UNIVERSITY COLLEGE CORK

A Study of Integer Sequences, Riordan Arrays, Pascal-like Arrays and Hankel Transforms

by
Paul Barry

A thesis submitted in partial fulfillment for the
degree of Doctor of Philosophy

in the
College of Science, Engineering and Food Science
Department of Mathematics
Head of Department: Professor Martin Stynes
Supervisor: Professor Patrick Fitzpatrick, Head of College of Science,
Engineering and Food Science

July 2009

Contents

Declaration of Authorship

I, PAUL BARRY, declare that this thesis titled, 'A Study of Integer Sequences, Riordan Arrays, Pascal-like Arrays and Hankel Transforms' and the work presented in it are my own. I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University.
2. Where I have consulted the published work of others, this is always clearly attributed.
3. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
4. I have acknowledged all main sources of help.
5. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:
____

Date:
____

Abstract

We study integer sequences and transforms that operate on them. Many of these transforms are defined by triangular arrays of integers, with a particular focus on Riordan arrays and Pascal-like arrays. In order to explore the structure of these transforms, use is made of methods coming from the theory of continued fractions, hypergeometric functions, orthogonal polynomials and most importantly from the Riordan groups of matrices. We apply the Riordan array concept to the study of sequences related to graphs and codes. In particular, we study sequences derived from the cyclic groups that provide an infinite family of colourings of Pascal's triangle. We also relate a particular family of Riordan arrays to the weight distribution of MDS error-correcting codes. The Krawtchouk polynomials are shown to give rise to many different families of Riordan arrays. We define and investigate Catalan-number-based transformations of integer sequences, as well as transformations based on Laguerre and related polynomials. We develop two new constructions of families of Pascal-like number triangles, based respectively on the ordinary Riordan group and the exponential Riordan group, and we study the properties of sequences arising from these constructions, most notably the central coefficients and the generalized Catalan numbers associated to the triangles. New exponential-factorial constructions are developed to further extend this theory. The study of orthogonal polynomials such as those of Chebyshev, Hermite, Laguerre and Charlier are placed in the context of Riordan arrays, and new results are found. We also extend results on the Stirling numbers of the first and second kind, using exponential Riordan arrays. We study the integer Hankel transform of many families of integer sequences, exploring links to related orthogonal polynomials and their coefficient arrays. Two particular cases of power series inversion are studied extensively, leading to results concerning the Narayana triangles.

Acknowledgements

I am indebted to Professor Patrick Fitzpatrick of University College Cork for his encouragement in this endeavour.

Special thanks must go to my wife, Mary, for her constant support and understanding during the writing of this work. Special thanks are also due to Nadine and Peter, both of whom provided many reasons for continuing when the path was less than clear.

I have gained much by working in collaboration with Prof. Dr. Predrag Rajković and Dr. Marko Petković, of the University of Niš, Serbia. It is a pleasure to acknowledge this.

All who work in the area of integer sequences are completely in the debt of Neil Sloane, whose Online Encyclopedia of Integer Sequences must stand as one of the greatest achievements of a single person in modern times. Gratitude is also expressed to Jeffrey Shallit, editor-in-chief of the Journal of Integer Sequences, for his continued promotion of the growing literature surrounding integer sequences.

Chapter 1

Introduction

1.1 Overview of this work

The central object of this work is the study of integer sequences, using both classical methods and methods that have emerged more recently, and in particular the methods that have been inspired by the concept of Riordan array. A leading theme is the use of transformations of integer sequences, many of them defined by Riordan arrays. In this context, a transformation that has attracted much attention in recent years stands out. This is the Hankel transform of integer sequences. This is not defined by Riordan arrays, but in this work we study some of the links that exist between this transformation and Riordan arrays. This link is determined by the nature of the sequences subjected to the Hankel transforms, and in the main, we confine ourselves to sequences which themselves are closely linked to Riordan arrays. This aids in the study of the algebraic and combinatorial nature of this transform, when applied to such sequences.

Many of the sequences that we will study in the context of the Hankel transform are moments sequences, defined by measures on the real line. This builds a bridge to the world of real analysis, and indeed to functional analysis. Associated to these sequences is the classical theory of orthogonal polynomials, continued fractions, and lattice paths.

An important aspect of this work is the construction of so-called "Pascal-like" number arrays. In many cases, we construct such arrays using ordinary, exponential or generalized Riordan arrays, which are found to give a uniform approach to certain of these constructions. We also look at other methods of construction of Pascal-like arrays where appropriate, to provide a contrast with the Riordan array inspired constructions.

The plan of this work is as follows. In this Introduction, we give an overview of the work and outline its structure.

In Chapter 2 we review many of the elements of the theory of integer sequences that will be important in ensuing chapters, including different ways of defining and describing an integer sequence. Preparatory ground is laid to study links between certain integer sequences, orthogonal polynomials and continued fractions, and the Hankel transform. This also includes a look at hypergeometric series. We finish this chapter by looking at different ways of defining triangular arrays of integers, some of which are simple Pascal-like arrays. Illustrative examples are to be found throughout this chapter.

In Chapter 3, based on the published work [19], we explore links between the cyclic groups, integer sequences, and decompositions of Pascal's triangle. The circulant nature of the associated adjacency matrices is exploited, allowing us to use Fourier analysis techniques to achieve our results. We finish by looking at the complete graphs as well.

In Chapter 4, we review the notion of Riordan group, and some of its generalizations. Examples are given that will be used in later chapters. The chapter ends by looking at the notion of production matrices.

In Chapter 5, we briefly introduce the topic of the so-called "Deleham DELTA construction." This method of constructing number triangles is helpful in the sequel. To our knowledge, this is the first time that this construction has been analyzed in the manner presented here.

In Chapter 6, based on the published article [15], we study certain transformations on integer sequences defined by Riordan arrays whose definitions are closely related to the generating function of the Catalan numbers. These transformations in many cases turn out to be well-known and important. Subsequent chapters explore links between these matrices and the structure of the Hankel transform of certain sequences.

In Chapter 7 we give an example of the application of the theory of Riordan arrays to the area of MDS codes. This chapter has appeared as [20].

In Chapter 8, based on the published paper [18], we apply the theory of exponential Riordan arrays to explore certain binomial and factorial-based transformation matrices. These techniques allow us to easily introduce generalizations of these transformations and to explore some of the properties of these new transformations. Links to classical orthogonal polynomials (e.g., the Laguerre polynomials) and classical number arrays are made explicit.

In Chapter 9.1 we continue to investigate links between certain Riordan arrays and orthogonal polynomials. We also study links between exponential Riordan arrays and the umbral calculus. This chapter has appeared as [22].

In Chapter 10 we use the formalism of Riordan arrays to define and analyze certain Pascal-like triangles. Links are drawn between sequences that emerge from this study and the reversion of certain simpler sequences. We finish this chapter by looking at alternative ways of constructing Pascal-like triangles, based on factorial and exponential methods. In this section we introduce and study the notion of sequence-specific generalized exponential arrays. An earlier version of this chapter has appeared as [16].

In Chapter 11 we continue the exploration of the construction of Pascal-like triangles, this time using exponential Riordan arrays as the medium of construction. In the final section we briefly indicate how some of the methods introduced in the final section of Chapter 10 can be used to build a family of generalized Narayana triangles. An earlier version of this chapter has appeared as [17].

In Chapter 12 we give a brief introduction to the theory of the Hankel transform of integer sequences, using relevant examples to prepare the ground for further chapters.

In Chapter 13 we extend the study already commenced in Chapter 11, and we also look at the Hankel transforms of some of the sequences that emerge from this extension.

In Chapter 14 we calculate the Hankel transform of sequences related to the central trinomial coefficients, and we conjecture the form of the Hankel transform of other associated sequences. Techniques related to Riordan arrays and orthogonal polynomials are used in this chapter. Elements of this chapter have been presented at the Applied Linear Algebra (ALA2008) conference in honour of Ivo Marek, held in the University of Novi Sad, May 2008. A forthcoming paper based on this in collaboration with Dr. Predrag Rajković and Dr. Marko Petković has been submitted to the Journal of Applied Linear Algebra.

The author wishes to acknowledge what he has learnt through collaborating with Dr. Predrag Rajković and Dr. Marko Petković, both of the University of Niš, Serbia. This collaboration centred initially on Hankel transform methods first deployed in [61], and subsequently used in [188], as well as in the chapters concerning the calculation of the Hankel transform of integer sequences.

Chapter 2

Preliminary Material

2.1 Integer sequences

We denote by $\mathbb{N}$ the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

When we include the element 0, we obtain the set of non-negative integers $\mathbb{N}_0$, or

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\}.$$

$\mathbb{N}_0$ is an ordered semigroup for the binary operation $+ : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$. $\mathbb{N}_0$ is a subset of the ring of integers $\mathbb{Z}$ obtained from $\mathbb{N}_0$ by adjoining to $\mathbb{N}_0$ the element $-n$ for each $n \in \mathbb{N}$, where $-n$ is the unique element such that $n + (-n) = 0$.
By an integer sequence we shall mean an element of the set $\mathbb{Z}^{\mathbb{N}_0}$. Regarded as an infinite group, the set $\mathbb{Z}^{\mathbb{N}_0}$ is called the Baer-Specker group [57, 199].
Thus a (one-sided) integer sequence $a(n)$ is a mapping

$$a : \mathbb{N}_0 \to \mathbb{Z}$$

where $a(n)$ denotes the image of $n \in \mathbb{N}_0$ under this mapping. The set of such integer sequences $\mathbb{Z}^{\mathbb{N}_0}$ inherits a ring structure from the image space $\mathbb{Z}$. Thus two sequences $a(n)$ and $b(n)$ define a new sequence $(a + b)(n)$ by the rule

$$(a + b)(n) = a(n) + b(n),$$

and similarly we obtain a sequence $(ab)(n)$ by the rule

$$(ab)(n) = a(n)b(n).$$

The additive inverse of the sequence $a(n)$ is the sequence with general term $-a(n)$.

An additional binary operation, called *convolution*, may be defined on sequences as follows:

$$(a * b)(n) = \sum_{k=0}^{n} a(k)b(n - k).$$

We then have $a * b(n) = b * a(n)$. In addition, the sequence $\delta_n = 0^n = (1, 0, 0, 0, \ldots)$ plays a special role for this operation, since we have $a * \delta(n) = a(n)$ for all n.
A related binary operation is that of the *exponential* convolution of two sequences, defined as $\sum_{k=0}^{n} \binom{n}{k} a(k) b(n-k)$.

Frequently we shall use the notation a_n for the term $a(n)$. For a sequence a_n, we define its *binomial* transform to be the sequence

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

This transformation has many interesting properties, some of which will be examined later. In the sequence, we shall use the notation $\mathbf{B}$ to denote the matrix with general term $\binom{n}{k}$.

Integer sequences may be characterized in many ways. In the sequel, we shall frequently use the following methods:

1. Generating functions.
2. Recurrences.
3. Moments.
4. Combinatorial definition.

We shall examine each of these shortly.

2.2 The On-Line Encyclopedia of Integer Sequences

Many integer sequences and their properties are to be found electronically on the On-Line Encyclopedia of Sequences [205, 206]. Sequences therein are referred to by their "A" number, which takes the form of A*nnnnnn*. We shall follow this practice, and refer to sequences by their "A" number, should one exist.

2.3 Polynomials

We let $\mathcal{R}$ denote an arbitrary ring. Let x denote an indeterminate. Then an expression of the form

$$P(x) = \sum_{k=0}^{n} a_k x^k,$$

where $a_i \in \mathcal{R}$ for $0 \leq i \leq n$ is called a *polynomial* in the unknown x over the ring $\mathcal{R}$. If $a_n \neq 0$ then n is called the *degree* of the polynomial P.

We denote by $\mathcal{R}[x]$ the set of polynomials over the ring $\mathcal{R}$. The set of polynomials over $\mathcal{R}$ inherits a ring structure from the base ring $\mathcal{R}$. For instance, if $P, Q \in \mathcal{R}[x]$, where

$$P(x) = \sum_{k=0}^{n_P} a_k x^k$$

and

$$Q(x) = \sum_{i=0}^{n_Q} b_i x^i,$$

then we define $P + Q \in \mathcal{R}[x]$ as the element

$$(P+Q)(x) = \sum_{j=0}^{max(n_P,n_Q)} (a_j + b_j)x^j,$$

where we extend either the a_k or the b_i by zero values as required.
A polynomial $P(x) = \sum_{k=0}^{n_P} a_k x^k$ is called *monic* if the coefficient of the highest order term is 1.
A *polynomial sequence* with values in $\mathcal{R}[x]$ is an element of $\mathcal{R}[x]^{\mathbb{N}_0}$. An example of an important sequence of polynomials is the family of Chebyshev polynomials of the second kind

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

The Chebyshev polynomials of the first kind $(T_n(x))_{n\geq 0}$ are defined by

$$T_n(x) = \frac{n + 2 \cdot 0^n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k+0^{n-k}} \binom{n-k}{k} (2x)^{n-2k}.$$

The Bessel polynomials $y_n(x)$ are defined by

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} x^k$$

(see [108]). The reverse Bessel polynomials are then given by

$$\Theta_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} x^{n-k}.$$

2.4 Orthogonal polynomials

By an *orthogonal polynomial sequence* $(p_n(x))_{n\geq 0}$ we shall understand [53, 99] an infinite sequence of polynomials $p_n(x)$, $n \geq 0$, with real coefficients (often integer coefficients) that are mutually orthogonal on an interval $[x_0, x_1]$ (where $x_0 = -\infty$ is allowed, as well as $x_1 = \infty$), with respect to a weight function $w : [x_0, x_1] \to \mathbb{R}$:

$$\int_{x_0}^{x_1} p_n(x)p_m(x)w(x)dx = \delta_{nm}\sqrt{h_n h_m},$$

where

$$\int_{x_0}^{x_1} p_n^2(x)w(x)dx = h_n.$$

We assume that w is strictly positive on the interval (x_0, x_1). Every such sequence obeys a so-called "three-term recurrence" :

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

for coefficients a_n, b_n and c_n that depend on n but not x. We note that if

$$p_j(x) = k_j x^j + k_j' x^{j-1} + \dots \qquad j = 0, 1, \dots$$

then

$$a_n = \frac{k_{n+1}}{k_n}, \qquad b_n = a_n\left(\frac{k_{n+1}'}{k_{n+1}} - \frac{k_n'}{k_n}\right), \qquad c_n = a_n\left(\frac{k_{n-1}h_n}{k_n h_{n-1}}\right).$$

Since the degree of $p_n(x)$ is n, the coefficient array of the polynomials is a lower triangular (infinite) matrix. In the case of monic orthogonal polynomials the diagonal elements of this array will all be 1. In this case, we can write the three-term recurrence as

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0.$$

The *moments* associated to the orthogonal polynomial sequence are the numbers

$$\mu_n = \int_{x_0}^{x_1} x^n w(x)dx.$$

We can find $p_n(x)$, α_n and β_n from a knowledge of these moments. To do this, we let Δ_n be the Hankel determinant $|\mu_{i+j}|_{i,j\geq 0}^n$ and $\Delta_{n,x}$ be the same determinant, but with the last row equal to $1, x, x^2, \dots$. Then

$$p_n(x) = \frac{\Delta_{n,x}}{\Delta_{n-1}}.$$

More generally, we let $H\begin{pmatrix} u_1 & \dots & u_k \\ v_1 & \dots & v_k \end{pmatrix}$ be the determinant of Hankel type with (i, j)-th term $\mu_{u_i+v_j}$. Let

$$\Delta_n = H\begin{pmatrix} 0 & 1 & \dots & n \\ 0 & 1 & \dots & n \end{pmatrix}, \qquad \Delta' = H\begin{pmatrix} 0 & 1 & \dots & n-1 & n \\ 0 & 1 & \dots & n-1 & n+1 \end{pmatrix}.$$

Then we have

$$\alpha_n = \frac{\Delta_n'}{\Delta_n} - \frac{\Delta_{n-1}'}{\Delta_{n-1}}, \qquad \beta_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}.$$

Given a family of monic orthogonal polynomials

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0,$$

we can write

$$p_n(x) = \sum_{k=0}^{n} a_{n,k}x^k.$$

Then we have

$$\sum_{k=0}^{n+1} a_{n+1,k}x^k = (x - \alpha_n)\sum_{k=0}^{n} a_{n,k}x^k - \beta_n \sum_{k=0}^{n-1} a_{n-1,k}x^k$$

from which we deduce

$$a_{n+1,0} = -\alpha_n a_{n,0} - \beta_n a_{n-1,0} \tag{2.1}$$

and

$$a_{n+1,k} = a_{n,k-1} - \alpha a_{n,k} - \beta_n a_{n-1,k} \tag{2.2}$$

2.5 Power Series

Again, we let $\mathcal{R}$ denote an arbitrary ring. An expression of the form

$$p(x) = \sum_{k=0}^{\infty} a_k x^k,$$

is called a (formal) power series in the indeterminate x. a_k is called the k-th coefficient of the power series. We denote by $\mathcal{R}[[x]]$ the set of formal power series in x over the ring $\mathcal{R}$ [210]. $\mathcal{R}(x)$ is a ring. For instance, if

$$q(x) = \sum_{k=0}^{\infty} b_i x^i,$$

then we can define the sum of p and q as

$$(p + q)(x) = \sum_{j=0}^{\infty}(a_j + b_j)x^j.$$

Example 1. We consider the power series $\sum_{k=0}^{\infty} x^k$. Here, the k-th coefficient of the power series is 1. If for instance $x \in \mathbb{C}$ is a complex number with $|x| < 1$, then it is known that

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}.$$

2.6 Ordinary generating functions

For a sequence a_n, we define its *ordinary generating function* (o.g.f.) to be the power series

$$f(x) = \sum_{k=0}^{\infty} a_n x^n.$$

Thus a_n is the coefficient of x^n in the power series $f(x)$. We often denote this by

$$a_n = [x^n]f(x).$$

Example 2. The sequence 0^n. The sequence with elements $1, 0, 0, 0, \ldots$ has o.g.f. given by $f(x) = 1$.

Example 3. The sequence 1. The sequence with elements $1, 1, 1, 1, \ldots$ has o.g.f.

$$f(x) = \sum_{k=0}^{\infty} x^k$$

which we can formally express as

$$\frac{1}{1-x}.$$

We shall on occasion refer to this as the sequence (1^n) or just 1^n. We note that we have

$$\sum_{k=0}^{n} \binom{n}{k} 0^k = 1.$$

Thus the binomial transform of 0^n is 1^n. Similarly,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 0^k = (-1)^n.$$

Thus the inverse binomial transform of 0^n is $(-1)^n$. In general, we have the following chain of binomial transforms:

$$\cdots \to (-2)^n \to (-1)^n \to 0^n \to 1^n \to 2^n \to \cdots$$

corresponding to the generating functions

$$\cdots \frac{1}{1+2x} \to \frac{1}{1+x} \to 1 = \frac{1}{1-0x} \to \frac{1}{1-x} \to \frac{1}{1-2x} \to \cdots$$

Example 4. Fibonacci numbers. The Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, \ldots$ A000045 with defining recurrence

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 1, \quad F_1 = 1,$$

have o.g.f.

$$\frac{x}{1-x-x^2}.$$

Example 5. Jacobsthal numbers. The Jacobsthal numbers $J_n = J(n) = \frac{2^n}{3} - \frac{(-1)^n}{3}$ A001045 have generating function

$$\frac{x}{1-x-2x^2}.$$

They begin $0, 1, 1, 3, 5, 11, 21, \ldots$. The sequence $J_1(n) = \frac{2^n}{3} + 2\frac{(-1)^n}{2}$ has o.g.f.

$$\frac{1-x}{1-x-2x^2}.$$

This sequence begins $1, 0, 2, 2, 6, 10, 22, \ldots$ A078008. The sequence with elements $J(n+1) + J_1(n)$ form the Jacobsthal-Lucas sequence A014551. This sequence has o.g.f. given by $\frac{2-x}{1-x-2x^2}$.

