On The Hurwitz Transform of Sequences

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Abstract

Based on classical concepts, we introduce and study the Hurwitz transform of sequences, relating this transform to the Hankel transform of sequences. We also define and study associated polynomials, including links to related families of orthogonal polynomials. Examples of these associated polynomials are given within the context of Riordan arrays.S

1 Introduction

Given a sequence a_n, we denote by h_n the general term of the sequence with $h_n = |a_{i+j}|_{0 \le i,j \le n}$. The sequence h_n is called the Hankel transform of a_n [19, 20, 21]. This sequence of Hankel determinants has attracted much attention of late amongst those working in the area of integer and polynomial sequences in particular [7, 18, 24, 32]. In this note we shall introduce the notion of a related Hurwitz transform, and we shall study some of its properties. As with the Hankel transform, this transform is based on classical results which have a rich literature. Part of this literature is captured in the review article by Holtz and Tyaglov [17], which forms a good background to this note. Our Hurwitz transform will give rise to a sequence of determinant values, which can be related to the Hankel transform.

In the sequel, we shall be mainly concerned with integer sequences. Known integer sequences are often referred to by their OEIS number [26, 27]. For instance, the sequence of Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ is A000108. Its generating function, defined by $\sum_{n=0}^{\infty} C_n x^n$, is equal to $c(x) = \frac{1-\sqrt{1-4x}}{2x}$. Its first elements are

$$1, 1, 2, 5, 14, 42, 132, \ldots.$$

This sequence finds many applications in combinatorics [29, 30]. It is the unique sequence whose Hankel transform, along with that of its first shift C_{n+1}, is the all 1's sequence [4, 22]. We use it in many of our examples, partly because of these properties.

We recall the following notational elements. For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{k=0}^{\infty} a_n x^n$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$ [23]. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \ldots,$

A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$. Here, we have used the Iverson bracket notation [13], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$.

2 Definition of the Hurwitz transform

We consider two sequences a_n and b_n, and define the Hurwitz matrix of order n defined by these sequences as follows. If n is even, $n = 2m$, then the Hurwitz matrix of order n is defined to be the matrix

$$\mathcal{H}_n = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_m & \cdots & a_{2m} \\ b_0 & b_1 & b_2 & \cdots & b_m & \cdots & b_{2m} \\ 0 & a_0 & a_1 & \cdots & a_{m-1} & \cdots & a_{2m-1} \\ 0 & b_0 & b_1 & \cdots & b_{m-1} & \cdots & b_{2m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & \cdots & a_m \end{pmatrix}.$$

If n is odd, $n = 2m + 1$, then the Hurwitz matrix of order n is defined to be the matrix

$$\mathcal{H}_n = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_m & \cdots & a_{2m+1} \\ b_0 & b_1 & b_2 & \cdots & b_m & \cdots & b_{2m+1} \\ 0 & a_0 & a_1 & \cdots & a_{m-1} & \cdots & a_{2m} \\ 0 & b_0 & b_1 & \cdots & b_{m-1} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & \cdots & a_{m+1} \\ 0 & 0 & 0 & \cdots & b_0 & \cdots & b_{m+1} \end{pmatrix}.$$

We shall call the sequence of determinants $\mathbb{H}_n = |\mathcal{H}_n|$ the *Hurwitz transform* of the sequences a_n and b_n (in that order). We shall sometimes write $\mathbb{H}_n(a_n, b_n)$ or $\mathbb{H}_n(a, b)$ for the transform of a_n and b_n, to make the dependence on a_n and b_n more explicit. By the definition, it is clear that if a_n and b_n are integer sequences, then $\mathbb{H}_n$ is an integer sequence. We have

$$\mathbb{H}_n(a, a) = a_0 0^n = \begin{cases} a_0, & \text{if } n = 0; \\ 0, & n > 0. \end{cases}$$

Note that we can express the general term $\mathcal{H}_{i,j}$ of the Hurwitz matrix in the following manner.

$$\mathcal{H}_{i,j} = \begin{cases} 0, & \text{if } 2j + 2 \leq i; \\ a_{j-\frac{i}{2}}, & \text{if } 2|i; \\ b_{j-\frac{i-1}{2}}, & \text{otherwise.} \end{cases} \tag{1}$$

We can associate a sequence s_n with the two sequences a_n and b_n in the following manner. We can define s_n implicity by the relations

$$a_n = \sum_{k=0}^{n} s_k b_{n-k},$$

which is a convolution equation for s_n. If $b_0 \neq 0$ (which we will assume henceforth), we have

$$s_n = [x^n] \frac{\sum_{j=0}^{\infty} a_j x^j}{\sum_{j=0}^{\infty} b_j x^j}.$$

That is, s_n is the sequence whose generating function is the quotient of the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ of a_n and of the generating function $g(x) = \sum_{n=0}^{\infty} b_n x^n$ of b_n. Using generating functions allows us to express the elements of the Hurwitz matrix as follows.

$$\mathcal{H}_{j,i} = \begin{cases} [x^i] x^{\frac{j}{2}} f(x), & \text{if } 2|j; \\ \\ [x^i] x^{\frac{j-1}{2}} g(x), & \text{otherwise.} \end{cases} \tag{2}$$

We let h_n denote the Hankel transform of s_n and we let h_n^* denote the Hankel transform of the shifted sequence $s_n^* = s_{n+1}$. Then we have the following proposition characterizing $\mathbb{H}_n$.

Proposition 1. *We have*

$$\mathbb{H}_{2n} = b_0^{2n+1} h_n, \qquad \mathbb{H}_{2n+1} = (-1)^{n+1} b_0^{2n+2} h_n^*.$$

Proof. We take the case $n = 2m$. Beginning with the original matrix, we carry out the following steps.

1. Factor b_0 out of column 0.
2. Subtract b_j times column 0 from column j for $1 \leq j \leq 2m$.
3. Factor b_0 out of column 1.
4. Subtract b_{j-1} times column 1 from column j for $2 \leq j \leq 2m$.
5. Factor b_0 out of column 2.
6. Subtract b_{j-2} times column 2 from column j for $3 \leq j \leq 2m$.
7. etc.

