IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 14029–14031 doi:10.1088/1751-8113/40/46/N01

COMMENT

Comment on 'Analytical results for a Bessel function times Legendre polynomials class integrals'

P J Cregg[1] **and P Svedlindh**[2]

[1] Materials Characterisation and Processing Group, Waterford Institute of Technology, Waterford, Ireland
[2] Department of Engineering Sciences, Uppsala University, Box 534, SE-751 21 Uppsala, Sweden

E-mail: pjcregg@wit.ie

Received 4 June 2007
Published 31 October 2007
Online at stacks.iop.org/JPhysA/40/14029

Abstract
A result is obtained, stemming from Gegenbauer, where the products of certain Bessel functions and exponentials are expressed in terms of an infinite series of spherical Bessel functions and products of associated Legendre functions. Closed form solutions for integrals involving Bessel functions times associated Legendre functions times exponentials, recently elucidated by Neves *et al* (*J. Phys. A: Math. Gen.* **39** L293), are then shown to result directly from the orthogonality properties of the associated Legendre functions. This result offers greater flexibility in the treatment of classical Heisenberg chains and may do so in other problems such as occur in electromagnetic diffraction theory.

PACS numbers: 71.70.Gm, 73.43.Cd, 41.20.−q, 42.25.Fx, 42.60.Jf

Recently, Neves *et al* presented a closed-form exact solution to an integral, given by equation (5), which they encountered in the solution of problems in electromagnetic vector diffraction theory [1]. Subsequently, Koumandos [2] drew attention to this integral as a special case of an integral result of Gegenbauer [3]. One of the present authors previously used a similar integral, also obtained as a special case of the same Gegenbauer integral, in treating the partition functions of magnetic particles with anisotropy [4]. More recently, the present authors have been making use of this and similar results in treating the partition functions of classical isotropic Heisenberg chains [5–7]. Historically, Podolsky and Pauling [8] presented the same result as dealt with by Neves *et al*. Nevertheless, the present authors share the view of Neves *et al* that results of this nature are of use, whilst not easily available in integral tables, and are not generally well known. A wealth of useful results, many due originally to Gegenbauer, involving associated Legendre functions are to be found in Watson's classic treatise on Bessel functions [9]. However, these are often overlooked, perhaps owing to their expression in terms of the more general Gegenbauer polynomials, even though these are easily related to associated Legendre functions. Here we obtain a result stemming from Gegenbauer, which expresses a specific product of a Bessel function and an exponential as an infinite series of products of spherical Bessel functions and Legendre polynomials, or more generally associated Legendre functions. In this, the result of Neves *et al* is seen as a direct consequence of the orthogonality properties of the associated Legendre functions, whereby the infinite series collapses to a single term. These underlying results have been of particular use to the present authors in their treatment of Heisenberg chains, applicable to molecular magnets [10], allowing the order of integration to be altered in multiple integrals. These or other advantages may also arise for the other problems in which these mathematical forms occur. In the case of Heisenberg chains these mathematical forms appear in terms of the modified spherical Bessel functions of the first kind. The derivations are presented for the Bessel form but can be easily replicated for the modified form.

From Watson [9] section 11.5 equation (9) we have an expansion due to Gegenbauer

$$\frac{J_{\nu-\frac{1}{2}}(R\sin\theta\sin\psi)}{(R\sin\theta\sin\psi)^{\nu-\frac{1}{2}}}\exp(\mathrm{i}R\cos\theta\cos\psi) = \frac{2^{2\nu}[\Gamma(\nu)]^2}{\sqrt{2\pi}} \times \sum_{n=0}^{\infty}\frac{\mathrm{i}^n n!(\nu+n)}{\Gamma(2\nu+n)}\frac{J_{\nu+n}(R)}{R^{\nu}}C_n^{\nu}(\cos\theta)C_n^{\nu}(\cos\psi) \tag{1}$$

valid for all ν, where $J_n(x)$ are the Bessel functions, $\Gamma(\nu)$ is the Gamma function and $C_n^{\nu}(\cos\theta)$ are the Gegenbauer polynomials which relate to the associated Legendre functions $P_n^m(\cos\theta)$ via [2]

$$P_n^m(\cos\theta) = (-1)^m\frac{(2m)!}{2^m m!}\sin^m\theta C_{n-m}^{m+\frac{1}{2}}(\cos\theta). \tag{2}$$