If $A(x)$, $B(x)$ and $C(x)$ are the ordinary generating functions of the sequences (a_n),(b_n) and (c_n) respectively, then

1. $A(x) = B(x)$ if and only if $a_n = b_n$ for all n.
2. Let $\lambda, \mu \in \mathbb{Z}$, such that $c_n = \lambda a_n + \mu b_n$ for all n. Then
$$C(x) = \lambda A(x) + \mu B(x).$$
3. If $c = a * b$ then $C(x) = A(x)B(x)$ and vice versa.

If the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is such that $a_0 = 0$ (and hence $f(0) = 0$), then we can define the *compositional inverse* $\bar{f}(x)$ of f to be the unique power series such that $f(\bar{f}(x)) = x$. $\bar{f}$ is also called the *reversion* of f. We shall use the notation $\bar{f} = \text{Rev} f$ for this. We note that necessarily $\bar{f}(0) = 0$.

Example 6. The generating function

$$f(x) = \frac{x}{1-x-x^2}$$

has compositional inverse $\bar{f}$ given by

$$\bar{f}(x) = \frac{\sqrt{1+2x+5x^2} - x - 1}{2x}.$$

This is obtained by solving the equation

$$\frac{u}{1-u-u^2} = x$$

where $u = u(x) = \bar{f}(x)$. We note that the equation

$$\frac{u}{1-u-u^2} = x$$

has two formal solutions; the one above, and

$$\tilde{u} = -\frac{\sqrt{1+2x+5x^2} + x + 1}{2x}.$$

We reject this solution as it does not have a power series expansion such that $\tilde{u}(0) = 0$.

2.7 Exponential generating functions

For a sequence $(a_n)_{n\geq 0}$, we define its *exponential generating function* (e.g.f.) to be the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

In other words, $f(x)$ is the o.g.f. of the sequence $\left(\frac{a_n}{n!}\right)$.

Example 7. $\exp(x) = e^x$ is the e.g.f. of the sequence $1, 1, 1, \ldots$.

Example 8. $\cosh(x)$ is the e.g.f. of the sequence $1, 0, 1, 0, 1, 0, \ldots$ with general term

$$\frac{(1 + (-1)^n)}{2}.$$

Example 9. $\frac{1}{1-x}$ is the e.g.f. of $n!$

If $A(x)$, $B(x)$ and $C(x)$ are the exponential generating functions of the sequences (a_n),(b_n) and (c_n) respectively, then

1. $A(x) = B(x)$ if and only if $a_n = b_n$ for all n.
2. Let $\lambda, \mu \in \mathbb{Z}$, such that $c_n = \lambda a_n + \mu b_n$ for all n. Then
$$C(x) = \lambda A(x) + \mu B(x).$$
3. If $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ then $C(x) = A(x)B(x)$ and vice versa.

Example 10. The Bessel function $I_0(2x)$ is the e.g.f. of the 'aerated' *central binomial numbers* $1, 0, 2, 0, 6, 0, 20, 0, 70, \ldots$ with general term

$$a_n = \binom{n}{\frac{n}{2}}(1 + (-1)^n)/2.$$

Then the product $\exp(x)I_0(2x)$ is the e.g.f. of the sequence

$$t_n = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot 1 = \sum_{k=0}^{n} \binom{n}{k} a_k$$

since $\exp(x)$ is the e.g.f. of the sequence $b_n = 1$. t_n is the sequence $1, 1, 3, 7, 19, 51, 141, \ldots$ of *central trinomial numbers*, where $t_n =$ coefficient of x^n in $(1 + x + x^2)^n$.

We note that for $n = 2m + 1$, the expression $\binom{n}{\frac{n}{2}}$ has the value

$$\frac{\Gamma(2m+2)}{\Gamma\left(\frac{2m+3}{2}\right)^2}.$$

2.8 Generalized generating functions

We follow [226] in this section. Given a sequence $(c_n)_{n\geq 0}$, the formal power series $f(t) = \sum_{k=0}^{\infty} \frac{f_k t^k}{c_k}$ is called the *generating function with respect to the sequence* c_n of the sequence $(f_n)_{n\geq 0}$, where c_n is a fixed sequence of non-zero constants with $c_0 = 1$. In particular, $f(t)$ is the ordinary generating function if $c_n = 1$ for all n, and $f(t)$ is the exponential generating function if $c_n = n!$.

2.9 The Method of Coefficients

The method of coefficients [158, 157] consists of the consistent application of a set of rules for the functional

$$[x^n] : \mathbb{C}[[x]] \to \mathbb{C}.$$

In the sequel, we shall usually work with the restriction

$$[x^n] : \mathbb{Z}[[x]] \to \mathbb{Z}.$$

For $f(x)$ and $g(x)$ formal power series, the following statements hold :

$$\begin{aligned}
[x^n](\alpha f(x) + \beta g(x)) &= \alpha[x^n]f(x) + \beta[x^n]g(x) && \text{K1 (linearity)}\\
[x^n]xf(x) &= [x^{n-1}]f(x) && \text{K2 (shifting)}\\
[x^n]f'(x) &= (n+1)[x^{n+1}]f(x) && \text{K3 (differentiation)}\\
[x^n]f(x)g(x) &= \sum_{k=0}^{n}([y^k]f(y))[x^{n-k}]g(x) && \text{K4 (convolution)}\\
[x^n]f(g(x)) &= \sum_{k=0}^{\infty}([y^k]f(y))[x^n]g(x)^k && \text{K5 (composition)}\\
[x^n]\bar{f}^k &= \frac{k}{n}[x^{n-k}]\left(\frac{x}{f(x)}\right)^n && \text{K6 (inversion)}
\end{aligned}$$

We note that the rule K3 may be written as

$$[t^n]f(t) = \frac{1}{n}[t^{n-1}]f'(t).$$

Example 11. We extend the following result :

$$\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n} \binom{k}{n-k}\frac{n}{k} = F_{n+1} + F_{n-1}$$

of [158]. This identity is based on two facts :

$$\log\left(\frac{1}{1-t}\right) = \sum_{k=1}^{\infty} \frac{1}{k}t^k$$

and the following identity, a consequence of the rules for the evaluation of Riordan arrays:

$$\sum_k \binom{m+ak}{n+bk} f_k = [t^n](1+t)^m f(t^{-b}(1+t)^a) \quad (b<0).$$

We use the following fact.

$$[t^n]\frac{1}{1-\alpha t-\beta t^2} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \alpha^{n-2k}\beta^k. \tag{2.3}$$

We now wish to evaluate the expression

$$\sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} \binom{k}{n-k} \alpha^k \beta^{n-k} \frac{n}{k}.$$

We have

$$\begin{aligned}
\sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} \binom{k}{n-k} \alpha^k \beta^{n-k} \frac{1}{k} &= \beta^n \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} \binom{k}{n-k} \left(\frac{\alpha}{\beta}\right)^k \frac{n}{k} \\
&= \beta^n [t^n] \left[\ln\left(\frac{1}{1-\frac{\alpha}{\beta}y}\right) |y=t(1+t) \right] \\
&= \beta^n [t^n] \ln\left(\frac{1}{1-\frac{\alpha}{\beta}t-\frac{\alpha}{\beta}t^2}\right) \\
&= \frac{\beta^n}{n}[t^{n-1}] \frac{\frac{\alpha}{\beta}(1+2t)}{1-\frac{\alpha}{\beta}t-\frac{\alpha}{\beta}t^2} \\
&= \frac{\beta^n}{n}\frac{\alpha}{\beta}[t^{n-1}] \frac{1+2t}{1-\frac{\alpha}{\beta}t-\frac{\alpha}{\beta}t^2}.
\end{aligned}$$

Thus using Eq. (2.3), we obtain

$$\sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} \binom{k}{n-k} \alpha^k \beta^{n-k} \frac{n}{k} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} \alpha^{n-k}\beta^k + 2\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-k-2}{k} \alpha^{n-k-1}\beta^{k+1}.$$

For $\alpha=\beta=1$, we retrieve the Fibonacci result above. For $\alpha=1$, $\beta=2$, we obtain that that for $n>0$, we have

$$\sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} \binom{k}{n-k} 2^{n-k} \frac{n}{k} = J(n+1)+J_1(n)$$

the Jacobsthal-Lucas numbers (A014551).

2.10 Lagrange inversion

Let $\mathcal{F} = \mathbb{C}[[t]]$. If $f(t) \in \mathcal{F}$ with $f(t) = \sum_{k=0}^{\infty} f_k t^k$ and r is the minimum integer for which $f_r \neq 0$, then r is called the *order* of $f(t)$. The set of formal power series of order r is denoted by $\mathcal{F}_r$. $\mathcal{F}_0$ is the set of invertible formal power series, that is, series $f(t)$ for which a series $f^{-1}(t)$ exists in $\mathcal{F}$ such that $f(t)f^{-1}(t) = 1$.

One version of Lagrange inversion [157] is given by rule K6:

$$[t^n]\bar{f}^k = \frac{k}{n}[t^{n-k}]\left(\frac{t}{f(t)}\right)^n.$$

If now we have

$$w(t) = t\phi(w(t))$$

where $\phi \in \mathcal{F}_0$, then if we define f by

$$f(y) = \frac{y}{\phi(y)},$$

we have $\bar{f} = w$ and so

$$[t^n]w(t) = \frac{1}{n}[t^{n-1}]\left(\frac{t}{f(t)}\right)^n = \frac{1}{n}[t^{n-1}]\phi(t)^n.$$

Now let $F \in \mathcal{F}$, and let $w(t) = t\phi(w(t))$. Then

$$[t^n]F(w(t)) = \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n.$$

Also, we have, for $F, \phi \in \mathcal{F}$,

$$[t^n]F(t)\phi(t)^n = [t^n]\left[\frac{F(w)}{1 - t\phi'(w)} | w = t\phi(w)\right].$$

Example 12. Generalized central trinomial coefficients. We wish to find the generating function of

$$[t^n](1 + \alpha t + \beta t^2)^n,$$

the central trinomial coefficients (for the parameters α, β). We let $F(t) = 1$, and $\phi(t) = 1 + \alpha t + \beta t^2$. We have $w = t(1 + \alpha w + \beta w^2)$, and so

$$w = \frac{1 - \alpha t - \sqrt{1 - 2\alpha t + (\alpha^2 - 4\beta)t^2}}{2\beta t}.$$

Thus

$$\begin{aligned}
[t^n](1 + \alpha t + \beta t^2)^n &= [t^n]\left[\frac{F(w)}{1 - t\phi'(w)} | w = t\phi(w)\right] \\
&= [t^n]\left[\frac{1}{1 - t(\alpha + 2\beta w)} | w = \frac{1 - \alpha t - \sqrt{1 - 2\alpha t + (\alpha^2 - 4\beta)t^2}}{2\beta t}\right] \\
&= [t^n]\left[\frac{1}{\sqrt{1 - 2\alpha x + (\alpha^2 - 4\beta)x^2}} | w = \frac{1 - \alpha t - \sqrt{1 - 2\alpha t + (\alpha^2 - 4\beta)t^2}}{2\beta t}\right]
\end{aligned}$$

This shows that the required generating function of the generalized central trinomial coefficients is given by

$$\frac{1}{\sqrt{1-2\alpha x+(\alpha^2-4\beta)x^2}}.$$

Example 13. The Riordan array $(1, xc(x))$. Anticipating the developments of Chapter 4, we seek to calculate the general terms of the Riordan array $(1, xc(x))$. Now

$$(1, xc(x)) = (1, x(1-x))^{-1}.$$

Thus we let

$$\phi(w) = \frac{1}{1-w}$$

and so $w = t\phi(w)$ implies

$$w(1-w) = t.$$

We also let $F(t) = t^k$ and so $F'(t) = kt^{k-1}$. Then

$$\begin{aligned}
[t^n](w(t))^k &= [t^n]F(w(t)) \\
&= \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n \\
&= \frac{1}{n}[t^{n-1}]kt^{k-1}\left(\frac{1}{1-t}\right)^n \\
&= \frac{1}{n}[t^{n-1}]kt^{k-1}\sum_{i=0}^{\infty}\binom{-n}{i}(-t)^i \\
&= \frac{1}{n}[t^{n-1}]k\sum_{i=0}^{\infty}\binom{n+i-1}{i}t^{i+k-1} \\
&= \frac{1}{n}k\binom{n+n-k-1}{n-k} \\
&= \frac{k}{n}\binom{2n-k-1}{n-k}.
\end{aligned}$$

Adjusting for the first row, we obtain that the general term of the "Catalan" array

$$(1, xc(x))$$

is given by

$$\frac{k+0^{n-k}}{n+0^{nk}}\binom{2n-k-1}{n-k}.$$

2.11 Recurrence relations

Recurrence relations allow us to express the general term of a sequence as a function of earlier terms. Thus we may be able to express the term a_n as a function of $a_0, a_1, \ldots, a_{n-1}$ for all $n \geq r$. r is called the *order* of the recurrence. The values $a_0, a_1, \ldots, a_{r-1}$ are called the initial values of the recurrence.

Example 14. The sequence defined by the recurrence

$$a_n = a_{n-1} + a_{n-2}$$

with initial values $a_0 = 0$, $a_1 = 1$ is the Fibonacci sequence A000045 given by

$$a_n = F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

It is easy to calculate the o.g.f. of this sequence. Letting $A(x) = \sum_{n=0}^{\infty} a_n x^n$, and multiplying both sides of the recurrence by x^n and summing for $n \geq 2$, we find that

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n.$$

Now

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - a_1 x - a_0 = A(x) - x,$$

while, for instance,

$$\sum_{n=2}^{\infty} a_{n-1} x^n = x \sum_{n=1}^{\infty} a_n x^n = x(A(x) - a_0) = xA(x).$$

Thus we obtain

$$A(x) - x = xA(x) + x^2 A(x)$$

or

$$A(x) = \frac{x}{1 - x - x^2}.$$

Thus the generating function of the Fibonacci numbers is $\frac{x}{1-x-x^2}$.

Example 15. The sequence defined by the recurrence

$$a_n = a_{n-1} + 2a_{n-2}$$

with initial values $a_0 = 0$, $a_1 = 1$ is the Jacobsthal sequence A001045 given by

$$a_n = J(n) = \frac{2^n}{3} - \frac{(-1)^n}{3}.$$

This sequence starts $0, 1, 1, 3, 5, 11, 21, \ldots$. The generating function of the Jacobsthal numbers is $\frac{x}{1-x-2x^2}$.

In the above two examples, the recurrence was linear, of order 2. The following example, defining the well-known *Catalan* numbers A000108, is of a different nature.

Example 16. The sequence defined by the recurrence

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$$

with $C_0 = 1$ is the sequence of Catalan numbers, which begins $1, 1, 2, 5, 14, 42, \ldots\ldots$ This sequence has been extensively studied and has many interesting properties.
The generating function of the Catalan numbers is the function

$$c(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

We note that the series reversion of $xc(x)$ is given by $x(1-x)$. One way to see this is to solve the equation

$$uc(u) = x.$$

We do this with the following steps.

$$\begin{aligned}
\frac{1 - \sqrt{1-4u}}{2} &= x \\
1 - \sqrt{1-4u} &= 2x \\
\sqrt{1-4u} &= 1 - 2x \\
1 - 4u &= (1-2x)^2 = 1 - 4x + 4x^2 \\
4u &= 4x - 4x^2 \\
u &= x(1-x).
\end{aligned}$$

2.12 Moment sequences

Many well-known integer sequences can be represented as the *moments* of measures on the real line. For example, we have

$$\begin{aligned}
C_n &= \frac{1}{2\pi} \int_0^4 x^n \frac{\sqrt{x(4-x)}}{x} dx, \\
C_{n+1} &= \frac{1}{2\pi} \int_0^4 x^n \sqrt{x(4-x)} dx, \\
\binom{2n}{n} &= \frac{1}{\pi} \int_0^4 \frac{x^n}{\sqrt{x(4-x)}} dx, \\
\binom{n}{\frac{n}{2}} \frac{1 + (-1)^n}{2} &= \frac{1}{\pi} \int_{-2}^2 \frac{x^n}{\sqrt{4-x^2}} dx.
\end{aligned}$$

It is interesting to study the binomial transform of such a sequence. If the sequence a_n has the moment representation

$$a_n = \int_\alpha^\beta x^n w(x) dx$$

then we have

$$\begin{aligned} b_n &= \sum_{k=0}^{n} \binom{n}{k} a_k \\ &= \sum_{k=0}^{n} \binom{n}{k} \int_{\alpha}^{\beta} x^k w(x) dx \\ &= \int_{\alpha}^{\beta} \sum_{k=0}^{n} x^k w(x) dx \\ &= \int_{\alpha}^{\beta} (1+x)^n w(x) dx. \end{aligned}$$

Note that the change of variable $y = x + 1$ gives us the alternative form

$$b_n = \int_{\alpha}^{\beta} (1+x)^n w(x) dx = \int_{\alpha+1}^{\beta+1} y^n w(y-1) dy.$$

Example 17. The central trinomial numbers $t_n = [x^n](1+x+x^2)^n$ are given by the binomial transform of the aerated sequence $\binom{n}{\frac{n}{2}} \frac{1+(-1)^n}{2}$. Thus

$$\begin{aligned} t_n &= \frac{1}{\pi} \int_{-2}^{2} \frac{(1+x)^n}{\sqrt{4-x^2}} dx \\ &= \frac{1}{\pi} \int_{-1}^{3} \frac{x^n}{\sqrt{3+2x-x^2}} dx. \end{aligned}$$

The r-th binomial transform of a_n is similarly given by

$$\int_{\alpha}^{\beta} (r+x)^n w(x) dx.$$

More generally, we have

$$\int_{\alpha}^{\beta} (r+sx)^n w(x) dx = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} s^k a_k.$$

Aspects of these general binomial transforms have been studied in a more general context in [207].

Example 18. We consider the sequence $1, 3, 12, 51, 222, 978, \ldots$ or A007854 with o.g.f. $\frac{2}{3\sqrt{1-4x}-1}$ and general term

$$\sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) 2^k = \sum_{k=0}^{n} \frac{2k+1}{n+k+1} \binom{2n}{n-k} 2^k.$$

We have

$$\begin{aligned} a_n &= \frac{3}{2\pi}\int_0^4 \frac{x^n}{9-2x}\sqrt{\frac{4-x}{x}}dx + \frac{1}{2}\left(\frac{9}{2}\right)^n \\ &= \frac{3}{2\pi}\int_0^4 \frac{x^n}{9-2x}\sqrt{\frac{4-x}{x}}dx + \frac{1}{2}\left\langle \delta_{\frac{9}{2}}, x^n \right\rangle. \end{aligned}$$

We note that this sequence is the image of 2^n by the Riordan array $(c(x), xc(x)^2)$. Thus a_n is defined by a so-called "Sobolev" measure [150].

Example 19. The sequence $1, 0, 1, 0, 3, 0, 15, 0, 105, 0, 945, \ldots$ with general term

$$a_n = (2(n/2)-1)!!\frac{1+(-1)^n}{2}$$

where the double factorials $(2n-1)!! = \prod_{k=1}^n (2k-1)$ is A001147, counts the number of perfect matchings in K_n, the complete graph on n vertices. We have [104]

$$a_n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}dx.$$

The e.g.f. of this sequence is $\frac{e^{x^2}}{2}$.
Note that we have

$$(2n-1)!! = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} x^n \frac{e^{-\frac{x}{2}}}{\sqrt{x}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2}}dx.$$

The binomial transform b_n of a_n is given by

$$\begin{aligned} b_n &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (1+x)^n e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-\frac{(x-1)^2}{2}}dx, \end{aligned}$$

which is A000085. This counts, for instance, the number of Young tableaux with n cells.