One has now factored b_0 out $2m+1$ times, and the resulting matrix is

$$\begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_{m-1} & s_m & \cdots & s_{2m} \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & s_0 & s_1 & \cdots & s_{m-2} & s_{m-1} & \cdots & s_{2m-1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & s_0 & \cdots & s_{m-3} & s_{m-2} & \cdots & s_{2m-2} \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & s_0 & \cdots & s_m \end{pmatrix}.$$

Expanding iteratively along the rows with a single 1 we get an "upside-down" Hankel determinant

$$\begin{vmatrix} s_m & \cdots & s_{2m} \\ s_{m-1} & \cdots & s_{2m-1} \\ \vdots & \vdots & \vdots \\ s_0 & \cdots & s_m \end{vmatrix}.$$

Now looking at the case $n = 2m+1$, we get the matrix

$$\begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_{m-1} & s_m & s_{m+1} & \cdots & s_{2m+1} \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & s_0 & s_1 & \cdots & s_{m-2} & s_{m-1} & s_m & \cdots & s_{2m} \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & s_0 & \cdots & s_{m-3} & s_{m-2} & s_{m-1} & \cdots & s_{2m-1} \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & s_0 & s_1 & \cdots & s_{m+1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Again, expanding iteratively along the rows with a single 1 we get an "upside-down" Hankel determinant

$$\begin{vmatrix} s_{m+1} & \cdots & s_{2m+1} \\ s_m & \cdots & s_{2m} \\ \vdots & \vdots & \vdots \\ s_1 & \cdots & s_{m+1} \end{vmatrix}.$$

Keeping track of signs now yields the result.

By looking at the transposed matrix, $(\mathcal{H}_{i,j})^T$, we can interpret the above operations as follows, where we use the notation of Riordan arrays (see the section on Riordan arrays).

We have

$$
\begin{aligned}
(g(x),x)^{-1}\cdot(\mathcal{H}_{i,j})^T &= \left(\frac{1}{g(x)},x\right)\cdot(\mathcal{H}_{j,i})\\
&= \left(\frac{1}{g(x)},x\right)\cdot\left(\begin{cases}[x^i]x^{\frac{j}{2}}f(x), & \text{if } 2|j;\\ \\ [x^i]x^{\frac{j-1}{2}}g(x), & \text{otherwise.}\end{cases}\right)\\
&= \left(\begin{cases}[x^i]x^{\frac{j}{2}}\frac{f(x)}{g(x)}, & \text{if } 2|j;\\ \\ [x^i]x^{\frac{j-1}{2}}\frac{g(x)}{g(x)}, & \text{otherwise.}\end{cases}\right)\\
&= \left(\begin{cases}[x^i]x^{\frac{j}{2}}\frac{f(x)}{g(x)}, & \text{if } 2|j;\\ \\ [x^i]x^{\frac{j-1}{2}}, & \text{otherwise.}\end{cases}\right)\\
&= \left(\begin{cases}[x^i]x^{\frac{j}{2}}s(x), & \text{if } 2|j;\\ \\ [x^i]x^{\frac{j-1}{2}}, & \text{otherwise.}\end{cases}\right).
\end{aligned}
$$

Transposing and taking the first $n+1$ rows and columns (for $n=2m$ and $n=2m+1$) brings us back to the above cases. Note that the determinant of $(g(x),x)_n^{-1}$ is $\frac{1}{b_0^{n+1}}$.

□

We thus have

$$
\begin{pmatrix} b_0 & & & & \cdots\\ b_1 & b_0 & & & \cdots\\ b_2 & b_1 & b_0 & & \cdots\\ b_3 & b_2 & b_1 & b_0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1}\cdot\begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots\\ a_1 & b_1 & a_0 & b_0 & \cdots\\ a_2 & b_2 & a_1 & b_1 & \cdots\\ a_3 & b_3 & a_2 & b_2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}=\begin{pmatrix} s_0 & 1 & 0 & 0 & \cdots\\ s_1 & 0 & s_0 & 1 & \cdots\\ s_2 & 0 & s_1 & 0 & \cdots\\ s_3 & 0 & s_2 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

and hence

$$
\begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots\\ a_1 & b_1 & a_0 & b_0 & \cdots\\ a_2 & b_2 & a_1 & b_1 & \cdots\\ a_3 & b_3 & a_2 & b_2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}=\begin{pmatrix} b_0 & & & & \cdots\\ b_1 & b_0 & & & \cdots\\ b_2 & b_1 & b_0 & & \cdots\\ b_3 & b_2 & b_1 & b_0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\cdot\begin{pmatrix} s_0 & 1 & 0 & 0 & \cdots\\ s_1 & 0 & s_0 & 1 & \cdots\\ s_2 & 0 & s_1 & 0 & \cdots\\ s_3 & 0 & s_2 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
$$

Example 2. It is possible to gain further insight into this result by using Gaussian elimination in the following way. Let $s(x)$ be the g.f. of s_n. Then we have

$$s(x)=\frac{f(x)}{g(x)}\Rightarrow f(x)=s(x)g(x).$$

That is

$$\sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=0}^{\infty} s_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} s_k b_{n-k} x^n,$$

or

$$a_n = \sum_{k=0}^{n} s_k b_{n-k}.$$

We now substitute for a_n in the definitions of $\mathcal{H}_n$. For instance, we get

$$\mathcal{H}_n = \begin{pmatrix} s_0b_0 & s_0b_1+s_1b_0 & s_0b_2+s_1b_1+s_2b_0 & \cdots & s_0b_m+\dots & \cdots & s_0b_{2m}+\dots \\ b_0 & b_1 & b_2 & \cdots & b_m & \cdots & b_{2m} \\ 0 & s_0b_0 & s_0b_1+s_1b_0 & \cdots & s_0b_{m-1}+\dots & \cdots & s_0b_{2m-1}+\dots \\ 0 & b_0 & b_1 & \cdots & b_{m-1} & \cdots & b_{2m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_0b_0 & \cdots & s_0b_{m+1}+\dots \end{pmatrix}$$

for $n = 2m$.
For instance, assuming that $s_0 \neq 0$, $s_1 \neq 0$, we have

$$\mathbb{H}_2 = \begin{vmatrix} s_0b_0 & s_0b_1+s_1b_0 & s_0b_2+s_1b_1+s_2b_0 \\ b_0 & b_1 & b_2 \\ 0 & s_0b_0 & s_0b_1+s_1b_0 \end{vmatrix}$$

$$= \frac{1}{s_0}\begin{vmatrix} s_0b_0 & s_0b_1+s_1b_0 & s_0b_2+s_1b_1+s_2b_0 \\ s_0b_0 & s_0b_1 & s_0b_2 \\ 0 & s_0b_0 & s_0b_1+s_1b_0 \end{vmatrix}$$

$$= \frac{1}{s_0}\begin{vmatrix} s_0b_0 & s_0b_1+s_1b_0 & s_0b_2+s_1b_1+s_2b_0 \\ 0 & -s_1b_0 & -s_1b_1-s_2b_0 \\ 0 & s_0b_0 & s_0b_1+s_1b_0 \end{vmatrix}$$

