle y^*\rangle$.