We note that the Hankel transform of this last sequence is given by

$$2^{\binom{n}{2}}\prod_{k=1}^{n-1} k! = \prod_{k=0}^{n} k!2^k.$$

This is A108400.

Anticipating Chapter 4 we can represent A001147 as the row sums of the exponential Riordan array $\left[e^{-x}, x(1+\frac{x}{2})\right]$ which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 1 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 1 & 2 & -3 & 2 & 1 & 0 & \ldots \\ -1 & -5 & 5 & -5 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A154556.

2.13 The Stieltjes transform of a measure

The *Stieltjes transform* of a measure μ on $\mathbb{R}$ is a function G_μ defined on $\mathbb{C} \setminus \mathbb{R}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} \mu(t).$$

If f is a bounded continuous function on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} f(x)\mu(x) = -\lim_{y \to 0^+} \int_{\mathbb{R}} f(x) \Im G_\mu(x+iy) dx.$$

If μ has compact support, then G_μ is holomorphic at infinity and for large z,

$$G_\mu(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}},$$

where $a_n = \int_{\mathbb{R}} t^n \mu(t)$ are the moments of the measure. If $\mu(t) = d\psi(t) = \psi'(t)dt$ then

$$\psi(t) - \psi(t_0) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_{t_0}^{t} \Im G_\mu(x+iy) dx.$$

If now $g(x)$ is the generating function of a sequence a_n, with $g(x) = \sum_{n=0}^{\infty} a_n x^n$, then we can define

$$G(z) = \frac{1}{z} g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}.$$

By this means, under the right circumstances we can retrieve the density function for the measure that defines the elements a_n as moments.

2.14 Orthogonal polynomials as moments

Many common orthogonal polynomials, suitably parameterized, can be shown to be moments of other families of orthogonal polynomials. This is the content of [119, 120]. This allows us to derive results about the moment sequences in a well-known manner, once the characteristics (for instance, the three term recurrence relation) of the generating family of orthogonal polynomials are known. Such characteristics of common orthogonal polynomials may be found in [126]. This approach has been emphasized in [132], for instance, in the context of the evaluation of the Hankel transform of sequences.

Example 20. A simple example [21, 134] of this technique is as follows. The reversion of the generating function $\frac{x}{1+\alpha x+\beta x^2}$ generates the sequence with general term

$$u_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{k} C_k \alpha^{n-2k-1} \beta^k$$

(see Chapter 10). We are interested for this example in the Hankel transform of u_{n+1}. For this, we cast u_{n+1} into hypergeometric form :

$$u_{n+1} = \alpha^n \,{}_2F_1\left(\frac{1}{2} - \frac{n}{2}, -\frac{n}{2}; \frac{4\beta}{\alpha^2}\right).$$

Applying the transformation

$${}_2F_1\left(\alpha, \frac{1}{2} + \alpha; \frac{1}{2} + \beta; z^2\right) = \frac{1}{(1-z)^{-2\alpha}} \,{}_2F_1\left(2\alpha, \beta; 2\beta; \frac{2z}{z-1}\right),$$

we obtain

$$u_{n+1} = (\alpha - 2\sqrt{\beta})^n \,{}_2F_1\left(-n, \frac{3}{2}; 3; \frac{4\sqrt{\beta}}{2\sqrt{\beta} - \alpha}\right).$$

This exhibits u_{n+1} as a Meixner polynomial. Meixner polynomials are moments for the Jacobi polynomials [119, 251]. Hence we can readily compute the Hankel determinant of u_{n+1} (it is equal to $(\alpha(\alpha-\beta))^{\binom{n+1}{2}}$).

2.15 Lattice paths

Many well-known integer sequences can be represented by the number of paths through a lattice, where various restrictions are placed on the paths - for example, the types of allowable steps. The best-known example is the Catalan numbers, which count Dyck paths in the plane.

Lattice paths can be defined in two distinct but equivalent ways - explicitly, as a sequence of points in the plane, or implicitly, as a sequence of steps of defined types (we can find the points in the plane by "following" the steps).

Thus we can think of a lattice path [143] as a sequence of points in the integer lattice $\mathbb{Z}^2$, where a pair of consecutive points is called a *step* of the path. A *valuation* is a function on the set of possible steps $\mathbb{Z}^2 \times \mathbb{Z}^2$. A valuation of a path is the product of the valuations of its steps.

Alternatively, given a subset S of $\mathbb{Z} \times \mathbb{Z}$ we can define a *lattice path with step set* S to be a finite sequence $\Gamma = s_1 s_2 \cdots s_k$ where $s_i \in S$ for all i [56].

Well known and important paths include Dyck paths, Motzkin paths and Schröder paths.

Example 21. A Dyck path is a path starting at $(0, 0)$ and ending at $(2n, 0)$ with allowable steps $(1, 1)$ (a "*rise*") and $(1, -1)$ (a "*fall*"), which does not go below the x-axis. Thus $S = \{(1, 1), (1, -1)\}$ Such paths are enumerated by the Catalan numbers C_n. The central binomial coefficients $\binom{2n}{n}$ count all such paths, when the restriction of not going below the x-axis is lifted (such paths are then called *Grand-Dyck* paths or binomial paths [178]).

Example 22. The n-th central binomial coefficient, $[x^n](1 + x + x^2)^n$, counts the number of lattice paths starting at $(0, 0)$ and ending at $(n, 0)$, whose allowed steps are $(1, 0)$, $(1, 1)$ and $(1, -1)$. Thus in this case $S = \{(1, 0), (1, 1), (1, -1)\}$. The Motzkin numbers $m_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k} = {}_2F_1(\frac{1-n}{2}, \frac{-n}{2}; 2; 4)$ count the number of such paths that do not descend below the x-axis. A Dyck path is clearly a special case of a Motzkin path.

Example 23. A Schröder path is a path that starts at $(0,0)$, ends at $(2n,0)$, and has allowable steps $(1,1)$, $(2,0)$ and $(1,-1)$. The Schröder numbers $S_n = {}_2F_1(1-n, n+2; 2; -1)$ count the number of such paths that do not go below the x-axis.

Paths may be "coloured", that is, for each step $s \in S$, we can assign it an element from a finite set of "colours".
Families of disjoint paths play an important role in the evaluation of certain important determinants, including Hankel determinants [223]. For instance in the case [151] of the Catalan numbers C_n, if we define $H_n^{(k)} = |C_{k+i+j}|_{0 \le i,j \le n-1}$ then this determinant is given by the number of n-tuples $(\gamma_0, \ldots, \gamma_{n-1})$ of vertex-disjoint paths in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ (with directed vertices from (i,j) to either $(i, j+1)$ or to $(i+1, j)$) never crossing the diagonal $x = y$, where the path γ_r is from $(-r, -r)$ to $(k+r, k+r)$.

2.16 Continued fractions

Continued fractions [227] play an important role in many areas of combinatorics. They are naturally associated to orthogonal polynomials and lattice path enumeration [89]. They play an important role in the computation of Hankel transforms. In this section we briefly define continued fractions and give examples of their application to integer sequence. A *generalized continued fraction* is an expression of the form

$$t = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots}}}}$$

where the a_n $(n > 0)$ are the *partial numerators*, the b_n are the *partial denominators*, and the leading term b_0 is the so-called whole or integer part of the continued fraction. The successive *convergents* (also called *approximants*) of the continued fraction are formed as follows:

$$t_0 = \frac{A_0}{B_0} = b_0, \qquad t_1 = \frac{A_1}{B_1} = \frac{b_1 b_0 + a_1}{b_1}, \qquad t_2 = \frac{A_2}{B_2} = \frac{b_1(b_1 b_0 + a_1) + a_2 b_0}{b_2 b_1 + a_2}, \ldots$$

where A_n is the *numerator* and B_n is the *denominator* (also called *continuant*) of the nth convergent, and where we have the following recurrence relations:

$$A_{-1} = 1, \qquad B_{-1} = 0, \qquad A_0 = b_0, \qquad B_0 = 1;$$

$$\begin{aligned} A_{p+1} &= b_{p+1} A_p + a_{p+1} A_{p-1}, \\ B_{p+1} &= b_{p+1} B_p + a_{p+1} B_{p-1} \end{aligned}$$

for $p = 0, 1, 2, \ldots$.

$\frac{A_n}{B_n}$ is called the nth convergent (approximant). We have

$$\frac{A_n}{B_n} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots + \frac{a_n}{b_n}}}}}.$$

The convergents of a continued fraction do not change when an *equivalence transformation* is effected as follows:

$$b_0 + \cfrac{c_1 a_1}{c_1 b_1 + \cfrac{c_1 c_2 a_2}{c_2 b_2 + \cfrac{c_2 c_3 a_3}{c_3 b_3 + \cfrac{c_3 c_4 a_4}{c_4 b_4 + \dots}}}}$$

Example 24.

$$c(x) = \frac{1}{1 - xc(x)} = \frac{1 - \sqrt{1-4x}}{2} = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \dots}}}}$$

is the generating function of the Catalan numbers. The denominator polynomials are then given by

$$1, 1, 1 - x, 1 - 2x, 1 - 3x + x^2, 1 - 4x + 3x^2, 1 - 5x + 6x^2 - x^3, \dots$$

Thus the nth denominator polynomial is given by

$$B_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-1)^k x^k.$$

The binomial transform of the Catalan numbers has generating function

$$\cfrac{1}{1 - x - \cfrac{x}{1 - \cfrac{x}{1 - x - \cfrac{x}{1 - \cfrac{x}{1 - x - \cfrac{x}{1 - \cdots}}}}}}.$$

The continued fraction

$$\cfrac{1}{1-mx-\cfrac{x}{1-\cfrac{x}{1-mx-\cfrac{x}{1-\cfrac{x}{1-mx-\cfrac{x}{1-\cdots}}}}}}$$

is the generating function of the m-th binomial transform of the Catalan numbers (i.e. it is equal to $\frac{1}{1-mx}c\left(\frac{x}{1-mx}\right)$).

More generally, we have

$$\frac{1}{1-rxc(x)}=\cfrac{1}{1-\cfrac{rx}{1-\cfrac{x}{1-\cfrac{x}{1-\ldots}}}}.$$

The denominator polynomials are then given by

$$1,1,1-rx,1-(r+1)x,1-(r+2)x+rx^2,1-(r+3)x+(2r+1)x^2,\ldots$$

Thus the nth denominator polynomial is given by

$$B_n(x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k\left(\binom{n-k-1}{k}+\binom{n-k-1}{k-1}r\right)x^k.$$

The Hankel transform of the sequence with g.f. $\frac{1}{1-rxc(x)}$ is r^n.

We have

$$c(rx)=\cfrac{1}{1-\cfrac{rx}{1-\cfrac{rx}{1-\cfrac{rx}{1-\ldots}}}}.$$

$c(rx)$ is the g.f. of the sequence r^nC_n which has Hankel transform $r^{n(n+1)}$.

Example 25. The continued fraction

$$g(x;r)=\cfrac{1}{1-\cfrac{rx}{1-\cfrac{x}{1-\cfrac{rx}{1-\cfrac{x}{1-\cfrac{rx}{1-\ldots}}}}}}$$

generates the sequence $a_n(r)$ which begins

$$1, r, r(r+1), r(r^2+3r+1), r(r^3+6r^2+6r+1), r(r^4+10r^3+20r^2+10r+1), \ldots$$

which is the Narayana transform (see Example 184) of the power sequence $1, r, r^2, r^3, r^4, \ldots$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 1 & 0 & \ldots \\ 0 & 1 & 10 & 20 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ r \\ r^2 \\ r^3 \\ r^4 \\ r^5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ r \\ r(r+1) \\ r(r^2+3r+1) \\ r(r^3+6r^2+6r+1) \\ r(r^4+10r^3+20r^2+10r+1) \\ \vdots \end{pmatrix},$$

$$a_n(r) = \sum_{k=0}^{n} N(n,k) r^k.$$

For $r = 0, 1, 2, \ldots$ we obtain the following sequences

$r=0$	1	0	0	0	0	0	0	$\ldots$	A000007
$r=1$	1	1	2	5	14	42	132	$\ldots$	A000108
$r=2$	1	2	6	22	90	394	1806	$\ldots$	A006318
$r=3$	1	3	12	57	300	1686	9912	$\ldots$	A047891
$r=4$	1	4	20	116	740	5028	35700	$\ldots$	A082298
$r=5$	1	5	30	205	1530	12130	100380	$\ldots$	A082301
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			

which include the Catalan numbers ($r = 1$) and the large Schröder numbers ($r = 2$). These sequences can be characterized as

$$a_n(r) = [x^{n+1}]\text{Rev}\frac{x(1-x)}{1+(r-1)x}.$$

The sequence $a_n(r)$ has Hankel transform $r^{\binom{n+1}{2}}$.

Example 26.

$$g(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{2x}{1 - \cfrac{3x}{1 - \cfrac{4x}{1 - \ldots}}}}}$$

is the g.f. of the double factorials $(2n-1)!! = \prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{2^n n!}$, whose e.g.f. is $\frac{1}{\sqrt{1-2x}}$. The denominator polynomials are then given by

$$1, 1, 1-x, 1-3x, 1-6x+3x^2, 1-10x+15x^2, 1-15x+45x^2-15x^3, \ldots$$

Thus the nth denominator polynomial is given by

$$B_n(x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k}(2k+1)!!(-1)^k x^k.$$

Example 27.

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{2x}{1-\cfrac{3x}{1-\ldots}}}}}}$$

is the g.f. of the factorial numbers $n!$ with e.g.f. $\frac{1}{1-x}$. The denominator polynomials are given by

$1, 1, 1-x, 1-2x, 1-4x+2x^2, 1-6x+6x^2, 1-9x+18x^2-6x^3, 1-12x+36x^2-24x^3, \ldots$

We obtain the following array of numbers as the coefficient array of these polynomials:

1				
1				
1	-1			
1	-2			
1	-4	2		
1	-6	6		
1	-9	18	-6	
1	-12	36	-24	
1	-16	72	-96	24

The second column is minus times the quarter squares $\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil$ A002620. The third column is given by $\sum_{k=0}^{n-1}(-1)^{n-k-1}\lfloor\frac{k-1}{2}\rfloor\lceil\frac{k-1}{2}\rceil\lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil$. We note that this last sequence appears to be twice A000241$(n+1)$, where A000241 gives the crossing number of K_n, the complete graph with n nodes. The formula appears to be consistent with Zarankiewicz's conjecture (which states the the graph crossing number of the complete bigraph $K_{n,m}$ is given by $\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ [247]). In fact (see below), we conjecture that A000241 is given by $\binom{\lfloor\frac{n}{2}\rfloor}{2}\binom{\lfloor\frac{n-1}{2}\rfloor}{2}$.

An alternately signed version of every second row of this array is given by

1				
-1	1			
1	-4	2		
-1	9	-18	6	
1	-16	72	-96	24

This is A021010, the triangle of coefficients of the Laguerre polynomials $L_n(x)$ (powers of x in decreasing order). It has general term $T^{(1)}_{n,k} = (-1)^{n-k}k!\binom{n}{k}^2$.

Taking the second embedded triangle of alternate rows, we obtain

1			
1	-2		
1	-6	6	
1	-12	36	-24

which has general term

$$T^{(2)}_{n,k} = \frac{(-1)^k(n+1)!\binom{n}{k}}{(n-k+1)!} = (-1)^k\binom{n+1}{k}\binom{n}{k}k!$$

We note that the quotient

$$\frac{T^{(2)}_{n,k}}{(-1)^k(k+1)!}$$

is the Narayana triangle (see A001263)

1			
1	1		
1	3	1	
1	6	6	1

...

Combining the terms for $T^{(1)}_{n,k}$ and $T^{(2)}_{n,k}$ we see that the coefficient array for the denominator polynomials is given by

$$T_{n,k} = (-1)^k\binom{\lfloor\frac{n+1}{2}\rfloor}{k}\binom{\lfloor\frac{n}{2}\rfloor}{k}k!$$

This triangle is A145118. We note that the triangle with general term $\binom{\lfloor\frac{n+1}{2}\rfloor}{k}\binom{\lfloor\frac{n}{2}\rfloor}{k}$ is A124428. This triangle has row sums equal to $\binom{n}{\lfloor\frac{n}{2}\rfloor}$. A variant is given by A104559, which counts the number of left factors of peakless Motzkin paths of length n having k number of U's and D's.

Example 28. The Bell numbers A000110 have g.f. given by

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{3x}{1-\cfrac{x}{1-\cdots}}}}}}}}$$

The bi-variate generating function

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{3x}{1-\cfrac{x}{1-\cdots}}}}}}}}$$

generates the array that starts

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 2 & 1 & 0 & 0 & \ldots \\ 0 & 6 & 5 & 3 & 1 & 0 & \ldots \\ 0 & 22 & 16 & 9 & 4 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with row sums equal to the Bell numbers A000110. This is the array

$$[0,1,1,2,1,3,1,4,1,\ldots] \quad \Delta \quad [1,0,0,0,\ldots]$$

(see Chapter 5 for notation).

Of particular interest for this work is the notion of J-fraction. We shall consider these in the context of a sequence $c_0, c_1, \ldots$ such that $H_n = |c_{i+j}|_{0\le i,j\le n} \neq 0, \quad n \geq 0$. Then there exists a family of orthogonal polynomials $P_n(x)$ that satisfy the recurrence

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x).$$

This means that the family $P_n(x)$ are the denominator polynomials for the "J-fraction"

$$\cfrac{1}{1-\alpha_0 x-\cfrac{\beta_1 x^2}{1-\alpha_1 x-\cfrac{\beta_2 x^2}{1-\alpha_2 x-\ldots}}}$$

The theory now tells us that in fact

$$g(x) = \sum_{n=0}^{\infty} c_n x^n = \cfrac{c_0}{1-\alpha_0 x-\cfrac{\beta_1 x^2}{1-\alpha_1 x-\cfrac{\beta_2 x^2}{1-\alpha_2 x-\ldots}}}$$

is the generating function of the sequence. In addition, the Hankel transform of c_n is then given by

$$h_n = \prod_{k=1}^{n} \beta_k^{n-k+1}$$

[227, 131, 132].

Example 29. The continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\dots}}}}$$

is the J-fraction generating function of the Catalan numbers. More generally, we have

$$\frac{c(x)}{1-kxc(x)} = \cfrac{1}{1-(k+1)x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\dots}}}}.$$

This is therefore the image of the power sequence k^n (with g.f. $\frac{1}{1-kx}$) under the Riordan array $(c(x), xc(x))$. Each of these sequences has Hankel transform $h_n = 1$. Thus

$$\sum_{k=0}^{n} \frac{k+1}{n+1}\binom{2n-k}{n-k} r^k = [x^n]\cfrac{1}{1-(r+1)x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\dots}}}}.$$

We note that the g.f. of C_{n+1} is given by

$$\cfrac{1}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}$$

while that of $\binom{2n+1}{n}$ A001700 is given by

$$\cfrac{1}{1-3x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}$$

The sequence with g.f. given by

$$\cfrac{1}{1-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}$$

is Fine's sequence A000957.