$$= \frac{1}{s_0^2 s_1}\begin{vmatrix} s_0b_0 & s_0b_1+s_1b_0 & s_0b_2+s_1b_1+s_2b_0 \\ 0 & -s_0s_1b_0 & -s_0s_1b_1-s_0s_2b_0 \\ 0 & 0 & (s_1^2-s_0s_2)b_0 \end{vmatrix}$$

$$= b_0^3(s_0s_2 - s_1^2)$$

$$= b_0^3\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}.$$

Similarly, we obtain

$$\mathbb{H}_3 = \frac{1}{s_0^4 s_1^2(s_1^2 - s_0s_2)}\begin{vmatrix} s_0b_0 & \dots & \dots & \dots \\ 0 & -s_0s_1b_0 & \dots & \dots \\ 0 & 0 & s_0s_1(s_1^2-s_0s_2)b_0 & \dots \\ 0 & 0 & 0 & s_0s_1(s_2^2-s_1s_3)b_0 \end{vmatrix},$$

and hence we have

$$\mathbb{H}_3 = b_0^4(s_2^2 - s_1 s_3) = b_0^4 \begin{vmatrix} s_1 & s_2 \\ s_2 & s_3 \end{vmatrix}.$$

Finally we note that

$$\mathbb{H}_n(\alpha a_n, \beta b_n) = \alpha^{\lfloor \frac{n+2}{2} \rfloor} \beta^{\lfloor \frac{n+1}{2} \rfloor} \mathbb{H}_n(a_n, b_n).$$

Example 3. It is well-known that the Hankel transform of the Catalan numbers C_n, along with that of the shifted sequence C_{n+1}, is given by the all 1's sequence. We thus turn to the Catalan numbers to provide an example of a pair of sequences whose Hurwitz transform is the all 1's sequence. We recall that the sequence C_n has generating function

$$c(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

We define $C_{-1} = 0$. Then the Hurwitz transform of the pair

$$a_n = (-1)^n (C_n + C_{n-1}), \qquad b_n = (-1)^n \binom{1}{n}$$

is such that

$$\mathbb{H}_n = 1 \quad \text{for all } n.$$

This follows since in this case,

$$f(x) = (1-x)c(-x), \qquad g(x) = 1 - x.$$

Then

$$\frac{f(x)}{g(x)} = \frac{(1-x)c(-x)}{1-x} = c(-x),$$

which is the generating function of $(-1)^n C_n$. The Hankel transform of $(-1)^n C_n$ is $1, 1, 1, \ldots$ while that of $(-1)^{n+1} C_{n+1}$ is $(-1)^{n+1}$, hence the result.

It is clear that any pair of sequences a_n, b_n such that $\frac{f(x)}{g(x)} = c(-x)$ will furnish a Hurwitz transform consisting of the all 1's sequence. Thus, as with the Hankel transform, the Hurwitz transform is not injective.

Example 4. We now look at an example where $b_0 \neq 1$. Thus we take

$$a_n = C_n, \quad b_n = C_n + C_{n+1},$$

with

$$f(x) = c(x), \quad g(x) = c(x) + c(x)^2, \quad s(x) = \frac{1}{1 + c(x)} = \frac{1 + 2x + \sqrt{1-4x}}{2(x+2)}.$$

We have $b_0 = 2$. In this case, we find that

$$s_n = \frac{1}{2^{n+1}} \sum_{k=0}^{n-1} \frac{n-k}{n} \binom{n+k-1}{k} 2^k,$$

which begins

$$\frac{1}{2}, -\frac{1}{4}, -\frac{3}{8}, -\frac{13}{16}, -\frac{67}{32}, -\frac{381}{64}, \ldots.$$

We find that

$$2^{2n+1}h_n = (-1)^n(n+1), \quad 2^{2n+2}(-1)^{n+1}h_n^* = 1,$$

and so the Hurwitz transform $\mathbb{H}_n(C_n, C_n + C_{n+1})$ is given by

$$1, 1, -2, 1, 3, 1, -4, 1, 5, 1, -6, \ldots.$$

Example 5. This example uses the Motzkin numbers A001006

$$M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k,$$

with generating function

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

We let

$$a_n = M_n, \quad b_n = M_n + M_{n+1}.$$

We find that s_n has the generating function

$$\frac{1 + 3x + \sqrt{1 - 2x - 3x^2}}{6x + 4},$$

and begins

$$\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{5}{16}, -\frac{17}{32}, -\frac{77}{64}, \ldots.$$

We find that $2^{2n+1}h_n$ is the periodic sequence that begins

$$1, -1, 0, 1, -1, 0, 1, -1, 0, \ldots,$$

while $2^{2n+2}(-1)^{n+1}h_n^*$ is the all 1's sequence. Thus we find that $\mathbb{H}_n(M_n, M_n + M_{n+1})$ is the periodic sequence

$$1, 1, -1, 1, 0, 1, 1, 1, -1, 1, 0, 1, 1, 1, -1, 1, 0, 1, 1, 1, -1, \ldots,$$

with generating function

$$\frac{1 + x - x^2 + x^3 + x^5}{1 - x^6}.$$

Example 6. We define the Hurwitz transform of a single sequence a_n to be the Hurwitz transform of the pair $(a_n, 0^n)$. In this example, we take a_n to be the sequence A025262$(n+1)$, which begins

$$1, 1, 3, 8, 23, 68, 207, 644, 2040, 6558, 21343, \ldots.$$

This sequence has generating function

$$f(x)=\frac{1-2x-\sqrt{1-4x+4x^3}}{2x^2}.$$

Its Hankel transform h_n is an example of a Somos-4 [8, 12, 33] sequence. This means that it satisfies the recurrence

$$h_{n-1}h_{n-3}+h_{n-2}^2=h_nh_{n-4}, \quad n\geq 3.$$

In this case, h_n begins

$$1,2,3,7,23,59,314,1529,8209,83313,620297,\ldots.$$

This is A006720$(n+3)$.

More generally, we say that a sequence e_n is a (α,β) Somos-4 sequence if we have

$$\alpha h_{n-1}h_{n-3}+\beta h_{n-2}^2=h_nh_{n-4}, \quad n\geq 4.$$

Now h_n^* begins

$$1,-1,-5,-4,29,129,-65,-3689,-16264,113689,2382785,\ldots,$$

and hence $\mathbb{H}_n$ begins

$$1,-1,2,-1,3,5,7,-4,23,-29,59,129,314,\ldots.$$

Numerical evidence suggests that $\mathbb{H}_n$ is then a $(-1,1)$ Somos-4 sequence.