For $m \neq 0, m \leqslant n$, making the substitution $m = \nu - \frac{1}{2}$ in the Gegenbauer expansion and noting the properties of the Gamma function [11] we can write in terms of the associated Legendre functions

$$J_m(R\sin\theta\sin\psi)\exp(\mathrm{i}R\cos\theta\cos\psi) = \sum_{n=0}^{\infty}\mathrm{i}^{n-m}(2n+1)j_n(R) \times \frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\psi), \tag{3}$$

where $j_n(x)$ are the spherical Bessel functions of the first kind. The reader's attention is drawn to the fact that whilst θ and ψ both appear in the arguments of each of the functions on the left, on the right they are decoupled. Multiplying by $P_n^m(\cos\theta)$ and applying the orthogonality condition for associated Legendre functions, namely

$$\int_0^{\pi}P_n^m(\cos\theta)P_l^m(\cos\theta)\sin\theta\,\mathrm{d}\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n,l}, \tag{4}$$

where $\delta_{n,l}$ is the Kronecker delta function, leads to the collapse of the series to a single term

$$\int_0^{\pi}J_m(R\sin\theta\sin\psi)\exp(\mathrm{i}R\cos\theta\cos\psi)P_n^m(\cos\theta)\sin\theta\,\mathrm{d}\theta = 2\mathrm{i}^{n-m}j_n(R)P_n^m(\cos\psi), \tag{5}$$

which is the result of Neves *et al* [1] or Podolsky and Pauling [8], or indeed a special case of Gegenbauer's finite integral [2, 3].

Acknowledgments

PJC wishes to thank the Royal Irish Academy for a Senior Visiting Fellowship in 2004 which enabled him to undertake work at Uppsala University and Enterprise Ireland for an

International Collaboration grant IC/2005/75 to return to Uppsala University to complete the work. PJC is also grateful to the WIT sabbatical programme that facilitated continued work at Uppsala University. Kieran Murphy of WIT is thanked for advice on preparation of the manuscript.

References

[1] Neves A A R, Padilha L A, Fontes A, Rodriguez E, Cruz C H B, Barbosa L C and Cesar C L 2006 Analytical results for a Bessel function times Legendre polynomials class integrals *J. Phys. A: Math. Gen.* **39** L293–L296

[2] Koumandos S 2007 On a class of integrals involving a Bessel function times Gegenbauer polynomials *Int. J. Math. Math. Sci.* **2007** 73750

[3] Gegenbauer L 1877 *Wien. Sitz.* **75** 221 ([9, section 12.14])

[4] Cregg P J and Bessais L 1999 Series expansions for the magnetisation of a solid superparamagnetic system of non-interacting particles with anisotropy *J. Magn. Magn. Mater.* **202** 554–64

[5] Joyce G S 1967 Classical Heisenberg model *Phys. Rev.* **155** 478–91

[6] Cregg P J, García-Palacios J L and Svedlindh P 2007 Low-field susceptibility of classical Heisenberg chains with arbitrary exchange constants 11 Int. Conf. on Magnetic Fluids (Košice, Slovakia, 2007) *J. Magn. Magn. Mater.* at press

[7] Cregg P J, García-Palacios J L and Svedlindh P 2007 Linear and non-linear susceptibility of classical Heisenberg chains with arbitrary exchange constants *Phys. Rev.* B in preparation

[8] Podolsky B and Pauling L 1929 The momentum distribution in hydrogen-like atoms *Phys. Rev.* **34** 109–16

[9] Watson G N 1995 *A Treatise on the Theory of Bessel Functions* (Cambridge: Cambridge University Press)

[10] Luban M 2004 New issues in zero dimensions: magnetic molecules *J. Magn. Magn. Mater.* **272–276** e635–e641

[11] Abramowitz M and Stegun I A (ed) 1965 *Handbook of Mathematical Functions* (New York: Dover)