Example 30. The sequence A026671 which begins $1, 3, 11, 43, 173, \ldots$, with g.f.

$$\frac{1}{\sqrt{1-4x}-x} = \cfrac{1}{1-3x-\cfrac{2x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}$$

has Hankel transform 2^n. The sequence $1, 1, 3, 11, 43, 173, \ldots$ which has g.f.

$$\cfrac{1}{1-x-\cfrac{2x^2}{1-3x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}}$$

also has Hankel transform 2^n. A026671 is a transform of $F(2n+2)$ by the Riordan array $(1, xc(x))$. The sequence $1, 1, 3, 11, 43, 173, \ldots$ is the image of A001519, or $F(2n-1)$, by $(1, xc(x))$.

Example 31. The sequence $1, 5, 28, 161, 934, 5438, \ldots$ with g.f.

$$\cfrac{1}{1-5x-\cfrac{3x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}}$$

is the image of A107839, or $[x^n]\frac{1}{1-5x+2x^2}$ under the Riordan array $(1, xc(x))$. It has Hankel transform 3^n. Looking now at the sequence $1, 1, 5, 28, 161, 934, 5438, \ldots$ we find that it has a g.f. given by

$$\cfrac{1}{1-x-\cfrac{4x^2}{1-\frac{19}{4}x-\cfrac{\frac{15}{16}x^2}{1-\frac{41}{20}x-\cfrac{\frac{24}{25}x^2}{1-\frac{61}{30}x-\cfrac{\frac{35}{36}x^2}{1-\frac{85}{42}x-\cfrac{\frac{48}{49}x^2}{1-\dots}}}}}}$$

This sequence has Hankel transform $(n+3)3^{n-1}$ with g.f. $\frac{1-2x}{(1-3x)^2}$ (A006234).

Example 32. In this example we characterize a family of sequences in a number of ways. The family, parameterized by r, is obtained by applying the Riordan array $\left(\frac{1-x}{1-rx}, \frac{x(1-x)}{1-rx}\right)$ to the Catalan numbers C_k. Thus let $a_n(r)$ denote the r-th element of this family. We have

$$a_n(r) = \sum_{k=0}^{n}\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}\binom{n-j}{n-k-j}r^{n-k-j}C_k,$$

with generating function

$$g(x;r) = \frac{1-x}{1-rx}c\left(\frac{x(1-x)}{1-rx}\right).$$

In terms of continued fractions we find that

$$g(x;r) = \cfrac{1}{1-\cfrac{rx}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{rx}{1-\cfrac{x}{1-\dots}}}}}},$$

where the coefficients follow the pattern $r, 1, 1, r, 1, 1, r, 1, 1, \ldots$. As a J-fraction, we have

$$g(x;r) = \cfrac{1}{1-rx-\cfrac{rx^2}{1-2x-\cfrac{rx^2}{1-(r+1)x-\cfrac{x^2}{1-(r+1)x-\cfrac{rx^2}{1-2x-\dots}}}}}$$

Here, the α sequence is $r, 2, r+1, r+1, 2, r+1, r+1, 2, \ldots$ while the β sequence is $r, r, 1, r, r, 1, r, r, 1, \ldots$ (starting at β_1). The Hankel transform of $a_n(r)$ is then given by

$$h_n = r^{\lfloor \frac{(n+1)^2}{3} \rfloor}.$$

$a_n(2)$ is A059279 with Hankel transform A134751.

Example 33. The continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{2x^2}{1-3x-\cfrac{3x^2}{1-4x-\dots}}}}$$

is the generating function of the Bell numbers (see also Example 28), which enumerate the total number of partitions of $[n]$. These are the numbers $1, 1, 2, 5, 15, 52, 203, \dots$ A000110 with e.g.f. e^{e^x-1}. They satisfy the recurrence $a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k$. From the above, we have that

$$h_n = \prod_{k=1}^{n} k^{n-k+1}$$

which is $1, 1, 2, 12, 288, 34560, \dots$ or A000178, the superfactorials.

Example 34. The continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-3x-\cfrac{x^2}{1-4x-\dots}}}}$$

is the generating function of the Bessel numbers, which count the non-overlapping partitions of $[n]$. These are the numbers $1, 1, 2, 5, 14, 43, 143, 509, \dots$ A006789. They have Hankel transform $h_n = 1$.

Example 35. The continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2}{1-x-\cfrac{2x^2}{1-x-\cfrac{3x^2}{1-x-\dots}}}}$$

is the generating function of the sequence I_n of involutions, where an *involution* is a permutation that is its own inverse. These numbers start $1, 1, 2, 4, 10, 26, 76, \dots$ A000085 with e.g.f. $e^{x(2+x)/2}$. Once again the Hankel transform of this sequence is given by the superfactorials.

Example 36. The continued fraction

$$\cfrac{1}{1-0x-\cfrac{x^2}{1-0x-\cfrac{x^2}{1-0x-\cfrac{x^2}{1-0x-\dots}}}}$$

or

$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-0x-\ldots}}}}$$

is equal to $c(x^2)$, the generating function of the aerated Catalan numbers $1,0,1,0,2,0,5,0,14,0,\ldots$. Since $\beta_n=1$, the Hankel transform is $h_n=1$.
The coefficient array for the associated orthogonal polynomials (denominator polynomials) is given by the Riordan array

$$\left(\frac{1}{1+x^2},\frac{x}{1+x^2}\right).$$

We note that the aerated Catalan numbers are given by

$$[x^{n+1}]\mathrm{Rev}\left(\frac{x}{1+x^2}\right).$$

Example 37. The continued fraction

$$\cfrac{1}{1-1x-\cfrac{x^2}{1-1x-\cfrac{x^2}{1-1x-\cfrac{x^2}{1-1x-\ldots}}}}$$

is the generating function $M(x)$ of the Motzkin numbers $\sum_{k=0}^{n}\binom{n}{2k}C_k$. This is the binomial transform of the last sequence (we note that the coefficients α_n are incremented by 1). The Hankel transform of the Motzkin numbers is given by $h_n=1$.

The coefficient array for the associated orthogonal polynomials (denominator polynomials) is given by the Riordan array

$$\left(\frac{1}{1+x+x^2},\frac{x}{1+x+x^2}\right).$$

Example 38. The continued fraction

$$\cfrac{1}{1-0x-\cfrac{2x^2}{1-0x-\cfrac{x^2}{1-0x-\cfrac{x^2}{1-0x-\ldots}}}}$$

is the generating function of the aeration of the central binomial numbers $\binom{2n}{n}$. Thus these are the numbers $\binom{n}{n/2}(1+(-1)^n)/2$ beginning $1,0,2,0,6,0,\ldots$ with g.f. $\frac{1}{\sqrt{1-4x^2}}$. Their Hankel transform is governed by the β-sequence $2,1,1,1,1,\ldots$ and hence we have $h_n=2^n$.

The coefficient array for the associated orthogonal polynomials (denominator polynomials) is given by the Riordan array

$$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right).$$

Example 39. The central trinomial numbers $1, 1, 3, 7, 19, \ldots$ A002426, or $t_n = [x^n](1+x+x^2)^n$, are given by the binomial transform of the last sequence. Their generating function is then

$$\frac{1}{\sqrt{1-2x-3x^2}} = \cfrac{1}{1-x-\cfrac{2x^2}{1-x-\cfrac{x^2}{1-x-\cfrac{x^2}{1-x-\ldots}}}}.$$

The Hankel transform is again $h_n = 2^n$.

The coefficient array for the associated orthogonal polynomials (denominator polynomials) is given by the Riordan array

$$\left(\frac{1-x^2}{1+x+x^2}, \frac{x}{1+x+x^2}\right).$$

Example 40. The sequence $1, 1, 2, 3, 6, 10, 20, \ldots$ A001405 of central binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ has the following continued fraction expression for its generating function

$$\cfrac{1}{1-1x-\cfrac{x^2}{1-0x-\cfrac{x^2}{1-0x-\cfrac{x^2}{1-0x-\ldots}}}}.$$

Thus $h_n = 1$.

The coefficient array for the associated orthogonal polynomials (denominator polynomials) is given by the Riordan array

$$\left(\frac{1-x}{1+x^2}, \frac{x}{1+x^2}\right).$$

Example 41. We have the following general result : If the α sequence is given by $\alpha, 0, 0, 0, 0, \ldots$ (i.e. $\alpha_n = \alpha 0^n$), and the β sequence is given by $0, \beta, \gamma, \gamma, \gamma, \ldots$ (i.e $\beta_0 = 0$, $\beta_1 = \beta$, $\beta_n = \gamma$ for $n > 1$), then the coefficient array for the associated orthogonal polynomials (denominator polynomials) is given by the Riordan array

$$\left(\frac{1-\alpha x-(\beta-\gamma)x^2}{1+\gamma x^2}, \frac{x}{1+\gamma x^2}\right).$$

In particular,

$$\frac{2\gamma}{2\gamma-\beta-2\alpha\gamma x+\beta\sqrt{1-4\gamma x^2}}=\cfrac{1}{1-\alpha x-\cfrac{\beta x^2}{1-0x-\cfrac{\gamma x^2}{1-0x-\cfrac{\gamma x^2}{1-0x-\ldots}}}}$$

In this case we have

$$h_n=\beta^n\gamma^{\binom{n}{2}}.$$

Example 42. The continued fraction

$$\cfrac{1}{1-ax-\cfrac{bx^2}{1-\cfrac{bx^2}{1-ax-\cfrac{bx^2}{1-\cfrac{bx^2}{1-ax-\cfrac{bx^2}{1-\cdots}}}}}}$$

is the generating function of the sequence

$$\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-k}{k}C_k a^{n-2k}b^k.$$

It is equal to

$$\frac{1}{1-ax}c\left(\frac{bx^2}{1-ax}\right)$$

where $c(x)$ is the generating function of the Catalan numbers.
This sequence represents the diagonal sums of the triangular array with general term

$$\binom{n}{k}C_k a^{n-k}b^k.$$

To prove the initial assertion, we can proceed as follows. $\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-k}{k}C_k a^{n-2k}b^k$ is the result of applying the matrix with general term

$$\binom{n-k}{k}a^{n-2k}$$

to the scaled Catalan numbers b^nC_n. Now the matrix with general term $\binom{n-k}{k}a^{n-2k}$ is the generalized Riordan array

$$\left(\frac{1}{1-ax},\frac{x^2}{1-ax}\right)$$

while the g.f. of $b^n C_n$ is expressible as the continued fraction

$$\cfrac{1}{1-\cfrac{bx}{1-\cfrac{bx}{1-\cdots}}}.$$

Anticipating results from the theory of Riordan arrays, we can thus say that the g.f. of $\binom{n}{k}C_k a^{n-k}b^k$ is expressible as

$$\frac{1}{1-ax}\cfrac{1}{1-\cfrac{b\frac{x^2}{1-x}}{1-\cfrac{b\frac{x^2}{1-x}}{1-\cdots}}}.$$

Simplifying this expression leads to the desired result.
The Hankel transform of this sequence is $b^{\binom{n+1}{2}}$. If $a=b=1$, we obtain the sequence $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-k}{k}C_k$ which is A090344. This enumerates the number of Motzkin paths of length n with no level steps at odd levels. It represents the diagonal sums of the array $\binom{n}{k}C_k$, A098474. Thus its generating function is given by

$$\cfrac{1}{1-x-\cfrac{x^2}{1-\cfrac{x^2}{1-x-\cfrac{x^2}{1-\cfrac{x^2}{1-x-\cfrac{x^2}{1-\cdots}}}}}}$$

This is equal to $\frac{1}{1-x}c\left(\frac{x^2}{1-x}\right)$. The first differences of this sequence have g.f. $c\left(\frac{x^2}{1-x}\right)$ which is equal to

$$\cfrac{1}{1-\cfrac{x^2}{1-x-\cfrac{x^2}{1-\cfrac{x^2}{1-x-\cfrac{x^2}{1-\cfrac{x^2}{1-x-\cfrac{x^2}{1-\cdots}}}}}}}.$$

This sequence enumerates the number of Motzkin paths of length n with no level steps at even levels. This is A121482.

2.17 Hypergeometric functions

A generalized hypergeometric function [98, 141, 204] ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ is a power series $\sum_{k=0}^{\infty} c_k x^k$ which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)\ldots(k+a_p)}{(k+b_1)\ldots(k+b_q)(k+1)}.$$

We have

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{k \geq 0} \frac{(a_1)_k \cdot (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdot (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$. We note that ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; 0) = 1$. For the important case $p = 2$, $q = 1$, we have

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.$$

Example 43. ${}_2F_1(1, 1; 1; x) = \frac{1}{1-x}$, the o.g.f. of the sequence $u_n = 1$ and the e.g.f. of the sequence $u_n = n!$.

Example 44. ${}_2F_1(1, 2; 1; x) ={}_2 F_1(2, 1; 1; x) = \frac{1}{(1-x)^2}$, the o.g.f. of the sequence $u_n = n + 1$, and the e.g.f. of $(n + 1)!$

Example 45. More generally ${}_2F_1(a, 1; 1; x) = \frac{1}{(1-x)^a}$ while ${}_2F_1(-a, b; b; x) = (1 + x)^a$.

Example 46. We have

1. ${}_2F_1(1/2, 1; 1; 4x) = \frac{1}{\sqrt{1-4x}}$, the o.g.f. of $\binom{2n}{n}$. This has Hankel transform 2^n.

2. ${}_2F_1(1/2, -1/2; 1/2; 4x) = {}_2F_1(-1/2, 1; 1); 4x) = \sqrt{1-4x}$, the o.g.f. of $\frac{\binom{2n}{n}}{1-2n}$. This has Hankel transform $(2n + 1)(-2)^n$.

3. ${}_2F_1(1/2, -1/2; 1/2; 4x/(1 - x)) = \sqrt{\frac{1-5x}{1-x}}$ is the o.g.f. of the binomial transform of $\sum_{k=0}^{n}(-1)^{n-k}\frac{\binom{2k}{k}}{1-2k}$. This has Hankel transform with o.g.f. $\frac{1-2x}{1+6x+4x^2}$ which gives the sequence $(-2)^n \cdot L(2n + 1)$, where $L(n)$ is the Lucas sequence A000032.

4. ${}_2F_1(1/2, 1; 1; 4x/(1-x)) = \sqrt{\frac{1-x}{1-5x}}$ is the binomial transform of $\sum_{k=0}^{n}(-1)^{n-k}\binom{2k}{k}$. This has Hankel transform A082761, the trinomial transform of the Fibonacci numbers, with o.g.f. $\frac{1-2x}{1-6x+4x^2}$.

5. ${}_2F_1(1/2, 1; 1; 4x/(1+x)) = \sqrt{\frac{1+x}{1-3x}}$ is the inverse binomial transform of the partial sums of the central binomial coefficients. This has Hankel transform with o.g.f. $(1-2x)/(1-2x + 4x^2)$ or A120580.

Example 47. We have

1. ${}_2F_1(1/2, 1; 2; 4x) = c(x)$, the o.g.f. of the Catalan numbers. In general, ${}_2F_1(k - 1/2, k; 2k; 4x) = c(x)^{2k-1}$.

2. ${}_2F_1(1/2, 1; 2; 4x/(1-x)) = \frac{1-x-\sqrt{1-6x+5x^2}}{2x}$, the o.g.f. of A002212. The Binomial transform has o.g.f. $\frac{c(x)}{1+x}$ and the Hankel transform of A002212 is $F(2n+1)$.

3. ${}_2F_1(1/2, 1; 2; 4x/(1+x)) = \frac{1+x-\sqrt{1-2x-3x^2}}{2x}$, the o.g.f. of A086246. The Hankel transform of this sequence has o.g.f. $\frac{1-x}{1-x+x^2}$. It is the inverse binomial transform of the partial sum of the Catalan numbers i.e. the inverse binomial transform of $\frac{c(x)}{1-x}$.

Example 48. We have the following:

1. $3\,{}_2F_1(1/2, 1; 3; 4x)$ is the o.g.f. of the super-ballot numbers A007054, or $\frac{6(2n)!}{n!(n+2)!} = 4C_n - C_{n+1}$ [101]. The Hankel transform of this sequence is $2n+3$.

2. $3\,{}_2F_1(1/2, 2; 3; 4x)$ is the o.g.f. of $C(n+1)+2C(n)$, or A038629. The Hankel transform of this sequence has o.g.f. $\frac{3-x}{1-4x+x^2}$ (A001835).

3. $10\,{}_2F_1(1/2, 1; 4; 4x)$ is the sequence A007272 with general term $\frac{60(2n)!}{n!(n+3)!}$. It has Hankel transform A000447$(n+2)$ with o.g.f. $\frac{10-5x+4x^2-x^3}{(1-x)^4}$.

4. $5\,{}_2F_1(1/2, 2; 4; 4x)$ yields a sequence with Hankel transform with o.g.f. $(\frac{1-x+x^2}{(1-3x+x^2)^2} - 1)/x$.

5. $10\,{}_2F_1(1/2, 3; 4; 4x)$ is the o.g.f. of the sequence with general term $\frac{30\binom{2n}{n}}{n+3}$. Its Hankel transform has o.g.f. given by $\frac{10-5x+14x^2-x^3}{1-14x+6x^2-14x^3+x^4}$.

Example 49. ${}_2F_1(3, 1; 1; 9x) = (1-9x)^{-1/3}$ is the o.g.f. of A004987 with general term $3^n n! \prod_{k=0}^{n-1} 3k+1$.

Example 50. ${}_2F_1(1/2, 1; 1; 4x(1-kx))$ is the o.g.f. of the sequence $[x^n](1+2x-(k-1)x^2)^n$.

Example 51. We have

$$\begin{aligned} {}_2F_1(-n, 1/2; 1; -4k) &= [x^n](1+(2k+1)x+k^2x^2)^n \\ &= n![x^n]e^{(2k+1)x}I_0(2kx) \\ &= [x^n]\frac{1}{\sqrt{1-2(2k+1)x+(4k+1)x^2}}. \end{aligned}$$

The coefficient array of this family of polynomials in k is the array with general term

$$T_{n,k} = \binom{2k}{k}\binom{n}{k}$$

or

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 1 & 4 & 6 & 0 & 0 & 0 & \dots \\ 1 & 6 & 18 & 20 & 0 & 0 & \dots \\ 1 & 8 & 36 & 80 & 70 & 0 & \dots \\ 1 & 10 & 60 & 200 & 350 & 252 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus

$${}_2F_1(-n, 1/2; 1; -4x) = \sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k} x^k.$$

In particular,

$$[x^n](1 + (2k+1)x + k^2x^2)^n = \sum_{j=0}^{n} \binom{2j}{j}\binom{n}{j} k^j.$$

The row sums of the coefficient array above yield $1, 3, 11, 45, \dots$ or $[x^n](1 + 3x + x^2)^n$.

Taking the reversal of the above array, we obtain the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 6 & 4 & 1 & 0 & 0 & 0 & \dots \\ 20 & 18 & 6 & 1 & 0 & 0 & \dots \\ 70 & 80 & 36 & 8 & 1 & 0 & \dots \\ 252 & 350 & 200 & 60 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The general term of this array is in fact

$$[x^{n-k}]\,{}_2F_1(1/2, k+1; 1; 4x) = \binom{2n-2k}{n-k}\binom{n}{k}.$$

Equivalently,

$$\binom{n+k}{k}\binom{2n}{n} = [x^n]\,{}_2F_1(1/2, k+1; 1; 4x).$$

Example 52. We have

$$[x^{n-k}]\,{}_2F_1(-rn, -n-s; 1; x) = \binom{n+s}{k+s}\binom{rn}{n-k}.$$

For instance, $[x^{n-k}]\,{}_2F_1(-2n, -n, 1, x) = \binom{n}{k}\binom{2n}{n-k}$ is A110608, which has row sums equal to $\binom{3n}{n}$.