Example 7. We let a_n be the sequence A160702$(n+1)$. This sequence begins

$$1,1,5,19,79,333,1441,6351,28451,129185,\ldots,$$

and its Hankel transform h_n is a $(4,24)$ Somos-4 sequence, as is the Hankel transform h_n^* of a_{n+1}. We can then conjecture that the Hurwitz transform of a_n is a $(-2,2)$ Somos-4 sequence. $\mathbb{H}_n$ begins

$$1,-1,4,-6,20,88,464,512,17024,-173568,1632256,\ldots,$$

and our claim is that

$$(-2)\mathbb{H}_{n-1}\mathbb{H}_{n-3}+2\mathbb{H}_{n-2}^2=\mathbb{H}_n\mathbb{H}_{n-4}, \quad n\geq 4.$$

We are not at present able to prove this assertion.

Example 8. We finish this section with an example which recalls the use of the Hurwitz matrix to determine if a polynomial is stable. We let $e_n=\binom{n}{\frac{n}{2}}$, and we set

$$a_n=e_{2n+1}=\binom{2n+1}{n+1}, \quad b_n=e_{2n}=\binom{2n}{n}.$$

We find that the Hurwitz transform $\mathbb{H}_n(e_{2n+1},e_{2n})$ in this case is given by

$$1,-1,1,1,1,-1,1,1,1,-1,1,\ldots.$$

3 Hurwitz associated polynomials

One important application of Hankel determinants is in the construction of orthogonal polynomials [10, 31], where the Hankel determinants in question have elements that are the moments of the density associated with the orthogonal polynomials. We now use the Hurwitz matrix to construct families of polynomials, which we then relate to the polynomials defined by s_n and s_n^* by the Hankel construction.

We let

$$P_n^{(s)}(x) = \Delta_n^{(s)}(1, x, x^2, \cdots, x^n) = \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_n \\ s_1 & s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix},$$

respectively

$$P_n^{(s^*)}(x) = \Delta_n^{(s^*)}(1, x, x^2, \cdots, x^n) = \begin{vmatrix} s_1 & s_2 & s_3 & \cdots & s_{n+1} \\ s_2 & s_3 & s_4 & \cdots & s_{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_{2n} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix},$$

and let $L^{(s)}$ (respectively $L^{(s^*)}$) be the coefficient array of the family of orthogonal polynomials $P_n^{(s)}(x)$ (respectively $P_n^{(s^*)}(x)$).

If n is even, $n = 2m$, we set

$$\mathcal{D}_n(1, x, \dots, x^m) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_m & \cdots & a_{2m} \\ b_0 & b_1 & b_2 & \cdots & b_m & \cdots & b_{2m} \\ 0 & a_0 & a_1 & \cdots & a_{m-1} & \cdots & a_{2m-1} \\ 0 & b_0 & b_1 & \cdots & b_{m-1} & \cdots & b_{2m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots & x^m \end{vmatrix}.$$

If n is odd, $n = 2m + 1$, then we let

$$\mathcal{D}_n(1, x, \dots, x^{m+1}) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_m & \cdots & a_{2m+1} \\ b_0 & b_1 & b_2 & \cdots & b_m & \cdots & b_{2m+1} \\ 0 & a_0 & a_1 & \cdots & a_{m-1} & \cdots & a_{2m} \\ 0 & b_0 & b_1 & \cdots & b_{m-1} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & \cdots & a_{m+1} \\ 0 & 0 & 0 & \cdots & 1 & \cdots & x^{m+1} \end{vmatrix}.$$

We can gain insight into this construction again by looking at the transpose of the underlying matrix. Using an obvious notation [23], we have

$$\mathcal{H}_{j,i}(x) = \begin{cases} [t^i]t^{\frac{j}{2}}f(t), & \text{if } 2|j \text{ and } j<n; \\ \\ [t^i]t^{\frac{j-1}{2}}g(t), & \text{if } 2\nmid j \text{ and } j<n; \\ \\ [t^i]\frac{t^{\lfloor \frac{j}{2}\rfloor}}{1-xt}, & \text{if } j=n. \end{cases} \tag{3}$$

Finally we let $M = (b_{i-j}[j \le i])$ be the sequence (or renewal) array of the sequence b_n, and M_n represent the matrix composed of the first $n+1$ rows and columns of M (where we use a similar subscript notation to denote the first $n+1$ rows and columns of other matrices). We have $M = (g(x), x)$ as a Riordan array.

Proposition 9. *We have*

$$\mathcal{D}_{2n}(1, x, \ldots, x^n) = (-1)^n b_0^{2n+1}\Delta_n^{(s^*)}(1, \phi_1(x), \phi_2(x), \ldots, \phi_n(x))$$

where

$$(1, \phi_1(x), \phi_2(x), \ldots, \phi_n(x))^t = M_n^{-1}(1, x, x^2, \ldots, x^n)^t,$$

and we have

$$\mathcal{D}_{2n+1}(1, x, \ldots, x^{n+1}) = b_0^{2n+2}\Delta_{n+1}^{(s)}(1, \phi_1(x), \phi_2(x), \ldots, \phi_n(x), \phi_{n+1}(x))$$

where

$$(1, \phi_1(x), \phi_2(x), \ldots, \phi_n(x), \phi_{n+1}(x))^t = M_{n+1}^{-1}(1, x, x^2, \ldots, x^n)^t.$$

Equivalently, if $\tilde{L}$ is the coefficient array of the family of polynomials $Q_n(x) = \mathcal{D}_{2n}(1, x, \ldots, x^n)$ (respectively $\tilde{L}^*$ is the coefficient array of the polynomials $R_{n+1}(x) = \mathcal{D}_{2n+1}(1, x, \ldots, x^{n+1})$, $R_0(x) = 1$) then we have

$$\tilde{L} = S \cdot L^{(s)} \cdot M^{-1}$$

(respectively

$$\tilde{L}^* = L^{(s^*)} \cdot M^{-1}),$$

where $S = diag(1, -1, 1, -1, \ldots)$.

Proof. Carrying out the eliminations on $\mathcal{D}_{2m}$ that we have used in the first proposition, we get, for $n = 2m$,

$$\begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_{m-1} & s_m & \cdots & s_{2m} \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & s_0 & s_1 & \cdots & s_{m-2} & s_{m-1} & \cdots & s_{2m-1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & s_0 & \cdots & s_{m-3} & s_{m-2} & \cdots & s_{2m-2} \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_0 & s_1 & \cdots & s_{m+1} \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & \phi_m(x) \end{pmatrix}.$$

Expanding iteratively along the rows with a single 1 we get an "upside-down" augmented Hankel determinant

$$\left|\begin{array}{ccc} s_{m+1} & \cdots & s_{2m} \\ \vdots & \vdots & \vdots \\ s_1 & \cdots & s_{m+1} \\ 1 & \cdots & \phi_m(x) \end{array}\right|.$$

Similarly for $n = 2m+1$, we obtain in $\mathcal{D}_{2m+1}$ upon elimination the following matrix.

$$\left(\begin{array}{cccccccc} s_0 & s_1 & s_2 & \cdots & s_{m-1} & s_m & \cdots & s_{2m+1} \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & s_0 & s_1 & \cdots & s_{m-2} & s_{m-1} & \cdots & s_{2m} \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & s_0 & \cdots & s_{m-3} & s_{m-2} & \cdots & s_{2m-1} \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & s_0 & \cdots & s_{m+1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & \phi_{m+1}(x) \end{array}\right).$$

Expanding iteratively along the rows with a single 1 we get an "upside-down" augmented Hankel determinant

$$\left|\begin{array}{ccc} s_m & \cdots & s_{2m+1} \\ \vdots & \vdots & \vdots \\ s_0 & \cdots & s_{m+1} \\ 1 & \cdots & \phi_{m+1}(x) \end{array}\right|.$$

The result follows from this. Note that in terms of the matrix elements $\mathcal{H}_{i,j}(x)$, we can interpret the above as

$$\begin{aligned} M_n^{-1} \cdot (\mathcal{H}_{j,i}(x))_n &= \left(\frac{1}{g(t)}, t\right)_n \cdot (\mathcal{H}_{j,i}(x))_n \\ &= \left(\begin{cases} [t^i]t^{\frac{j}{2}} f(t)/g(t), & \text{if } 2|j \text{ and } j<n; \\ [t^i]t^{\frac{j-1}{2}} g(t)/g(t), & \text{if } 2\nmid j \text{ and } j<n; \\ [t^i]t^{\lfloor\frac{j}{2}\rfloor} \frac{1}{g(t)}\frac{1}{1-xt}, & \text{if } j=n. \end{cases}\right) \\ &= \left(\begin{cases} [t^i]t^{\frac{j}{2}} s(t), & \text{if } 2|j \text{ and } j<n; \\ [t^i]t^{\frac{j-1}{2}}, & \text{if } 2\nmid j \text{ and } j<n; \\ [t^i]t^{\lfloor\frac{j}{2}\rfloor} \frac{1}{g(t)}\frac{1}{1-xt}, & \text{if } j=n. \end{cases}\right), \end{aligned}$$

where

$$\phi_n(x) = [t^n]\frac{1}{g(t)}\frac{1}{1-xt}.$$

□

Note that we have

$$\left(\begin{array}{cccccc} b_0 & & & & & \\ b_1 & b_0 & & & & \\ \vdots & \vdots & \vdots & & & \\ b_m & b_{m-1} & b_{m-2} & \vdots & & \\ b_{m+1} & b_m & b_{m-1} & \vdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ b_{2m} & b_{2m-1} & b_{2m-2} & b_{2m-3} & \cdots & b_0 \end{array}\right)^{-1} \cdot \left(\begin{array}{cccccc} a_0 & b_0 & 0 & 0 & \cdots & 0 \\ a_1 & b_1 & a_0 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_m & b_m & a_{m-1} & b_{m-1} & \cdots & 1 \\ a_{m+1} & b_{m+1} & a_m & b_m & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2m} & b_{2m} & a_{2m-1} & b_{2m-1} & \cdots & x^m \end{array}\right)$$

$$= \left(\begin{array}{cccccc} s_0 & 1 & 0 & 0 & \cdots & 0 \\ s_1 & 0 & s_0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ s_m & 0 & s_{m-1} & 0 & \cdots & 1 \\ s_{m+1} & 0 & s_m & 0 & \cdots & \phi_1(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{2m} & 0 & s_{2m-1} & 0 & \cdots & \phi_m(x) \end{array}\right)$$

with a corresponding matrix equation for the case $n = 2m+1$.

4 The Riordan array case

We recall elements of the theory of Riordan arrays before giving examples of Hurwitz associated polynomials, where the relevant coefficient matrices happen to be Riordan arrays. For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$.

For a lower triangular matrix $(a_{n,k})_{n,k\geq 0}$ the row sums give the sequence with general term $\sum_{k=0}^{n} a_{n,k}$. More generally, a lower triangular matrix $(a_{n,k})_{n,k\geq 0}$ is the coefficient array of the polynomials

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k.$$

The *Riordan group* [25, 28], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \cdots$ and $f(x) = f_1 x + f_2 x^2 + \cdots$ where $f_1 \neq 0$ [28]. We often require in addition that $f_1 = 1$, but this is not the case in this note. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

Elements of the form $(g(x), x)$ form a subgroup called the *Appell subgroup*.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)^t$ is an integer sequence (expressed as an infinite column vector) with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

This action is often referred to as the fundamental theorem of Riordan arrays.

Example 10. The so-called *binomial matrix* $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

Example 11. The Riordan array $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$ has inverse $(c(x), xc(x)^2)$. It is the coefficient array of the unique family of orthogonal polynomials

$$P_n(x) = (x - 2)P_{n-1}(x) - P_{n-2}(x)$$

for which the Catalan numbers C_n are the moments.

The row sums of the matrix (g, f) have generating function

$$(g, f) \cdot \frac{1}{1 - x} = \frac{g(x)}{1 - f(x)},$$

while the polynomial sequence $P_n(t)$ for which (g, f) is the coefficient array will have g.f.

$$\frac{g(x)}{1 - tf(x)}.$$

For an invertible matrix M, its production matrix (also called its Stieltjes matrix) [14, 15] is the matrix

$$P_M = M^{-1}\hat{M},$$

where $\hat{M}$ is the matrix M with its first row removed. A Riordan array M is the inverse of the coefficient array of a family of orthogonal polynomials [10, 31] if and only if P_M is tri-diagonal [2, 3]. Necessarily, the Jacobi coefficients (i.e., the coefficients of the three-term recurrence [10] that defines the polynomials) of these orthogonal polynomials are then constant.

Example 12. The production matrix of $(c(x), xc(x)^2)$ is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 2 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

An important feature of Riordan arrays is that they have a number of sequence characterizations [9, 16]. The simplest of these is as follows.