Example 53. We have

$$[x^{n-k}]\,{}_2F_1(k+1,k;2;x)=\tilde{\tilde{N}}(n,k),$$

where $\tilde{\tilde{N}}(n,k)=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ for $n,k\geq 1$. This is the matrix

$$\tilde{\tilde{N}}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 3 & 1 & 0 & 0 & \dots \\ 0 & 1 & 6 & 6 & 1 & 0 & \dots \\ 0 & 1 & 10 & 20 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

a version of the Narayana numbers, A001263. Note that

$$[x^{n-k}]\,{}_2F_1(k+1,k;1;x)=(n-k+1)\tilde{\tilde{N}}(n,k),$$

or

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 6 & 1 & 0 & 0 & \dots \\ 0 & 4 & 18 & 12 & 1 & 0 & \dots \\ 0 & 5 & 40 & 60 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with row sums $\frac{\binom{2n}{n}+0^n}{2}$, or A088218. The Hankel transform of this latter sequence is $n+1$. The above triangle has general term $0^{n+k}+\binom{n}{k}\binom{n-1}{k-1}$. It is a left column augmented version of A103371.

Example 54. We have

$$[x^{n-k}]\,{}_2F_1(k+1,-k;1;x)=(-1)^{n-k}\binom{n}{k}\binom{k}{n-k}$$

which is the general term of the array that starts

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -6 & 1 & 0 & 0 & \dots \\ 0 & 0 & 6 & -12 & 1 & 0 & \dots \\ 0 & 0 & 0 & 30 & -20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of this array are $1, 1, -1, -5, -5, \ldots$, A098331, equal to

$$[x^n]\frac{1}{\sqrt{1-2x+5x^2}} = [x^n](1+x-x^2)^n.$$

The binomial transform of this array is the Pascal-like array

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 1 & -3 & -3 & 1 & 0 & 0 & \ldots \\ 1 & -8 & -12 & -8 & 1 & 0 & \ldots \\ 1 & -15 & -20 & -20 & -15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

A version of this triangle (A099037) is related to the Krawtchouk polynomials.

We also have

$$[x^{n-k}]\,{}_2F_1(k+1,-k;2;x) = (-1)^{n-k}\frac{\binom{n}{k}\binom{k}{n-k}}{n-k+1}$$

which is the general term of the array that starts

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 2 & -6 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 10 & -10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This array has row sums $1, 1, 0, -2, -3, 1, 11, \ldots$ or

$$[x^n]\frac{\sqrt{5x^2-2x+1}+x-1}{2x^2} = [x^{n+1}]\mathrm{Rev}\frac{x}{1+x-x^2}.$$

Multiplying this array on the left by Pascal's triangle $\mathbf{B}$ yields the Pascal-like matrix

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 1 & -2 & -4 & -2 & 1 & 0 & \ldots \\ 1 & -5 & -10 & -10 & -5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Finally

$$[x^{n-k}]\,{}_2F_1(k+1,-k;1/2;x) = (-4)^{n-k}\binom{n}{2k-n}$$

which is the general term of the array A117411 which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -4 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -12 & 1 & 0 & 0 & \dots \\ 0 & 0 & 16 & -24 & 1 & 0 & \dots \\ 0 & 0 & 0 & 80 & -40 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This has row sums $1, 1, -3, -11, -7, 41, 117, 29, \ldots$, A006495 which is $[x^n]\frac{1-x}{1-2x+5x^2}$. Multiplying this array by Pascal's triangle $\mathbf{B}$ yields the Pascal-like array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & 0 & 0 & \dots \\ 1 & -9 & -9 & 1 & 0 & 0 & \dots \\ 1 & -20 & -26 & -20 & 1 & 0 & \dots \\ 1 & -35 & -30 & -30 & -35 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

2.18 Transformations on integer sequences

In this section, we shall use the notation $\mathfrak{Seq}$ to denote the set $\mathbb{Z}^{\mathbb{N}_0}$ of integer sequences. By an integer sequence transformation T we shall mean a mapping

$$T : \mathfrak{Seq} \to \mathfrak{Seq}$$

such that

$$T(r_1(a_n) + r_2(b_n)) = r_1 T(a_n) + r_2 T(b_n)$$

for $r_1, r_2 \in \mathcal{R}$.

Example 55. We define the *binomial transform*

$$B : \mathfrak{Seq} \to \mathfrak{Seq}$$

to be the transformation of integer sequences defined by

$$a_n \to b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

Thus $B\{a_n\} = \{b_n\}$ where b_n is defined above. Since $\binom{n}{k}$ is always an integer, this is a mapping from $\mathfrak{Seq}$ to $\mathfrak{Seq}$. It is a classical result that this transformation is invertible, with the inverse, which we denote by B^{-1}, given by

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k.$$

We have already seen above that if $A(x)$ is the e.g.f. of the sequence $(a_n)_{n\geq 0}$, then $\exp(x)A(x)$ is the e.g.f. of the binomial transform (b_n). It is not difficult to show that the k-th iterate of B, denoted by B^k, is given by

$$B^k : a_n \to b_n = \sum_{j=0}^{n} k^{n-j} \binom{n}{j} a_j.$$

We shall see later that if $A(x)$ is the o.g.f. of $(a_n)_{n\geq 0}$, then

$$\frac{1}{1-x} A\left(\frac{x}{1-x}\right)$$

is the o.g.f. of the binomial transform of (a_n).

Example 56. The partial sum of the sequence $(a_n)_{n\geq 0}$ is the sequence $(b_n)_{n\geq 0}$ given by

$$b_n = \sum_{k=0}^{n} a_k.$$

This obviously defines a mapping

$$\mathcal{P} : \mathfrak{Seq} \to \mathfrak{Seq}.$$

If $A(x)$ is the o.g.f. of (a_n) then the o.g.f. of (b_n) is simply given by $\frac{1}{1-x}A(x)$.

Example 57. Define a transform of the sequence $(a_n)_{n\geq 0}$ by the formula

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} a_k.$$

Then the o.g.f. of (b_n) is given by $\frac{1}{1+x}A(x)$. This corresponds to the fact that (b_n) is the convolution of (a_n) and the sequence with general term $(-1)^n$.

Example 58. Define a transform of the sequence $(a_n)_{n\geq 0}$ by the formula

$$b_n = \sum_{k=0}^{n} (-1)^k a_k.$$

Then the o.g.f. of (b_n) has generating function $\frac{1}{1-x}A(-x)$. $(b_n)_{n\geq 0}$ is called the *alternating sum* of (a_n).

Example 59. In this example, we define a transformation on a subspace of $\mathfrak{Seq}$. Thus let $\mathfrak{Seq}_0$ denote the set of integer sequences $(a_n)_{n\geq 0}$ with $a_0 = 0$. For such a sequence (a_n) with o.g.f. $A(x)$, we define the INVERT transform of (a_n) to be the sequence with o.g.f. $B(x) = \frac{A(x)}{1+A(x)}$. We note that since $a_0 = 0$, we also have $b_0 = 0$. Hence we have

$$\text{INVERT} : \mathfrak{Seq}_0 \to \mathfrak{Seq}_0.$$

A number of consequences follow from this. For instance, we have

$$A(x) = \frac{B(x)}{1 - B(x)}.$$

Similarly, we have

$$A(x) - B(x) = A(x)B(x)$$

and hence

$$a_n - b_n = \sum_{k=0}^{n} a_j b_{n-j}.$$

Finally, since $a_0 = 0$ and $b_0 = 0$, we can calculate the b_n according to

$$b_n = a_n - \sum_{j=1}^{n-1} a_j b_{n-j}.$$

As an example, the Fibonacci numbers $F(n)$ with o.g.f. $\frac{x}{1-x-x^2}$ have INVERT transform equal to the Pell numbers $Pell(n)$ with o.g.f. $\frac{x}{1-2x-x^2}$. Thus

$$F(n) - Pell(n) = \sum_{k=0}^{n} F(k)Pell(n-k) = F * Pell(n)$$

.

Example 60. Let $(c_{n,k})_{n,k\geq 0}$ be an element of $\mathbb{Z}^{\mathbb{N}_0\times\mathbb{N}_0}$. Then $(c_{n,k})$ defines a transformation from $\mathfrak{Seq}$ to $\mathfrak{Seq}$ in the following natural way. Given $\{a_n\} \in \mathfrak{Seq}$ we can define $\{b_n\} \in \mathfrak{Seq}$ by

$$b_n = \sum_{k=0} c_{n,k} a_k,$$

where we place appropriate conditions on a_n or $c_{n,k}$ to ensure that the sum is finite for finite n. Thus we consider the element $(c_{n,k})_{n,k\geq 0} \in \mathbb{Z}^{\mathbb{N}_0\times\mathbb{N}_0}$ as an infinite integer matrix.

2.19 The Hankel transform of integer sequences

For a general (integer) sequence $(a_n)_{n\geq 0}$, we define

$$h_n = \Delta_n = \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_{n-1} & a_n & \dots & a_{2n-1} \\ a_n & a_{n+1} & \dots & a_{2n} \end{vmatrix}$$

Then the sequence of numbers h_n is called the Hankel transform of a_n. The Hankel transform is closely associated to the theory of orthogonal polynomials, moment sequences and continued fractions [53, 99]. The link to orthogonal polynomials may be derived as follows. We let

$$h_n(x) = \Delta_n(x) = \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_{n-1} & a_n & \dots & a_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}$$

Then letting

$$P_n(x) = \frac{\Delta_n(x)}{\Delta_{n-1}}$$

defines a family of orthogonal polynomials when $h_n \neq 0$ for all n. This family will then obey a three-term recurrence:

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x).$$

The Hankel transform can be calculated [132, 227] as

$$h_n = a_0^{n+1}\beta_1^n\beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n.$$

The generating function of the sequence a_n is given by

$$\mathcal{G}(x) = \sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1 + \alpha_0 x - \cfrac{\beta_1 x^2}{1 + \alpha_1 x - \cfrac{\beta_2 x^2}{1 + \alpha_2 x - \cdots}}}.$$

2.20 Simple Pascal-like triangles

We shall be concerned in Chapters 10 and 11 with the construction of generalized Pascal-like triangles. In this section, we will study several simple families of Pascal-like triangles, as well as looking at three Pascal-like triangles with hypergeometric definitions. By a *Pascal-like* triangle we shall mean a lower-triangular array $T_{n,k}$ such that $T_{n,0} = 1$, $T_{0,k} = 0^k$ and $T_{n,k} = T_{n,n-k}$.

The first family of Pascal-like triangles corresponds to integer sequences a_n with $a_0 = 0$ and $a_1 = 1$. We define the triangle T_a by

$$T_a(n,k) = [k \leq n](1 + a(k)a(n-k)).$$

Clearly this triangle is Pascal-like. We note that the row sums are given by $n + 1 + \sum_{k=0}^n a(k)a(n-k)$ while the central coefficients $T_a(2n,n)$ are given by $1 + a(n)^2$.

Example 61. Taking $a(n) = F(n)$, we obtain the triangle

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 2 & 3 & 1 & 0 & \dots \\ 1 & 4 & 3 & 3 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with row sums equal to $n+1+\sum_{k=0}^{n} F(k)F(n-k)$ with g.f. $\frac{1-2x+2x^4}{(1-x)^2(1-x-x^2)^2}$. The central coefficients in this case are given by $1+F(n)^2$.

Example 62. The case of $a(n) = 1-0^n$ is of some interest. We obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 1 & 0 & 0 & \dots \\ 1 & 2 & 2 & 2 & 1 & 0 & \dots \\ 1 & 2 & 2 & 2 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term $[k \le n](2-0^{k(n-k)})$. Taking the matrix obtained by removing the first row as a production matrix (see Chapter 4), we find that this matrix generates the array A132372, which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 3 & 1 & 0 & 0 & 0 & \dots \\ 6 & 10 & 5 & 1 & 0 & 0 & \dots \\ 22 & 38 & 22 & 7 & 1 & 0 & \dots \\ 90 & 138 & 98 & 38 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In terms of the so-called Deleham construction (see Chapter 5) this is the array

$$[1,1,2,1,2,1,2,1,\ldots] \quad \Delta \quad [1,0,0,0,\ldots]$$

with generating function

$$\cfrac{1}{1-\cfrac{x(1+y)}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cdots}}}}}}$$

The inverse of this array is the following signed version of the Delannoy triangle A008288

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -3 & 1 & 0 & 0 & 0 & \dots \\ -1 & 5 & -5 & 1 & 0 & 0 & \dots \\ 1 & -7 & 13 & -7 & 1 & 0 & \dots \\ -1 & 9 & -25 & 25 & -9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is the Riordan array

$$\left(\frac{1}{1+x}, \frac{x(1-x)}{1+x}\right).$$

In general, the Pascal-like array with general term $[k \le n](r-(r-1)*0^{k(n-k)})$ leads in a similar manner to the array

$$\left(\frac{1+(r-2)x}{1+(r-1)x}, \frac{x(1-x)}{1+(r-1)x}\right).$$

The next family of Pascal-like triangles derive many of their properties from Pascal's triangle $\mathbf{B}$ with general term $\binom{n}{k}$. Thus we define the triangle $\mathbf{B}_r$ to be the triangle with general term

$$T_{n,k}(r) = [k \le n]\left(\binom{n}{k}(r-(r-1)0^{n-k}) - (r-1)0^k + (r-1)0^{n+k}\right).$$

Clearly, $\mathbf{B}_1 = \mathbf{B}$, while $T_{n,n-k}(r) = T_{n,k}(r)$ shows that this is a Pascal-like family of matrices. We have

$$\mathbf{B}_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2r & 1 & 0 & 0 & 0 & \dots \\ 1 & 3r & 3r & 1 & 0 & 0 & \dots \\ 1 & 4r & 6r & 4r & 1 & 0 & \dots \\ 1 & 5r & 10r & 10r & 5r & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus

$$\mathbf{B}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with inverse

$$\mathbf{B}_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & 1 & 0 & \dots \\ -1 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

while

$$\mathbf{B}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 4 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 8 & 12 & 8 & 1 & 0 & \dots \\ 1 & 10 & 20 & 20 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Interesting characteristics of these triangle are given below.

Sequence	Formula	g.f.
Row sums	$r2^n-(r-1)(2-0^n)$	$\frac{1-x+(r-1)x^2}{(1-x)(1-2x)}$
Diagonal sums	$rF(n+1)-(r-1)\left(\frac{3+(-1)^n}{2}-0^n\right)$	$\frac{1-x^2+2(r-1)x^3+(r-1)x^4}{(1-x^2)(1-x-x^2)}$
Central coefficients	$\binom{2n}{n}(r-(r-1)0^n)$	$-$
Central coefficients	$\binom{n}{\lfloor\frac{n}{2}\rfloor}\left(r-(r-1)\binom{1}{n}\right)$	$-$

Example row sums are presented in the next table, along with their inverse binomial transforms, which have g.f. $\frac{1+x+(r-1)2x^2}{1-x^2}$.

r	Row sum sequence	A-number	Inverse binomial transform
0	$1, 2, 2, 2, 2, 2, 2, \dots$	A040000	$1, 1, -1, 1, -1, 1, \dots$
1	$1, 2, 4, 8, 16, 32, 64, \dots$	A000079	$1, 1, 1, 1, 1, 1, \dots$
2	$1, 2, 6, 14, 30, 62, 126, \dots$	A095121	$1, 1, 3, 1, 3, 1, \dots$
3	$1, 2, 8, 20, 44, 92, 188, \dots$	A131128	$1, 1, 5, 1, 5, 1, \dots$
4	$1, 2, 10, 26, 58, 122, 250, \dots$	A131130	$1, 1, 7, 1, 7, 1, \dots$

These row sums have the interesting property that when we apply a certain inverse "Chebyshev" transform to them then the resulting sequences have interesting Hankel transforms. Anticipating results of later chapters, we have the following. Consider the family of sequences

$$a_n(r) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n}{k}(r2^{n-2k}-(r-1)(2-0^{n-2k})).$$

The sequence $a_n(r)$ is obtained from the corresponding row sum sequence by applying to it the Riordan array $\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}$. Letting $h_n(r)$ denote the Hankel transform of $a_n r$, we find that the ratio

$$\frac{h_n(r)}{(2r)^n}$$

has a form whose formula is easy to conjecture. The first few of these sequences is as follows:

r	First elements of sequence $\frac{h_n(r)}{(2r)^n}$
0	$1, 1, 2, 5, 5, -2, -11, -11, 2, 17, 17, \ldots$
1	$1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, \ldots$
2	$1, 1, -2, -7, 2, 13, 13, -2, -19, -19, \ldots$
3	$1, 1, -4, -13, -13, 4, 25, 25, -4, -37, -37, \ldots$

The indicated form of the Hankel transform of $a_n(r)$ is then

$$\begin{aligned}\frac{h_n(r)}{(2r)^n} = -\frac{4}{3}(r-1)\left(\cos\left(\frac{2}{3}(n-2)\pi\right)+\frac{1}{2}\right)+(-1)^{\frac{n}{3}}(2n(r-1)+1)\left(\frac{2}{3}\cos\left(\frac{2n\pi}{3}\right)+\frac{1}{3}\right)\\ +(-1)^{\frac{n-1}{3}}(2(n-1)(r-1)+1)\left(\frac{2}{3}\cos\left(\frac{2(n-1)\pi}{3}\right)+\frac{1}{3}\right).\end{aligned}$$

where the sequence on the right has generating function

$$\frac{(1+x)(1-x^3(6r-7))}{(1+x^3)^2}-\frac{2(r-1)x^2}{1-x^3}.$$

The next family $T_{n,k}^{(r)}$ is simple to describe: the rth member of the family has general (n,k)-th element given by

$$T_{n,k}^{(r)} = [x^{2k}](1+\sqrt{r}x+x^2)^n.$$

T^0 is Pascal's triangle, so we obtain a parameterized family of Pascal-like triangles $T^{(r)}$ with

$$T^{(0)} = \mathbf{B}.$$

The first few members of the family are

$$\begin{array}{cccccc|cccccc|cccccc}
1 &&&&&& 1 &&&&&& 1 &&&&& \\
1 & 1 &&&&& 1 & 1 &&&&& 1 & 1 &&&& \\
1 & 2 & 1 &&&& 1 & 3 & 1 &&&& 1 & 4 & 1 &&& \\
1 & 3 & 3 & 1 &&& 1 & 6 & 6 & 1 &&& 1 & 9 & 9 & 1 && \\
1 & 4 & 6 & 4 & 1 && 1 & 10 & 19 & 10 & 1 && 1 & 16 & 34 & 16 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1 & 1 & 15 & 45 & 45 & 15 & 1 & 1 & 25 & 90 & 90 & 25 & 1
\end{array}$$

$$\begin{array}{cccccc|cccccc|cccccc}
1 &&&&&& 1 &&&&&& 1 &&&&& \\
1 & 1 &&&&& 1 & 1 &&&&& 1 & 1 &&&& \\
1 & 5 & 1 &&&& 1 & 6 & 1 &&&& 1 & 7 & 1 &&& \\
1 & 12 & 12 & 1 &&& 1 & 15 & 15 & 1 &&& 1 & 18 & 18 & 1 && \\
1 & 22 & 51 & 22 & 1 && 1 & 28 & 70 & 28 & 1 && 1 & 34 & 91 & 34 & 1 & \\
1 & 35 & 145 & 145 & 35 & 1 & 1 & 45 & 210 & 210 & 45 & 1 & 1 & 55 & 285 & 285 & 55 & 1
\end{array}$$

These triangles exist for negative values of r as well. For instance,

$$
\begin{array}{cccccc|cccccc|cccccc}
1 & & & & & & 1 & & & & & & 1 & & & & & \\
1 & 1 & & & & & 1 & 1 & & & & & 1 & 1 & & & & \\
1 & 1 & 1 & & & & 1 & 0 & 1 & & & & 1 & -1 & 1 & & & \\
1 & 0 & 0 & 1 & & & 1 & -3 & -3 & 1 & & & 1 & -6 & -6 & 1 & & \\
1 & -2 & -5 & -2 & 1 & & 1 & -8 & -14 & -8 & 1 & & 1 & -14 & -21 & -14 & 1 & \\
1 & -5 & -15 & -15 & -5 & 1 & 1 & -15 & -30 & -30 & -15 & 1 & 1 & -25 & -35 & -35 & -25 & 1
\end{array}
$$

correspond to the values $r = -1$, $r = -2$ and $r = -3$, respectively. The general triangle $T^{(r)}$ is given by

$$
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & r+2 & 1 & & & \\
1 & 3r+3 & 3r+3 & 1 & & \\
1 & 6r+4 & r^2+12r+6 & 6r+4 & 1 & \\
1 & 10r+5 & 5r^2+30r+10 & 5r^2+30r+10 & 10r+5 & 1
\end{array}
$$

with bivariate generating function

$$G(x,y) = \frac{1 - x(1+y)}{1 - 2x(1+y) + x^2(1-(r-2)y+y^2)}$$

and general term

$$T_{n,k}^{(r)} = \sum_{j=0}^{n} \binom{n}{j} \binom{j}{2(j-k)} r^{j-k}.$$

The central coefficients of the triangle $T^{(r)}$ are given by

$$
\begin{aligned}
T^{(r)}(2n,n) &= \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2n}{n+k} r^k \\
&= \sum_{k=0}^{n} \binom{2n}{2k} \binom{2n-2k}{n-k} r^k.
\end{aligned}
$$

The coefficient array $\binom{n+k}{2k}\binom{2n}{n+k}$ begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
2 & 1 & 0 & 0 & 0 & 0 & \ldots \\
6 & 12 & 1 & 0 & 0 & 0 & \ldots \\
20 & 90 & 30 & 1 & 0 & 0 & \ldots \\
70 & 560 & 420 & 56 & 1 & 0 & \ldots \\
252 & 3150 & 4200 & 1260 & 90 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).
$$

The k-th column of this array has g.f. given by

$$\frac{x^k \sum_{j=0}^{k} \binom{k+j}{2j}\binom{2k}{k+j} x^{k-j}}{(1-4x)^{(4k+1)/2}}.$$

$T^{(0)}(2n,n) = \binom{2n}{n}$ while $T^{(1)}(2n,n)$ is A082758.