Proposition 13. *[16, Theorem 2.1, Theorem 2.2]. Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan array if and only if there exist two sequences $A = [\alpha_0, \alpha_1, \alpha_2, \ldots]$ and $Z = [z_0, z_1, z_2, \ldots]$ with $\alpha_0 \neq 0$, $z_0 \neq 0$ such that*

- $d_{n+1,k+1} = \sum_{j=0}^{\infty} \alpha_j d_{n,k+j}, \quad (k, n = 0, 1, \ldots)$
- $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, \ldots).$

The coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots$ and $z_0, z_1, z_2, \ldots$ are called the A-sequence and the Z-sequence of the Riordan array $M = (g(x), f(x))$, respectively. Letting $A(x)$ be the generating function of the A-sequence and $Z(x)$ be the generating function of the Z-sequence, we have

$$A(x) = \frac{x}{\bar{f}(x)}, \quad Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{d_{0,0}}{g(\bar{f}(x))}\right). \tag{4}$$

The first column of P_M is then generated by $Z(x)$, while the k-th column is generated by $x^{k-1}A(x)$ (taking the first column to be indexed by 0). There is a close link between orthogonal polynomials whose defining three term recurrences have constant coefficients and Riordan arrays whose inverses have tri-diagonal production matrices [2, 3]. We devote this section to examples where the coefficient array $L^{(s)}$ is a Riordan array

$$L^{(s)} = (u(x), v(x)).$$

We shall also assume that $a_0 = b_0 = 1$ for the rest of this section. Note that we have, in this case,

$$L^{(s)^{-1}} = (u(x), v(x))^{-1} = \left(\frac{1}{u(\bar{v}(x))}, \bar{v}(x)\right) = (s(x), \bar{v}(x)).$$

In particular,

$$s(x) = \frac{1}{u(\bar{v}(x))} \Rightarrow u(x) = \frac{1}{s(v(x))}.$$

We have

$$L^{(s)} \cdot M^{-1} = L^{(s)} \cdot (g(x), x)^{-1} = L^{(s)} \cdot \left(\frac{1}{g(x)}, x\right) = \left(\frac{u(x)}{g(x)}, v(x)\right).$$

Looking at inverses, we get

$$(L^{(s)} \cdot M^{-1})^{-1} = M \cdot (L^{(s)})^{-1} = \left(\frac{g(x)}{u(\bar{v}(x))}, \bar{v}(x)\right).$$

We wish to compare the production array of the matrix $(L^{(s)})^{-1}$ with that of the matrix $(L^{(s)} \cdot M^{-1})^{-1}$. For the matrix $(L^{(s)})^{-1} = (u, v)^{-1} = \left(\frac{1}{u(\bar{v}(x))}, \bar{v}(x)\right)$, we have

$$A(x) = \frac{x}{v(x)}, \quad Z(x) = \frac{1}{v}(1 - u(x)), \tag{5}$$

since by assumption $d_{0,0} = 1$. For $(L^{(s)} \cdot M^{-1})^{-1} = \left(\frac{g(x)}{u(\bar{v}(x))}, \bar{v}(x)\right)$ we find that

$$\tilde{A}(x) = \frac{x}{v(x)}, \quad \tilde{Z}(x) = \frac{1}{v(x)}\left(1 - \frac{u(x)}{g(v(x))}\right). \tag{6}$$

In our case (that of $L^{(s)}$ being the coefficient array of a family of orthogonal polynomials), we have

$$L^{(s)} = \left(\frac{1 + \alpha' x + \beta' x^2}{1 + \alpha x + \beta x^2}, \frac{x}{1 + \alpha x + \beta x^2}\right),$$

for suitable values of $\alpha, \beta, \alpha', \beta'$. Then

$$A(x) = \tilde{A}(x) = 1 + \alpha x + \beta x^2,$$

and

$$Z(x) = \frac{1 + \alpha x + \beta x^2}{x}\left(1 - \frac{1 + \alpha' x + \beta' x^2}{1 + \alpha x + \beta x^2}\right) = (\alpha - \alpha') + (\beta - \beta')x,$$

along with

$$\tilde{Z}(x) = \frac{1}{x}\left(1 + \alpha x + \beta x^2 - (1 + \alpha' x + \beta' x^2) \cdot \frac{1}{g\left(\frac{x}{1+\alpha x+\beta x^2}\right)}\right).$$

Thus as expected, $(L^{(s)})^{-1}$ has a tri-diagonal production matrix, which begins

$$\left(\begin{array}{ccccccc} \alpha - \alpha' & 1 & 0 & 0 & 0 & 0 & \cdots \\ \beta - \beta' & \alpha & 1 & 0 & 0 & 0 & \cdots \\ 0 & \beta & \alpha & 1 & 0 & 0 & \cdots \\ 0 & 0 & \beta & \alpha & 1 & 0 & \cdots \\ 0 & 0 & 0 & \beta & \alpha & 1 & \cdots \\ 0 & 0 & 0 & 0 & \beta & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

However, given the form of $\tilde{Z}(x)$, only in the exceptional case when

$$\tilde{Z}(x) = \frac{1}{x}\left(1 + \alpha x + \beta x^2 - \frac{1 + \alpha' x + \beta' x^2}{g\left(\frac{x}{1+\alpha x+\beta x^2}\right)}\right)$$

is of the form $\gamma + \delta x$ will the production matrix of $(L^{(s)} \cdot M^{-1})^{-1}$ be tri-diagonal. Only in this case are the polynomials $Q_n(x)$ orthogonal.

In the case when

$$L^{(s^*)} = (u^*(x), v^*(x)) = (s^*(x), \bar{v^*}(x))^{-1}$$

is a Riordan array, a similar analysis is valid.

Example 14. We look at the case where $a_n = \sum_{k=0}^{n} \binom{2k}{k} C_{n-k} = \binom{2n+1}{n+1}$ and $b_n = \binom{2n}{n}$. In this case, we have

$$f(x) = \frac{c(x)}{\sqrt{1-4x}}, \quad g(x) = \frac{1}{\sqrt{1-4x}}, \quad s(x) = \frac{f(x)}{g(x)} = c(x),$$

and hence $s_n = C_n$.
We find that the coefficient array $\tilde{L}$ of the polynomials $Q_n(x)$ is the array

$$\left(\frac{1-x}{(1+x)^2}, \frac{x}{(1+x)^2} \right),$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 5 & -5 & 1 & 0 & 0 & 0 & \cdots \\ -7 & 14 & -7 & 1 & 0 & 0 & \cdots \\ 9 & -30 & 27 & -9 & 1 & 0 & \cdots \\ -11 & 55 & -77 & 44 & -11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have

$$Q_n(x) = \sum_{k=0}^{n} \frac{2n+1}{2k+1} \binom{n+k}{2k} (-1)^{n-k} x^k.$$

The inverse array $\tilde{L}^{-1}$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 10 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 35 & 21 & 7 & 1 & 0 & 0 & \cdots \\ 126 & 84 & 36 & 9 & 1 & 0 & \cdots \\ 462 & 330 & 165 & 55 & 11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (\text{A111418}).$$

By its form, we see that $\left(\frac{1-x}{(1+x)^2}, \frac{x}{(1+x)^2} \right)$ is in this case the coefficient array of a family of orthogonal polynomials. This is verified by noting that

$$\tilde{Z}(x) = \frac{(1+x)^2}{x} \left(1 - \frac{\frac{1}{1+x}}{\frac{1}{\sqrt{1-\frac{4x}{(1+x)^2}}}} \right) = 3 + x.$$

In fact, the production matrix of $\left(\frac{1-x}{(1+x)^2}, \frac{x}{(1+x)^2}\right)^{-1}$ is given by

$$\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 2 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and so we have

$$Q_n(x) = (x-2)Q_{n-1}(x) - Q_{n-2}(x), \quad Q_0(x) = 1,\, Q_1(x) = x-3.$$

Looking at $M^{-1}\tilde{L}^{-1}$, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 20 & 6 & 2 & 1 & 0 & 0 & \cdots \\ 70 & 20 & 6 & 2 & 1 & 0 & \cdots \\ 252 & 70 & 20 & 6 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 10 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 35 & 21 & 7 & 1 & 0 & 0 & \cdots \\ 126 & 84 & 36 & 9 & 1 & 0 & \cdots \\ 462 & 330 & 165 & 55 & 11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 9 & 5 & 1 & 0 & 0 & \cdots \\ 14 & 28 & 20 & 7 & 1 & 0 & \cdots \\ 42 & 90 & 75 & 35 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{(A039599)}.$$

This latter array is the inverse of $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)^{-1}$. The array $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$ is the coefficient array of the orthogonal polynomials whose moments are the Catalan numbers C_n.

Turning now to s_n^*, we have $s_n^* = C_{n+1}$. In this case, we find that

$$\tilde{L}^* = L^{(s^*)} \cdot M^{-1} = (u^*(x), v^*(x)) = \left(\frac{1-x}{(1+x)^3}, \frac{x}{(1+x)^2}\right).$$

This array represents the coefficients of a family of polynomials $R_n(x)$ that are "almost orthogonal", in the sense that the production matrix of the inverse of this matrix is of the form

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 2 & 1 & 0 & 0 & \cdots \\ -2 & 0 & 1 & 2 & 1 & 0 & \cdots \\ 2 & 0 & 0 & 1 & 2 & 1 & \cdots \\ -2 & 0 & 0 & 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus we have

$$R_n(x) = (x-2)R_{n-1}(x) - R_{n-2}(x) - \tilde{z}_{n-1}, \quad R_0(x) = 1,\ R_1(x) = x-4,$$

where $\tilde{z}_n$ is the sequence $0, 0, 2, -2, 2, -2, 2, \ldots$.

The inverse matrix $(u^*(x), v^*(x))^{-1}$, which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 15 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 56 & 28 & 8 & 1 & 0 & 0 & \cdots \\ 210 & 120 & 45 & 10 & 1 & 0 & \cdots \\ 792 & 495 & 220 & 66 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

has general term $\binom{2n+2}{n+k+2}$, and satisfies

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 20 & 6 & 2 & 1 & 0 & 0 & \cdots \\ 70 & 20 & 6 & 2 & 1 & 0 & \cdots \\ 252 & 70 & 20 & 6 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 15 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 56 & 28 & 8 & 1 & 0 & 0 & \cdots \\ 210 & 120 & 45 & 10 & 1 & 0 & \cdots \\ 792 & 495 & 220 & 66 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 14 & 14 & 6 & 1 & 0 & 0 & \cdots \\ 42 & 48 & 27 & 8 & 1 & 0 & \cdots \\ 132 & 165 & 110 & 44 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(c(x)^2, xc(x)^2\right),$$

where the first column of this last matrix (which is A039598) is given by $s_n^* = C_{n+1}$. Note that the first column elements of $(u^*(x), v^*(x))^{-1}$ are given by

$$\sum_{k=0}^{n} \binom{2k}{k} C_{n-k+1} = \sum_{k=0}^{n} \binom{2n-2k}{n-k} C_{k+1} = \binom{2n+2}{n+2}, \quad \text{(A001791)}.$$

The inverse of this last matrix is the coefficient array

$$L^{(s^*)} = \left(\frac{1}{(1+x)^2}, \frac{x}{(1+x)^2}\right) = (c(x)^2, xc(x)^2)^{-1}$$

for the family of orthogonal polynomials

$$\tilde{R}_n(x) = (x-2)\tilde{R}_{n-1}(x) - \tilde{R}_{n-2}(x), \quad \tilde{R}_0(x) = 1,\ \tilde{R}_1(x) = x-2.$$

These polynomials have moments given by C_{n+1}. We note that we have

$$\tilde{Z}^*(x) = \frac{(1+x)^2}{x}\left(1 - \frac{\frac{1}{(1+x)^2}}{\frac{1}{\sqrt{1-\frac{4x}{(1+x)^2}}}}\right) = \frac{4+3x+x^2}{1+x},$$

where $\frac{4+3x+x^2}{1+x}$ expands to give $4, -1, 2, -2, 2, -2, 2, \ldots$.