We further note that the inverse binomial transform of $T^{(r)}$ has a special form. We find in fact that $\mathbf{B}^{-1}T^{(r)}$ is the array

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & r & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3r & 1 & 0 & 0 & \dots \\ 0 & 0 & r^2 & 6r & 1 & 0 & \dots \\ 0 & 0 & 0 & 5r^2 & 10r & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with general term

$$\binom{n}{2k-n} r^{n-k} = \binom{n}{2n-2k} r^{n-k}.$$

This observation allows us to generalize the foregoing construction. Thus if a_n is a sequence with $a_0 = 1$, we form the matrix with general term

$$T_{n,k}^{(a)} = \sum_{j=0}^{n} \binom{n}{j} \binom{j}{2(j-k)} a_{j-k}.$$

We have

$$T_{n,k}^{(a)} = \sum_{j=0}^{n} \binom{n}{j} \binom{n-j}{2(k-j)} a_{k-j}$$

since $\binom{n}{j} = \binom{n}{n-j}$.

Thus for $a_n = F(n+1)$ we obtain the matrix A162745 which begins

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 & \dots \\ 1 & 15 & 50 & 50 & 15 & 1 & 0 & \dots \\ 1 & 21 & 105 & 173 & 105 & 21 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with row sums starting $1, 2, 5, 14, 42, 132, 427, \ldots$. This is the sequence A162746 which is equal to the second binomial transform of the aerated Fibonacci numbers starting at index

1. This sequence has general term b_n given by

$$\begin{aligned} b_n &= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{2k} F(k+1) \\ &= \sum_{k=0}^{n} 2^{n-k} \binom{n}{k} F(k/2+1) \frac{1+(-1)^k}{2} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{n-2k} F(k+1). \end{aligned}$$

The case $a_n = C_n$ is a special case, yielding the matrix with general term

$$T_{n,k} = \sum_{j=0}^{n} \binom{n}{j} \binom{j}{2(j-k)} [k \le j] C_{j-k}$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 & \ldots \\ 1 & 15 & 50 & 50 & 15 & 1 & 0 & \ldots \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is the Narayana triangle A001263. The row sums of this matrix are C_{n+1}. We obtain

$$C_{n+1} = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{2k} C_k = \sum_{k=0}^{n} 2^{n-k} \binom{n}{k} C_{\frac{k}{2}} \frac{1+(-1)^k}{2} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{n-2k} C_k.$$

Example 63. We let $a_n = \frac{1+(-1)^n}{2}$, with e.g.f. $\cosh(x)$. Then we obtain the matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 7 & 4 & 1 & 0 & 0 & \ldots \\ 1 & 5 & 15 & 15 & 5 & 1 & 0 & \ldots \\ 1 & 6 & 30 & 50 & 30 & 6 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with row sums that start $1, 2, 4, 8, 17, 42, 124, 408, \ldots$ with general term

$$\sum_{k=0}^{n} 2^{n-k} \binom{n}{k} \frac{1+(-1)^{\frac{k}{2}}}{2} \frac{1+(-1)^k}{2}.$$

The Hankel transform of this sequence is $1, 0, 0, -1, 0, 0, \ldots$.

Example 64. We now take $a_n = n!$. We obtain the triangle A162747 which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 & \ldots \\ 1 & 15 & 50 & 50 & 15 & 1 & 0 & \ldots \\ 1 & 21 & 105 & 176 & 105 & 21 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with row sums that start $1, 2, 5, 14, 42, 132, 430, 1444, 4984, \ldots$. This is the sequence A162748, equal to the second binomial transform of the aerated factorial numbers. It has general term

$$\sum_{k=0}^{n} 2^{n-k} \binom{n}{k} \left(\frac{k}{2}\right)! \frac{1 + (-1)^k}{2} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{n-2k} k!$$

and generating function expressible as the continued fraction

$$\cfrac{1}{1 - 2x - \cfrac{x^2}{1 - 2x - \cfrac{x^2}{1 - 2x - \cfrac{2x^2}{1 - 2x - \cfrac{2x^2}{1 - 2x - \cfrac{3x^2}{1 - 2x - \cdots}}}}}}.$$

The Hankel transform of this sequence is A137704.

Example 65. The following example will be returned to in Chapter 11. We let $a_n = (2n-1)!! = \frac{(2n)!}{2^n n!}$. These are the double factorial numbers, A001147, with e.g.f. $\frac{1}{\sqrt{1-2x}}$. This gives us the Pascal-like triangle that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 10 & 21 & 10 & 1 & 0 & 0 & \ldots \\ 1 & 15 & 55 & 55 & 15 & 1 & 0 & \ldots \\ 1 & 21 & 120 & 215 & 120 & 21 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with row sums starting $1, 2, 5, 14, 43, 142, 499, \ldots$, A005425, with general term

$$\sum_{k=0}^{n} 2^{n-k}\binom{n}{k}\frac{k!}{2^{\frac{k}{2}}\left(\frac{k}{2}\right)!}\frac{1+(-1)^k}{2}.$$

The e.g.f. of this sequence is $e^{2x+\frac{x^2}{2}}$. The Hankel transform of this sequence is the sequence of superfactorials A000178.

We close this section by looking at three special arrays whose binomial transforms are Pascal-like, along with three closely related triangular arrays. Thus we consider the expressions

$$_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; r; \frac{4}{x}\right) \qquad \text{and} \qquad _2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; r; \frac{4}{x^2}\right)$$

for $r = \frac{1}{2}$, $r = 1$ and $r = 2$.
Starting with the value $r = \frac{1}{2}$, we find that $x^n\, {}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \frac{4}{x}\right)$ defines the sequence of polynomials

$$1, x, x^2+4x, x^3+12x^2, x^4+24x^3+16x^2, x^5+40x^4+80x^3, \ldots$$

with coefficient array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 12 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 16 & 24 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 80 & 40 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the matrix $\binom{n}{2k-n}4^{n-k}$. We find that its binomial transform is the matrix $T^{(4)}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 15 & 15 & 1 & 0 & 0 & \ldots \\ 1 & 28 & 70 & 28 & 1 & 0 & \ldots \\ 1 & 45 & 210 & 210 & 45 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term $\binom{2n}{2k}$. This is the coefficient array of the polynomials

$$x^n\, {}_2F_1(-n, -n+1/2; 1/2; 1/x).$$

Thus

$$x^n\, {}_2F_1(-n, -n+1/2; 1/2; 1/x) = \sum_{k=0}^{n}\binom{2n}{2k}x^k.$$

We have

$$x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};\frac{4}{x}\right)=\sum_{k=0}^{n}\binom{n}{2k-n}4^{n-k}x^k=\sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{2j}{2k}x^k.$$

We note further that the polynomials ${}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};4x\right)$ or

$$1,1,4x+1,12x+1,16x^2+24x+1,80x^2+40x+1,\ldots$$

are given by

$${}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};4x\right)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}4^kx^k.$$

For example, the sequence $a_n={}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};4\right)$ is A046717 with

$$a_n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}4^k=(3^n+(-1)^n)/2.$$

Associated with these arrays is the coefficient array of the family of polynomials

$$x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};\frac{4}{x^2}\right)$$

or

$$1,x,x^2+4,x^3+12x,x^4+24x^2+16,x^5+40x^3+80x,\ldots$$

The coefficient array is then given by

$$\left(\begin{array}{ccccccc}1&0&0&0&0&0&\ldots\\0&1&0&0&0&0&\ldots\\4&0&1&0&0&0&\ldots\\0&12&0&1&0&0&\ldots\\16&0&24&0&1&0&\ldots\\0&80&0&40&0&1&\ldots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{array}\right)$$

with general term

$$\binom{n}{k}4^{(n-k)/2}(1+(-1)^{n-k})/2,$$

indicating that

$$x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};\frac{4}{x^2}\right)=\sum_{k=0}^{n}\binom{n}{k}4^{(n-k)/2}\frac{1+(-1)^{n-k}}{2}x^k.$$

We now consider the family $x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;\frac{4}{x}\right)$ of polynomials:

$$1,x,x^2+2x,x^3+6x^2,x^4+12x^3+6x^2,x^5+20x^4+30x^3,\ldots$$

with coefficient array A105868:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 6 & 1 & 0 & 0 & \dots \\ 0 & 0 & 6 & 12 & 1 & 0 & \dots \\ 0 & 0 & 0 & 30 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term

$$\binom{k}{n-k}\binom{n}{k}.$$

This signifies that

$$x^n \, {}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; 1; \frac{4}{x}\right) = \sum_{k=0}^n \binom{k}{n-k}\binom{n}{k} x^k.$$

Taking the binomial transform of this coefficient array, we obtain the number triangle

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 4 & 1 & 0 & 0 & 0 & \dots \\ 1 & 9 & 9 & 1 & 0 & 0 & \dots \\ 1 & 16 & 36 & 16 & 1 & 0 & \dots \\ 1 & 25 & 100 & 100 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which has general term $\binom{n}{k}^2$ A008459. This is the coefficient array of the family of polynomials defined by

$$x^n \, {}_2F_1(-n, -n; 1; 1/x),$$

so that

$$x^n \, {}_2F_1(-n, -n; 1; 1/x) = \sum_{k=0}^n \binom{n}{k}^2 x^k.$$

We also have

$$x^n \, {}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; 1; \frac{4}{x}\right) = \sum_{k=0}^n \sum_{j=0}^n (-1)^{n-j}\binom{n}{j}\binom{j}{k}^2 x^k.$$

We note further that the polynomials ${}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; 1; 4x\right)$ or

$$1, 1, 2x+1, 6x+1, 6x^2+12x+1, 30x^2+20x+1, \dots$$

are given by

$$ {}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;4x\right)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{2k}{k}x^k. $$

In particular, we have

$$ {}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;4\right)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{2k}{k}=\sum_{k=0}^{n}\binom{k}{n-k}\binom{n}{k}=t_n $$

where $t_n=[x^n](1+x+x^2)^n$ denotes the n-th central trinomial coefficient A002426. Similarly

$$ \begin{aligned} [x^n](1+x-x^2)^n &= (-1)^n\sum_{k=0}^{n}\binom{k}{n-k}\binom{n}{k}(-1)^k \\ &= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{2k}{k}(-1)^k \\ &= {}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;-4\right) \end{aligned} $$

is A098331.

We consider now the family $x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;\frac{4}{x^2}\right)$, or

$$ 1,x,x^2+2,x^3+6x,x^4+12x^2+6,x^5+20x^3+30x,\ldots $$

with coefficient array

$$ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 6 & 0 & 1 & 0 & 0 & \ldots \\ 6 & 0 & 12 & 0 & 1 & 0 & \ldots \\ 0 & 30 & 0 & 20 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} $$

A109187 with general term

$$ \binom{\frac{n+k}{2}}{k}\binom{n}{\frac{n+k}{2}}\frac{1+(-1)^{n-k}}{2}. $$

This array counts Motzkin paths of length n having k $(1,0)$-steps. We have

$$ x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;\frac{4}{x^2}\right)=\sum_{k=0}^{n}\binom{\frac{n+k}{2}}{k}\binom{n}{\frac{n+k}{2}}\frac{(1+(-1)^{n-k}}{2}x^k. $$

Finally, we consider the family $x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;\frac{4}{x}\right)$, or

$$ 1,x,x^2+x,x^3+3x^2,x^4+6x^3+2x^2,x^5+10x^4+10x^3,\ldots $$

with coefficient array A107131

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 6 & 1 & 0 & \dots \\ 0 & 0 & 0 & 10 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which has general term

$$[k \le n]\binom{n}{2k-n}C_{n-k}.$$

Again, this array is associated to the counting of Motzkin paths, and it has as row sums the Motzkin numbers. We have

$$x^n \, {}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;\frac{4}{x}\right) = \sum_{k=0}^{n}\binom{n}{2k-n}C_{n-k}x^k.$$

Multiplying this coefficient array by the binomial matrix $\mathbf{B}$, we obtain the Narayana numbers

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ 1 & 15 & 50 & 50 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This triangle is the coefficient array of $(x-1)^n \, {}_2F_1\left(-n, n+3; 2; \frac{1}{1-x}\right)$. Thus

$$(x-1)^n \, {}_2F_1\left(-n,n+3;2;\frac{1}{1-x}\right) = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n+1}{k}x^k.$$

We note further that the polynomials ${}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;4x\right)$ or

$$1, 1, x+1, 3x+1, 2x^2+6x+1, 10x^2+10x+1, \dots$$

are given by

$${}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;4x\right) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}C_k x^k.$$

In particular, we have

$${}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;4\right) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}C_k = M_n,$$

the n-th Motzkin number (A001006). From the above, we also see that

$$M_n = \sum_{k=0}^{n} \binom{n}{2k-n} C_{n-k}.$$

We look now at the family $x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; 2; \frac{4}{x^2}\right)$, or

$$1, x, x^2+1, x^3+3x, x^4+6x^2+2, x^5+10x^3+10x, \ldots$$

with coefficient array A097610 which starts

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 1 & 0 & 0 & \ldots \\ 2 & 0 & 6 & 0 & 1 & 0 & \ldots \\ 0 & 10 & 0 & 10 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with general term

$$\binom{n}{k} C_{\frac{n-k}{2}} \frac{1+(-1)^{n-k}}{2}.$$

This array counts Motzkin paths of length n having k horizontal steps. The row sums of the array are indeed the Motzkin numbers. We have

$$x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; 2; \frac{4}{x^2}\right) = \sum_{k=0}^{n} \binom{n}{k} C_{\frac{n-k}{2}} \frac{1+(-1)^{n-k}}{2} x^k.$$

It is interesting to note that the family of polynomials given by

$$x^n\,{}_2F_1(-n, -n+1; 2; 1/x)$$

which begins

$$1, x, x(x+1), x(x^2+3x+1), x(x^3+6x^2+6x+1), x(x^4+10x^3+20x^2+10x+1), \ldots$$

has coefficient array given by the Narayana numbers

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 1 & 0 & \ldots \\ 0 & 1 & 10 & 20 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with general term

$$N(n,k) = 0^{n+k} + \frac{1}{n+0^{nk}}\binom{n}{k}\binom{n}{k-1} = (-1)^k\binom{1}{k}0^n + \binom{n-1}{k-1}\binom{n}{k-1}\frac{1}{k+0^k}.$$

Thus $x^n\,{}_2F_1(-n,-n+1;2;1/x) = \sum_{k=0}^n N(n,k)x^k$. In particular, we find

$$C_n = {}_2F_1(-n,-n+1;2;1).$$

The family of polynomials

$$x^n\,{}_2F_1(-n,-n+1;1;1/x)$$

has coefficient array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 6 & 1 & 0 & 0 & \ldots \\ 0 & 4 & 18 & 12 & 1 & 0 & \ldots \\ 0 & 5 & 40 & 60 & 20 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term $N(n,k)(n-k+1)$. Thus

$$x^n\,{}_2F_1(-n,-n+1;1;1/x) = \sum_{k=0}^n N(n,k)(n-k+1)x^k.$$

In particular we find that ${}_2F_1(-n,-n+1;1;1) = \binom{2n-1}{n}$, A088218.

We summarize these findings as follows.

Polynomial family	Coefficient array	Polynomial family	Coefficient array
$x^n\,{}_2F_1(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};\frac{4}{x})$	$\binom{n}{2k-n}4^{n-k}$	$x^n\,{}_2F_1(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};\frac{4}{x^2})$	$\binom{n}{k}4^{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}$
$x^n\,{}_2F_1(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;\frac{4}{x})$	$\binom{n}{k}\binom{k}{n-k}$	$x^n\,{}_2F_1(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;\frac{4}{x^2})$	$\binom{n}{k}\binom{n-k}{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}$
$x^n\,{}_2F_1(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;\frac{4}{x})$	$\binom{n}{2k-n}C_{n-k}$	$x^n\,{}_2F_1(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;\frac{4}{x^2})$	$\binom{n}{k}C_{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}$

We note the following. We have

$$\begin{aligned} r^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};\frac{4}{r^2}\right) &= \sum_{k=0}^n \binom{n}{k}4^{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}r^k \\ &= \sum_{k=0}^n \binom{n}{k}r^{n-k}4^{\frac{k}{2}}\frac{1+(-1)^k}{2}. \end{aligned}$$

Thus this is the r-th binomial transform of the aerated sequence $1,0,4,0,16,0,64,0,\ldots$ with g.f. $\frac{1}{1-4x^2}$. Thus the sequence with general term $r^n\,{}_2F_1(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};\frac{4}{r^2})$ has g.f. $\frac{1-rx}{1-2rx-(4-r^2)x^2}$, that is,

$$r^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2};\frac{4}{r^2}\right) = [x^n]\frac{1-rx}{1-2rx-(4-r^2)x^2}.$$

Similarly we have

$$r^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;\frac{4}{r^2}\right) = \sum_{k=0}^n \binom{n}{k}\binom{n-k}{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}r^k$$
$$= \sum_{k=0}^n \binom{n}{k} r^{n-k}\binom{k}{k/2}\frac{1+(-1)^k}{2}$$

which shows that $r^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;\frac{4}{r^2}\right)$ is the r-th binomial transform of the aerated sequence of central binomial coefficients $1,0,2,0,6,0,\ldots$. Hence this sequence has g.f. $\frac{1}{\sqrt{1-2rx-(4-r^2)x^2}}$, that is,

$$r^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};1;\frac{4}{r^2}\right) = [x^n]\frac{1}{\sqrt{1-2rx-(4-r^2)x^2}}.$$

Finally, we have

$$r^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;\frac{4}{r^2}\right) = \sum_{k=0}^n \binom{n}{k} C_{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}r^k$$
$$= \sum_{k=0}^n \binom{n}{k} r^{n-k} C_{k/2}\frac{1+(-1)^k}{2}$$

where the last expression is the r-th binomial transform of the aerated Catalan numbers. Thus

$$r^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;\frac{4}{r^2}\right) = [x^n]\frac{1-rx-\sqrt{1-2kx-(4-r^2)x^2}}{2x^2}.$$

Chapter 3

Integer sequences and graphs [1]

Integer sequences arise naturally in the study of graphs. They are closely related to the notion of walk or path on a graph, and their enumeration. In this chapter we shall study the cyclic groups, and find links to a family of decompositions of Pascal's triangle. The role of the circulant matrix is central to much of the work of this note. We first recall some definitions and then study some sequences associated to particular types of graph.