Example 15. In the last example, it happened that the family of polynomials $Q_n(x)$ constituted a family of orthogonal polynomials. This is not true in general. In this example, we let

$$a_n = \sum_{k=0}^{n} F_{k+1} C_{n-k}, \quad b_n = F_{k+1},$$

where F_n A000045 denotes the n-th Fibonacci number. Then we have

$$f(x) = \frac{c(x)}{1-x-x^2}, \quad g(x) = \frac{1}{1-x-x^2}, \quad s(x) = \frac{f(x)}{g(x)}$$

and $s_n = C_n$. Thus again, $L^{(s)} = \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$ and hence

$$\tilde{L} = \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right) \cdot \left(\frac{1}{1-x-x^2}, x\right)^{-1} = \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right) \cdot (1-x-x^2, x).$$

Thus

$$\tilde{L} = \left(\frac{1+3x+3x^2+3x^3+x^4}{(1+x)^5}, \frac{x}{(1+x)^2}\right),$$

whose inverse has production matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 0 & 1 & 2 & 1 & 0 & \cdots \\ -9 & 0 & 0 & 1 & 2 & 1 & \cdots \\ 14 & 0 & 0 & 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This follows since

$$\tilde{Z}(x)=\frac{1}{x}\left((1+x)^2-\frac{1+x}{g\left(\frac{x}{(1+x)^2}\right)}\right)=\frac{2+7x+7x^2+4x^3+x^4}{(1+x)^3},$$

which expands to

$$2,1,-2,5,-9,14,-20,27,-35,\ldots.$$

This means that $\tilde{L}$ is the coefficient array of the "almost-orthogonal" polynomials $Q_n(x)$ that satisfy

$$Q_n(x)=(x-2)Q_{n-1}-Q_{n-2}(x)-\tilde{z}_{n-1},$$

where $\tilde{z}_n$ is the sequence $0,0,-2,5,-9,14,-20,27,-35,\ldots$.
Looking now at s^*, we have

$$\tilde{L}^*=\left(\frac{1}{(1+x)^2},\frac{x}{(1+x)^2}\right)\cdot(1-x-x^2,x)=\left(\frac{1+3x+3x^2+3x^3+x^4}{(1+x)^6},\frac{x}{(1+x)^2}\right).$$

The inverse of $\tilde{L}^*$ then has production array

$$\left(\begin{array}{ccccccc} 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 2 & 1 & 0 & 0 & \cdots \\ 6 & 0 & 1 & 2 & 1 & 0 & \cdots \\ -15 & 0 & 0 & 1 & 2 & 1 & \cdots \\ 29 & 0 & 0 & 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We have

$$\tilde{Z}^*(x)=\frac{1}{x}\left((1+x)^2-\frac{1}{g\left(\frac{x}{(1+x)^2}\right)}\right)=\frac{3+12x+17x^2+14x^3+6x^4+x^5}{(1+x)^3},$$

which expands to $3,0,-1,6,-15,29,\ldots$. Thus in this case we have

$$R_n(x)=(x-2)R_{n-1}(x)-R_{n-2}(x)-\tilde{z}^*_{n-1},\quad R_0(x)=1,\ R_1(x)=x-3.$$

Example 16. For any element $T=(u(x),x)$ of the Appell subgroup of Riordan arrays [25], it is clear that

$$\mathbb{H}_n(Ta,Tb)=\mathbb{H}_n(a,b).$$

This is so because for the pair Ta, Tb, we have

$$s(x)=\frac{u(x)f(x)}{u(x)g(x)}=\frac{f(x)}{g(x)},$$

by the fundamental theorem of Riordan arrays. Thus for instance we have

$$\mathbb{H}_n(a_n, b_n) = \mathbb{H}_n(\sum_{k=0}^{n} a_k, \sum_{k=0}^{n} b_k),$$

since the partial sum operator is equal to the Riordan array $\left(\frac{1}{1-x}, x\right)$.

Similarly we have

$$\mathbb{H}_n(a_n, b_n) = \mathbb{H}_n(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k}, \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} b_{n-2k}),$$

where in this case the Riordan array is $\left(\frac{1}{1-x^2}, x\right)$.

5 Further examples

We list below a table showing a small sample of Hurwitz transforms for the pairs of sequences shown.

a_n	b_n	$s(x)$	Hurwitz transform
$(-1)^n C_{n+1}$	$(-1)^n C_n$	$c(-x)$	$1, 1, 1, 1, 1, 1, \ldots$
C_n	C_{n+1}	$\frac{1}{c(x)}$	$1, 1, -2, 1, 3, 1, -4, 1, 5, 1, \ldots$
C_{n+1}	C_n	$c(x)$	$1, -1, 1, 1, 1, -1, 1, 1, 1, -1, 1, \ldots$
$(-1)^n \binom{1}{n}$	$(-1)^n (C_n + C_{n-1})$	$\frac{1}{c(-x)}$	$1, -1, -2, 1, 3, -1, -4, 1, 5, \ldots$
C_n	0^n	$c(x)$	$1, -1, 1, 1, 1, -1, 1, 1, 1, -1, 1, \ldots$
C_n	$(-1)^n \binom{1}{n}$	$\frac{c(x)}{1-x}$	$1, -2, 0, 2, -1, 1, -1, -5, 0, 5, 1, \ldots$
C_n	1	$(1-x)c(x)$	$1, 0, 1, -1, -1, 1, -4, 0, -4, -1, 1, \ldots$
C_n	$\frac{1}{1-2n}\binom{2n}{n}$	$\frac{c(x)}{\sqrt{1-4x}}$	$1, -3, 1, 5, 1, -7, 1, 9, 1, -11, 1, \ldots$
$C_n + C_{n+1}$	C_n	$1 + c(x)$	$2, -1, 3, 1, 4, -1, 5, 1, 6, -1, 7, \ldots$
$C_{\frac{n}{2}} \frac{1+(-1)^n}{2}$	2^n	$(1-2x)c(x^2)$	$1, 2, -3, 3, -3, 4, 5, 5, 5, 6, -7, \ldots$
$T_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k}$	T_{n+1}	$\frac{\sqrt{1-2x-3x^2}+3x-1}{x(1-3x)}$	$1, 2, -2, 4, 0, 8, 8, 16, -16, 32, 0, \ldots$

6 Conclusion

Since the notion of Hurwitz transform proposed here has been linked to the Hankel transform in an easily understood way, it may be said that this notion does not add much to what is already known. What it does add, however, is a fresh perspective, both on the Hankel transform, and on the applications of the Hurwitz matrix. Some natural questions arise in this context. What known sequences are the Hurwitz transforms of pairs of sequences? Are there sequences which cannot be the Hurwitz transform of a sequence? Given that many pairs of sequences may have the same Hurwitz transform, what notion of inverse transform can we formulate?

With regard to new perspectives, the Hurwitz transform makes us look at the pair (h_n, h_n^*) whenever we wish to study h_n. An interesting example of this is the case of the Narayana polynomials,

$$s_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n+1}{k}\binom{n}{k}x^k.$$

It is well known that in this case, we have

$$h_n = x^{\binom{n+1}{2}}.$$

In looking at the pair (h_n, h_n^*), we discover that

$$h_n^* = x^{\binom{n+1}{2}}\frac{1-x^{n+2}}{1-x}.$$

Other examples of the pairing (h_n, h_n^*) have been studied in different contexts [1, 5, 6, 20].

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