A graph X is a triple consisting of a vertex set $V = V(X)$, an edge set $E = E(X)$, and a map that associates to each edge two vertices (not necessarily distinct) called its endpoints. A *loop* is an edge whose endpoints are equal. To any graph, we may associate the *adjacency matrix*, which is an $n \times n$ matrix, where $|V| = n$ with rows and columns indexed by the elements of the vertex set V and the (x, y)-th element is the number of edges connecting x and y. As defined, graphs are undirected, so this matrix is symmetric. We will restrict ourselves to *simple* graphs, with no loops or multiple edges.

The *degree* of a vertex v, denoted $\deg(v)$, is the number of edges incident with v. A graph is called k-regular if every vertex has degree k. The adjacency matrix of a k*-regular* graph will then have row sums and column sums equal to k.

A *matching* M in a graph X is a set of edges such that no two have a vertex in common. The *size* of a matching is the number of edges in it. An r*-matching* in a graph X is a set of r edges, no two of which have a vertex in common. A vertex contained in an edge of M is *covered* by M. A matching that covers every vertex of X is called a *perfect matching*. We note that a graph that contains a perfect matching has an even number of vertices. A *maximum matching* is a matching with the maximum possible number of edges.

If x, $y \in V$ then an x-y *walk* in X is a (loop-free) finite alternating sequence

$$x = x_0, e_1, x_1, e_2, x_2, e_3, \ldots, e_{r-1}, x_{r-1}, e_r, x_r = y$$

of vertices and edges from X, starting at the vertex x and ending at the vertex y and involving the r edges $e_i = \{x_{i-1}, x_i\}$, where $1 \le i \le r$. The *length* of this walk is r. If $x = y$, the walk is *closed*, otherwise it is *open*. If no edge in the x-y walk is repeated, then the walk is called an x-y *trail*. A closed x-x trail is called a *circuit*. If no vertex of the x-y walk is repeated, then the walk is called an x-y *path*. When $x = y$, a closed path is called a *cycle*.

[1] This chapter reproduces the content of the published article "P. Barry, On Integer Sequences Associated with the Cyclic and Complete Graphs, J. Integer Seq., **10** (2007), Art. 07.4.8" [19].

The number of walks from x to y of length r is given by the x, y-th entry of $\mathbf{A}^r$, where $\mathbf{A}$ is the adjacency matrix of the graph X.
The *cyclic graph* C_r on r vertices is the graph with r vertices and r edges such that if we number the vertices $0, 1, \ldots, r-1$, then vertex i is connected to the two adjacent vertices $i+1$ and $i-1 (\bmod r)$. The *complete graph* K_r on r vertices is the loop-free graph where for all $x, y \in V, x \neq y$, there is an edge $\{x, y\}$.
We note that C_3=K_3.
A final graph concept that will be useful is that of the *chromatic polynomial* of a graph. If $X = (V, E)$ is an undirected graph, a *proper colouring* of X occurs when we colour the vertices of X so that if $\{x, y\}$ is an edge in X, then x and y are coloured with different colours. The minimum number of colours needed to properly colour X is called the *chromatic number* of X and is written $\chi(X)$. For $k \in \mathbf{Z}^+$, we define the polynomial $P(X, k)$ as the number of different ways that we can properly colour the vertices of X with k colours.
For example,

$$P(K_r, k) = k(k-1)\ldots(k-r+1) \tag{3.1}$$

and

$$P(C_r, k) = (k-1)^r + (-1)^r(k-1). \tag{3.2}$$

3.1 Notation

In this chapter, we shall employ the following notation: r will denote the number of vertices in a graph. Note that the adjacency matrix A of a graph with r vertices will then be an $r \times r$ matrix. We shall reserve the number variable n to index the elements of a sequence, as in a_n, the n-th element of the sequence $\mathbf{a} = (a_n)_{n \geq 0}$, or as the n-th power of a number or a matrix (normally this will be related to the n-th term of a sequence). The notation 0^n signifies the integer sequence with generating function 1, which has elements $1, 0, 0, 0, \ldots$.
Note that the Binomial matrix $\mathbf{B}$ and the Fourier matrix $\mathbf{F}_r$ (see below) are indexed from $(0, 0)$, that is, the leftmost element of the first row is the $(0, 0)$-th element. This allows us to give the simplest form of their general (n, k)-th element ($\binom{n}{k}$ and $e^{-\frac{2\pi i n k}{r}}$ respectively).
The adjacency matrix of a graph, normally denoted by $\mathbf{A}$, will be indexed as usual from $(1, 1)$. Similarly the eigenvalues of the adjacency matrix will be labelled $\lambda_1, \lambda_2, \ldots, \lambda_r$.
In the sequel, we will find it useful to use the machinery of circulant matrices.

3.2 Circulant matrices

We now provide a quick overview of the theory of circulant matrices [64], as these will be encountered shortly. An $r \times r$ circulant matrix $\mathbf{C}$ is a matrix whose rows are obtained by shifting the previous row one place to the right, with wraparound, in the following precise sense. If the elements of the first row are $(c_1, \ldots, c_r)$ then

$$c_{jk} = c_{k-j+1}$$

where subscripts are taken modulo n. Circulant matrices are diagonalised by the discrete Fourier transform, whose matrix $\mathbf{F}_r$ is defined as follows : let $\omega(r) = e^{-2\pi i/r}$ where $i = \sqrt{-1}$. Then $\mathbf{F}_r$ has i, j-th element $\omega^{i \cdot j}$, $0 \le i, j \le r - 1$.
We can write the above matrix as $\mathbf{C} = \mathrm{circ}(c_1, \ldots, c_r)$. The permutation matrix $\pi = \mathrm{circ}(0, 1, 0, \ldots, 0)$ plays a special role. If we let p be the polynomial $p(x) = c_1 + c_2 x + \ldots + c_r x^{r-1}$, then $\mathbf{C} = p(\pi)$.
We have, for $\mathbf{C}$ a circulant matrix,

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^{-1} \mathbf{\Lambda} \mathbf{F}, \\ \mathbf{\Lambda} &= \mathrm{diag}(p(1), p(\omega), \ldots, p(\omega^{r-1})). \end{aligned}$$

The i-th eigenvalue of $\mathbf{C}$ is $\lambda_i = p(\omega^i)$, $0 \le i \le r - 1$.

3.3 The graph C_3 and Jacobsthal numbers

We let $\mathbf{A}$ be the adjacency matrix of the cyclic graph C_3. We have

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Here, $p(x) = x(1 + x)$. We note that this matrix is circulant. We shall be interested both in the powers $\mathbf{A}^n$ of $\mathbf{A}$ and its eigenvalues. There is the following connection between these entities:

$$\mathrm{trace}(\mathbf{A}^n) = \sum_{j=1}^{r} \lambda_j^n$$

where $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of $\mathbf{A}$. Here, $r = 3$. In order to obtain the eigenvalues of $\mathbf{A}$, we use $\mathbf{F}_3$ to diagonalize it. We obtain

$$\mathbf{F}_3^{-1} \mathbf{A} \mathbf{F}_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We immediately have

$$\mathrm{trace}(\mathbf{A}^n) = 2^n + 2(-1)^n.$$

Since $J_1(n) = (2^n + 2(-1)^n)/3$, we obtain

Proposition 66. $J_1(n) = \frac{1}{3} \mathit{trace}(\mathbf{A}^n)$

Our next observation relates $J_1(n)$ to 3-colourings of C_r. For this, we recall that $P(C_r, k) = (k-1)^r + (-1)^r(k-1)$. Letting $k = 3$, we get $P(C_r, 3) = 2^r + 2(-1)^r$.

Proposition 67. $J_1(r) = \frac{1}{3} P(C_r, 3)$.

Since $\mathbf{A}$ is circulant, it and its powers $\mathbf{A}^n$ are determined by the elements of their first rows. We shall look at the integer sequences determined by the first row elements of $\mathbf{A}^n$ - that is, we shall look at the sequences $a_{1j}^{(n)}$, for $j = 1, 2, 3$.

Theorem 68.

$$a_{11}^{(n)} = J_1(n),\ a_{12}^{(n)} = J(n),\ a_{13}^{(n)} = J(n).$$

Proof. We use the fact that

$$\mathbf{A}^n = \mathbf{F}^{-1} \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix} \mathbf{F}. \tag{3.3}$$

Then

$$\begin{aligned}
\mathbf{A}^n &= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 2^n & (-1)^n & (-1)^n \\ \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 2^n + 2(-1)^n & (2^n + (-1)^n\omega_3 + (-1)^n\omega_3^2) & (2^n + (-1)^n\omega_3^2 + (-1)^n\omega_3^4) \\ \vdots & \vdots & \vdots \end{pmatrix}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
a_{11}^{(n)} &= (2^n + 2(-1)^n)/3 \\
a_{12}^{(n)} &= (2^n + (-1)^n\omega_3 + (-1)^n\omega_3^2)/3 \\
a_{13}^{(n)} &= (2^n + (-1)^n\omega_3^2 + (-1)^n\omega_3^4)/3.
\end{aligned}$$

The result now follows from the fact that

$$1 + \omega_3 + \omega_3^2 = 1 + \omega_3^2 + \omega_3^4 = 0.$$

□

Corollary 69. *The Jacobsthal numbers count the number of walks on C_3. In particular, $J_1(n)$ counts the number of closed walks of length n on the edges of a triangle based at a vertex. $J(n)$ counts the number of walks of length n starting and finishing at different vertices.*

An immediate calculation gives

Corollary 70.

$$2^n = a_{11}^{(n)} + a_{13}^{(n)} + a_{13}^{(n)}.$$

The identity

$$2^n = 2J(n) + J_1(n)$$

now becomes a consequence of the identity

$$2^n = a_{11}^{(n)} + a_{13}^{(n)} + a_{13}^{(n)}.$$

This is a consequence of the fact that C_3 is 2-regular. We have arrived at a link between the Jacobsthal partition (or colouring) of Pascal's triangle and the cyclic graph C_3. We recall that this comes about since $2^n = J_1(n) + 2J(n)$, and the fact that $J_1(n)$ and $J(n)$ are expressible as sums of binomial coefficients.
We note that although $C_3 = K_3$, it is the cyclic nature of the graph and the fact that it is 2-regular that links it to this partition. We shall elaborate on this later in the chapter.
In terms of ordinary generating functions, we have the identity

$$\frac{1}{1-2x} = \frac{1-x}{(1+x)(1-2x)} + \frac{x}{(1+x)(1-2x)} + \frac{x}{(1+x)(1-2x)}$$

and in terms of exponential generating functions, we have

$$\exp(2x) = \frac{2}{3}\exp(-x)\left(1+\exp(\frac{3x}{2})\sinh(\frac{3x}{2})\right) + 2\frac{2}{3}\exp(\frac{x}{2})\sinh(\frac{3x}{2})$$

or more simply,

$$\exp(2x) = \frac{1}{3}(\exp(2x) + 2\exp(-x)) + 2\frac{1}{3}(\exp(2x) - \exp(-x)).$$

An examination of the calculations above and the fact that $\mathbf{F}$ is symmetric allows us to state

Corollary 71.

$$\begin{pmatrix} a_{11}^{(n)} \\ a_{12}^{(n)} \\ a_{13}^{(n)} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}\begin{pmatrix} 2^n \\ (-1)^n \\ (-1)^n \end{pmatrix}.$$

In fact, this result can be easily generalized to give the following

$$\begin{pmatrix} a_{11}^{(n)} \\ a_{12}^{(n)} \\ \vdots \\ a_{1r}^{(n)} \end{pmatrix} = \frac{1}{r}\mathbf{F}_r\begin{pmatrix} \lambda_1^n \\ \lambda_2^n \\ \vdots \\ \lambda_r^n \end{pmatrix} \tag{3.4}$$

so that

$$\mathbf{A}^n = \mathrm{circ}(a_{11}^{(n)}, a_{12}^{(n)}, \dots, a_{1r}^{(n)}) = \mathrm{circ}(\frac{1}{r}\mathbf{F}_r(\lambda_1^n, \dots, \lambda_r^n)').$$

It is instructive to work out the generating function of these sequences. For instance, we have

$$a_{12}^{(n)} = \frac{1.2^n + \omega.(-1)^n + \omega^2.(-1)^n}{3}.$$

This implies that $a_{12}^{(n)}$ has generating function

$$\begin{aligned}
g_{12}(x) &= \frac{1}{3}\left(\frac{1}{1-2x}+\frac{\omega}{1+x}+\frac{\omega^2}{1+x}\right) \\
&= \frac{1}{3}\left(\frac{(1+x)^2+\omega(1+x)(1-2x)+\omega^2(1-2x)(1+x)}{(1-2x)(1+x)(1+x)}\right) \\
&= \frac{3x(1+x)}{3(1-2x)(1+x)^2} \\
&= \frac{x(1+x)}{1-3x^2-2x^3} \\
&= \frac{x}{1-x-2x^2}.
\end{aligned}$$

This is as expected, but it also highlights the importance of

$$(1-2x)(1+x)(1+x) = (1-\lambda_1 x)(1-\lambda_2 x)(1-\lambda_3 x) = 1-3x^2-2x^3.$$

Hence each of these sequences not only obeys the Jacobsthal recurrence

$$a_n = a_{n-1} + 2a_{n-2}$$

but also

$$a_n = 3a_{n-2} + 2a_{n-3}, \quad n \geq 3.$$

Of course,

$$1-3x^2-2x^3 = (1-p(\omega_3^0)x)(1-p(\omega_3^1)x)(1-p(\omega_3^2)x)$$

where $p(x) = x + x^2$.

3.4 The case of C_4

For the cyclic graph on four vertices C_4 we have the following adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{3.5}$$

Here, $p(x) = x(1+x^2)$. We can carry out a similar analysis as for the case $n = 3$. Using $\mathbf{F}$ to diagonalize $\mathbf{A}$, we obtain

$$\mathbf{F}^{-1}\mathbf{A}\mathbf{F} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this, we can immediately deduce the following result.

Theorem 72.

$$\frac{1}{4}\text{trace}\mathbf{A}^n = \frac{1}{4}(2^n + (-2)^n + 2.0^n) = 1, 0, 2, 0, 8, 0, 32, \ldots$$

Theorem 73.

$$\begin{aligned}
a_{11}^{(n)} &= (2^n + (-2)^n + 2.0^n)/4 = 1, 0, 2, 0, 8, 0, 32, \ldots \\
a_{12}^{(n)} &= (2^n - (-2)^n)/4 = 0, 1, 0, 4, 0, 16, 0 \ldots \\
a_{13}^{(n)} &= (2^n + (-2)^n - 2.0^n)/4 = 0, 0, 2, 0, 8, 0, 32, \ldots \\
a_{14}^{(n)} &= (2^n - (-2)^n)/4 = 0, 1, 0, 4, 0, 16, 0, \ldots
\end{aligned}$$

Proof. We have

$$\mathbf{A}^n = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 & 0 \\ 0 & 0^n & 0 & 0 \\ 0 & 0 & (-2)^n & 0 \\ 0 & 0 & 0 & 0^n \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

From this we obtain

$$\begin{aligned}
a_{11}^{(n)} &= (2^n + 0^n + (-2)^n + 0^n)/4 = (2^n + (-2)^n + 2.0^n)/4 \\
a_{12}^{(n)} &= (2^n - i.0^n - (-2)^n + i.0^n)/4 = (2^n - (-2)^n)/4 \\
a_{13}^{(n)} &= (2^n - 1.0^n + (-2)^n - 1.0^n)/4 = (2^n + (-2)^n - 2.0^n)/4 \\
a_{14}^{(n)} &= (2^n + i.0^n - (-2)^n - i.0^n)/4 = (2^n - (-2)^n)/4.
\end{aligned}$$

□

Corollary 74. *The sequences above count the number of walks on the graph C_4. In particular, $a_{11}^{(n)}$ counts the number of closed walks on the edges of a quadrilateral based at a vertex.*

An easy calculation also gives us the important

Corollary 75.

$$2^n = a_{11}^{(n)} + a_{12}^{(n)} + a_{13}^{(n)} + a_{14}^{(n)}.$$

In terms of ordinary generating functions of the sequences $a_{11}^{(n)}$, $a_{12}^{(n)}$, $a_{13}^{(n)}$, $a_{14}^{(n)}$, we have the following algebraic expression

$$\frac{1}{1-2x} = \frac{1}{1-2x}\left(\frac{1-2x^2}{1+2x} + \frac{x}{1+2x} + \frac{2x^2}{1+2x} + \frac{x}{1+2x}\right).$$

Anticipating the general case, we can state

Theorem 76. *There exists a partition*

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} + \mathbf{B_2} + \mathbf{B_3}$$

where the $\mathbf{B_i}$ are zero-binomial matrices with row sums equal to $a_{11}^{(n)}$, $a_{12}^{(n)}$, $a_{13}^{(n)}$, $a_{14}^{(n)}$, respectively, for $i = 0 \ldots 3$.

In fact, we have

$$\begin{aligned}
a_{11}^{(n)} &= \sum_{2k-n \equiv 0 \bmod 4} \binom{n}{k} \\
a_{12}^{(n)} &= \sum_{2k-n \equiv 1 \bmod 4} \binom{n}{k} \\
a_{13}^{(n)} &= \sum_{2k-n \equiv 2 \bmod 4} \binom{n}{k} \\
a_{14}^{(n)} &= \sum_{2k-n \equiv 3 \bmod 4} \binom{n}{k}.
\end{aligned}$$

We shall provide a proof for this later, when we look at the general case. Each of these sequences satisfy the recurrence

$$a_n = 4a_{n-2}.$$

We can see the decomposition induced from C_4 in the following coloured rendering of $\mathbf{B}$.

$$\mathbf{B} = \left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\
1 & 3 & 3 & 1 & 0 & 0 & 0 & \dots \\
1 & 4 & 6 & 4 & 1 & 0 & 0 & \dots \\
1 & 5 & 10 & 10 & 5 & 1 & 0 & \dots \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
\quad
\begin{array}{cccc}
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\
\mathbf{2} & \mathbf{0} & \mathbf{2} & \mathbf{0} \\
\mathbf{0} & \mathbf{4} & \mathbf{0} & \mathbf{4} \\
\mathbf{8} & \mathbf{0} & \mathbf{8} & \mathbf{0} \\
\mathbf{0} & \mathbf{16} & \mathbf{0} & \mathbf{16} \\
\mathbf{32} & \mathbf{0} & \mathbf{32} & \mathbf{0} \\
. & . & . & .
\end{array}$$

3.5 The case of C_5

This case is worth noting, in the context of integer sequences, as there is a link with the Fibonacci numbers. For the cyclic graph on five vertices C_5 we have the following adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.6}$$

Here, $p(x) = x(1 + x^3)$.
Diagonalizing with $\mathbf{F}$, we obtain

$$\mathbf{\Lambda} = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{5}}{2} - \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & -\frac{\sqrt{5}}{2} - \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{5}}{2} - \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2} - \frac{1}{2}
\end{pmatrix}$$

or

$$\mathbf{\Lambda} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\phi} & 0 & 0 & 0 \\ 0 & 0 & -\phi & 0 & 0 \\ 0 & 0 & 0 & -\phi & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\phi} \end{pmatrix}$$

where $\phi = \frac{\sqrt{5}+1}{2}$.

Proposition 77. $\frac{1}{5}\text{trace}(\mathbf{A}^n) = (2^n + 2(-1)^n(F(n+1) + F(n-1)))/5$.

Proof. We have $\frac{1}{5}\text{trace}(\mathbf{A}^n) = \left(2^n + 2\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^n + 2\left(-\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^n\right)/5$. An easy manipulation produces the result. □

This sequence is A054877.

Theorem 78.

$$\begin{aligned}
a_{11}^{(n)} &= \left(2^n + 2\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^n + 2\left(-\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^n\right)/5 \\
a_{12}^{(n)} &= \left(2^n + \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^{n+1} + \left(-\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^{n+1}\right)/5 \\
a_{13}^{(n)} &= \left(2^n - \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^n\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right) + \left(-\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^n\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)\right)/5 \\
a_{14}^{(n)} &= \left(2^n - \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^n\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right) + \left(-\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^n\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)\right)/5 \\
a_{15}^{(n)} &= \left(2^n + \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^{n+1} + \left(-\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^{n+1}\right)/5.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
a_{11}^{(n)} &= \left(2^n + 2\phi^{-n} + 2(-\phi)^n\right)/5 \\
a_{12}^{(n)} &= \left(2^n + \phi^{-n-1} + (-\phi)^{n+1}\right)/5 \\
a_{13}^{(n)} &= \left(2^n - \phi^{-n+1} + (-1)^n\phi^{n+1}\right)/5 \\
a_{14}^{(n)} &= \left(2^n - \phi^{-n+1} + (-1)^n\phi^{n+1}\right)/5 \\
a_{15}^{(n)} &= \left(2^n + \phi^{n+1} + (-\phi)^{n+1}\right)/5.
\end{aligned}$$

Proof. The result follows from the fact that the first row of $\mathbf{A}^n$ is given by $\frac{1}{5}\mathbf{F}(\lambda_1^n, \lambda_2^n, \lambda_3^n, \lambda_4^n, \lambda_5^n)'$. □

Corollary 79. *The sequences in Theorem 78 count walks on C_5. In particular, the sequence $a_{11}^{(n)}$ counts closed walks of length n along the edges of a pentagon, based at a vertex.*

We note that $a_{12}^{(n)} = a_{15}^{(n)}$. This is A052964. Similarly $a_{13}^{(n)} = a_{14}^{(n)}$. This is (the absolute value of) A084179.
An easy calculation gives us the important result

Corollary 80.
$$2^n = a_{11}^{(n)} + a_{12}^{(n)} + a_{13}^{(n)} + a_{14}^{(n)} + a_{15}^{(n)}.$$

In terms of the ordinary generating functions for these sequences, we obtain the following algebraic identity

$$\frac{1}{1-2x} = \frac{1}{1-2x}\left(\frac{1-x-x^2}{1+x-x^2} + \frac{2x(1-x)}{1+x-x^2} + \frac{2x^2}{1+x-x^2}.\right)$$

Anticipating the general case, we can state the

Theorem 81. *There exists a partition*

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} + \mathbf{B_2} + \mathbf{B_3} + \mathbf{B_4}$$

where the $\mathbf{B_i}$ *are zero-binomial matrices with row sums equal to* $a_{11}^{(n)}$, $a_{12}^{(n)}$, $a_{13}^{(n)}$, $a_{14}^{(n)}$, $a_{15}^{(n)}$, *respectively, for* $i = 0 \dots 4$.

In fact, we have

$$\begin{aligned}
a_{11}^{(n)} &= \sum_{2k-n \equiv 0 \bmod 5} \binom{n}{k} \\
a_{12}^{(n)} &= \sum_{2k-n \equiv 1 \bmod 5} \binom{n}{k} \\
a_{13}^{(n)} &= \sum_{2k-n \equiv 2 \bmod 5} \binom{n}{k} \\
a_{14}^{(n)} &= \sum_{2l-n \equiv 3 \bmod 5} \binom{n}{k} \\
a_{15}^{(n)} &= \sum_{2k-n \equiv 4 \bmod 5} \binom{n}{k}.
\end{aligned}$$

We note that

$$(1-p(\omega_5^0)x)(1-p(\omega_5^1)x)(1-p(\omega_5^2)x)(1-p(\omega_5^3)x)(1-p(\omega_5^4)x) = 1 - 5x^3 + 5x^4 - 2x^5$$

for $p(x) = x + x^4$ which implies that each of the sequences satisfies the recurrence

$$a_n = 5a_{n-3} - 5a_{n-4} + 2a_{n-5}.$$

3.6 The General Case of C_r

We begin by remarking that since C_r is a 2-regular graph, its first eigenvalue is 2. We have seen explicit examples of this in the specific cases studied above. We now let $\mathbf{A}$ denote the adjacency matrix of C_r. We have $\mathbf{A} = p(\pi)$ where $p(x) = x + x^{r-1}$, so $\mathbf{A} = \mathrm{circ}(0, 1, 0, \ldots, 0, 1)$.

Theorem 82.

$$2^n = \sum_{j=1}^{r} a_{1j}^{(n)}$$

where $a_{1j}^{(n)}$ is the j-th element of the first row of $\mathbf{A}^n$.

Proof. We have

$$\begin{aligned}
(a_{1j}^{(n)})_{1 \le j \le r} &= \frac{1}{r}\mathbf{F}(\lambda_1^n, \lambda_2^n, \ldots, \lambda_r^n)' \\
&= \frac{1}{r}(\sum_{k=1}^{r} \lambda_k^n \omega^{(j-1)(k-1)}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\sum_{j=1}^{r} a_{1j}^{(n)} &= \frac{1}{n}\sum_{j=1}^{r}\sum_{k=1}^{r} \lambda_k^n \omega^{(j-1)(k-1)} \\
&= \frac{1}{r}\sum_{k=1}^{r} \lambda_k^n \sum_{j=1}^{r} \omega^{(j-1)(k-1)} \\
&= \frac{1}{r} r \lambda_1^n = \frac{1}{r} r 2^n = 2^n.
\end{aligned}$$

□

We can now state the main result of this section.

Theorem 83. *Let $\mathbf{A}$ be the adjacency matrix of the cyclic graph on r vertices C_r. Let $a_{1j}^{(n)}$ be the first row elements of the matrix $\mathbf{A}^n$. There exists a partition*

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} + \ldots + \mathbf{B_{r-1}}$$

where the $\mathbf{B_i}$ are zero-binomial matrices with row sums equal to the sequences $a_{1,i+1}^{(n)}$, respectively, for $i = 0 \ldots r-1$.

Proof. We have already shown that

$$2^n = \sum_{i=0}^{n} \binom{n}{i} = \sum_{j=1}^{r} a_{1j}^{(n)}.$$

We shall now exhibit a partition of this sum which will complete the proof. For this, we recall that $\mathbf{A} = p(\pi)$, where $p(x) = x + x^{r-1}$. Then

$$
\begin{aligned}
(a_{1j}^{(n)})_{1\le j\le r} &= \frac{1}{r}\mathbf{F}_r \begin{pmatrix} p(\omega_r^0)^n \\ p(\omega_r^1)^n \\ p(\omega_r^2)^n \\ \vdots \\ p(\omega_r^{r-1})^n \end{pmatrix} \\
&= \frac{1}{r}\mathbf{F}_r \begin{pmatrix} (\omega^0 + \omega^{0.(r-1)})^n \\ (\omega^1 + \omega^{1.(r-1)})^n \\ (\omega^2 + \omega^{2.(r-1)})^n \\ \vdots \\ (\omega^{r-1} + \omega^{(r-1).(r-1)})^n \end{pmatrix} \\
&= \frac{1}{r}\mathbf{F}_r \begin{pmatrix} (\omega^0 + \omega^{-0})^n \\ (\omega^1 + \omega^{-1})^n \\ (\omega^2 + \omega^{-2})^n \\ \vdots \\ (\omega^{r-1} + \omega^{-(r-1)})^n \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
a_{1j}^{(n)} &= \frac{1}{r}\sum_{l=0}^{r-1}(\omega_r^l + \omega_r^{-l})^n \omega_r^{(j-1)l} \\
&= \frac{1}{r}\sum_{l=0}^{r-1}\sum_{k=0}^{n}\binom{n}{k}\omega_r^{kl}\omega_r^{-l(n-k)}\omega_r^{(j-1)l} \\
&= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{r}\sum_{l=0}^{r-1}\omega_r^{kl}\omega_r^{-l(n-k)}\omega_r^{(j-1)l}\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{r}\sum_{l=0}^{r-1}\omega_r^{2kl+l(j-1-n)}\right) \\
&= \sum_{r|2k+(j-1-n)}\binom{n}{k} \\
&= \sum_{2k\equiv(n+1-j) \bmod r}\binom{n}{k}.
\end{aligned}
$$

We thus have

$$(B_i)_{n,k} = [2k \equiv n+1-i \bmod r]\binom{n}{k}.$$

$\square$

Corollary 84.

$$\mathbf{A}^n = \operatorname{circ}\left(\frac{1}{r}\mathbf{F}_r\left(2^n\cos(\frac{2\pi j}{r})^n\right)_{0\le j\le r-1}\right).$$

Proof. This comes about since $(\omega^j + \omega^{-j}) = e^{2\pi i j/k} + e^{-2\pi i j/k} = 2\cos(2\pi j/r)$. $\square$

This verifies the well-known fact that the eigenvalues of C_r are given by $2\cos(2\pi j/r)$, for $0 \le j \le r-1$ [248]. It is clear now that if σ_i = i-th symmetric function in $2\cos(2\pi j/r)$, $0 \le j \le r-1$, then the sequences $a_{1j}^{(n)}$, $1 \le j \le r$, satisfy the recurrence

$$a_n = \sigma_2 a_{n-2} - \sigma_3 a_{n-3} + \cdots + (-1)^{r-1}\sigma_{r-1}a_{r-1}.$$

We thus have

Corollary 85. *The sequences*

$$a_{1j}^{(n)} = \sum_{2k \equiv n+1-j \bmod r} \binom{n}{k},$$

which satisfy the recurrence

$$a_n = \sigma_2 a_{n-2} - \sigma_3 a_{n-3} + \cdots + (-1)^{r-1}\sigma_{r-1}a_{r-1}$$

count the number of walks of length n from vertex 1 to vertex j of the cyclic graph C_r.

3.7 A worked example

We take the case $r = 8$. We wish to characterize the 8 sequences $\sum_{8|2k+(j-1-n)} \binom{n}{k}$ for $j = 1 \ldots 8$. We give details of these sequences in the following table.

sequence	a_n	binomial expression
$1, 0, 2, 0, 6, 0, 20 \ldots$	$(1 + (-1)^n)(0^n + 2.2^{n/2} + 2^n)/8$	$\sum_{n-2k\equiv 0 \bmod 8} \binom{n}{k}$
$0, 1, 0, 3, 0, 10, 0 \ldots$	$(1 - (-1)^n)(2^n + \sqrt{2}(\sqrt{2})^n)/8$	$\sum_{2k-n\equiv 1 \bmod 8} \binom{n}{k}$
$0, 0, 1, 0, 4, 0, 16, \ldots$	$(1 + (-1)^n)2^n/8 - 0^n/4$	$\sum_{2k-n\equiv 2 \bmod 8} \binom{n}{k}$
$0, 0, 0, 1, 0, 6, 0 \ldots$	$(1 - (-1)^n)(2^n - \sqrt{2}(\sqrt{2})^n)/8$	$\sum_{2k-n\equiv 3 \bmod 8} \binom{n}{k}$
$0, 0, 0, 0, 2, 0, 12, \ldots$	$(1 + (-1)^n)(2^n - 2(\sqrt{2})^n)/8 + 0^n/4$	$\sum_{2k-n\equiv 4 \bmod 8} \binom{n}{k}$
$0, 0, 0, 1, 0, 6, 0 \ldots$	$(1 - (-1)^n)(2^n - \sqrt{2}(\sqrt{2})^n)/8$	$\sum_{2k-n\equiv 5 \bmod 8} \binom{n}{k}$
$0, 0, 1, 0, 4, 0, 16, \ldots$	$(1 + (-1)^n)2^n/8 - 0^n/4$	$\sum_{2k-n\equiv 6 \bmod 8} \binom{n}{k}$
$0, 1, 0, 3, 0, 10, 0 \ldots$	$(1 - (-1)^n)(2^n + \sqrt{2}(\sqrt{2})^n)/8$	$\sum_{2k-n\equiv 7 \bmod 8} \binom{n}{k}$

In terms of ordinary generating functions, we have

$$\begin{aligned}
1,0,2,0,6,0,20\ldots &: \frac{1-4x+2x^2}{(1-2x^2)(1-4x^2)} \\
0,1,0,3,0,10,0\ldots &: \frac{x(1-3x^2)}{(1-2x^2)(1-4x^2)} \\
0,0,1,0,4,0,16,\ldots &: \frac{x^2(1-2x^2)}{(1-2x^2)(1-4x^2)} \\
0,0,0,1,0,6,0\ldots &: \frac{x^3}{(1-2x^2)(1-4x^2)} \\
0,0,0,0,2,0,12,\ldots &: \frac{2x^4}{(1-2x^2)(1-4x^2)} \\
0,0,0,1,0,6,0\ldots &: \frac{x^3}{(1-2x^2)(1-4x^2)} \\
0,0,1,0,4,0,16,\ldots &: \frac{x^2(1-2x^2)}{(1-2x^2)(1-4x^2)} \\
0,1,0,3,0,10,0\ldots &: \frac{x(1-3x^2)}{(1-2x^2)(1-4x^2)}
\end{aligned}$$

All these sequences satisfy the recurrence

$$a_n = 6a_{n-2} - 8a_{n-4}$$

with suitable initial conditions. In particular, the sequence $1,0,2,0,6,\ldots$ has general term

$$a_{11}^{(n)} = \frac{0^n}{4} + (1+(-1)^n)\left(\frac{2^n}{8} + \frac{(\sqrt{2})^n}{4}\right).$$

This counts the number of closed walks at a vertex of an octagon.

The sequences are essentially A112798, A007582, A000302, A006516, A020522, with interpolated zeros.

3.8 The case $n \to \infty$

We recall that the modified Bessel function of the first kind [231, 249] is defined by the integral

$$I_k(z) = \int_0^\pi e^{z\cos(t)}\cos(kt)dt.$$

$I_n(z)$ has the following generating function

$$e^{z(t+1/t)/2} = \sum_{k=-\infty}^{\infty} I_k(z)t^k.$$

Letting $z = 2x$ and $t = 1$, we get

$$e^{2x} = \sum_{k=-\infty}^{\infty} I_k(2x) = I_0(2x) + 2\sum_{k=1}^{\infty} I_k(2x).$$

The functions $I_k(2x)$ are the exponential generating functions for the columns of Pascal's matrix (including 'interpolated' zeros). For instance, $I_0(2x)$ generates the sequence of central binomial coefficients $1, 0, 2, 0, 6, 0, 20, 0, 70, \ldots$ with formula $\binom{n}{n/2}(1 + (-1)^n)/2$. This gives us the limit case of the decompositions of Pascal's triangle - in essence, each of the infinite matrices that sum to $\mathbf{B}_\infty$ corresponds to a matrix with only non-zero entries in a single column.

The matrix

$$\mathbf{A}_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

corresponds to the limit cyclic graph C_∞. We can characterize the (infinite) set of sequences that correspond to the row elements of the powers $\mathbf{A}_\infty^n$ as those sequences with exponential generating functions given by the family $I_k(2x)$. We also obtain that $\mathrm{trace}(\mathbf{A}_\infty^n)$ is the set of central binomial numbers (with interpolated zeros) generated by $I_0(2x)$.

3.9 Sequences associated to K_r

By way of example for what follows, we look at the adjacency matrix $\mathbf{A}$ for K_4. We note that K_4 is 3-regular. $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Again, this matrix is circulant, with defining polynomial $p(x) = x + x^2 + x^3 = x(1 + x + x^2)$. Using $\mathbf{F}$ to diagonalize it, we obtain

$$\mathbf{\Lambda} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This leads us to the sequence $\frac{1}{4}\mathrm{trace}(\mathbf{A}^n) = (3^n + 3(-1)^n)/4 = 1, 0, 3, 6, 21, 60, \ldots$, which is A054878. Comparing this result with the expression $P(C_n, 4) = 3^n + 3(-1)^n$ we see that $\mathrm{trace}(\mathbf{A}^n) = P(C_n, 4)$.

Proposition 86.

$$\begin{aligned} a_{11}^{(n)} &= (3^n + 3(-1)^n)/4 = 1, 0, 3, 6, 21, 60, \ldots \\ a_{12}^{(n)} &= (3^n - (-1)^n)/4 = 0, 1, 2, 7, 20, 61, \ldots \\ a_{13}^{(n)} &= (3^n - (-1)^n)/4 = 0, 1, 2, 7, 20, 61, \ldots \\ a_{14}^{(n)} &= (3^n - (-1)^n)/4 = 0, 1, 2, 7, 20, 61, \ldots \end{aligned}$$

Proof. Using

$$(a_{1j}^{(n)})_{1 \le j \le n} = \frac{1}{n} \mathbf{F}(\lambda_1^n, \lambda_2^n, \ldots, \lambda_n^n)'$$

we obtain, for instance,

$$\begin{aligned} a_{12}^{(n)} &= (3^n + \omega(-1)^n + \omega^2(-1)^n + \omega^3(-1)^n)/4 \\ &= (3^n + (-1)^n(\omega + \omega^2 + \omega^3))/4 = (3^n - (-1)^n)/4. \end{aligned}$$

□

We note that $a_{11}^{(n)}$ is A054878, while $a_{12}^{(n)} = a_{13}^{(n)} = a_{14}^{(n)}$ are all equal to A015518.

Corollary 87.

$$3^n = \mathbf{a}_{11}^{(n)} + \mathbf{a}_{12}^{(n)} + \mathbf{a}_{13}^{(n)} + \mathbf{a}_{14}^{(n)}.$$

Corollary 88. *The sequences $a_{1j}^{(n)}$ satisfy the linear recurrence*

$$a_n = 2a_{n-1} + 3a_{n-2}$$

with initial conditions

$$\begin{aligned} a_0 = 1,\ a_1 = 0,\ j = 0 \\ a_0 = 0,\ a_1 = 1,\ j = 2 \ldots 4. \end{aligned}$$

This result is typical of the general case, which we now address. Thus we let $\mathbf{A}$ by the adjacency matrix of the complete graph K_r on r vertices.

Lemma 89. *The eigenvalues of* $\mathbf{A}$ *are* $r-1, -1, \ldots, -1$.

Proof. We have $\mathbf{A} = p(\pi)$, where $p(x) = x + x^2 + \ldots + x^{r-1}$. The eigenvalues of $\mathbf{A}$ are $p(1), p(\omega), p(\omega^2), \ldots, p(\omega^{r-1})$, where $\omega^r = 1$. Then $p(1) = 1 + \ldots + 1 = r - 1$. Now

$$p(x) = x + \ldots + x^{r-1} = 1 + x + \ldots + x^{r-1} - 1 = \frac{1 - x^r}{1 - x} - 1.$$

Then

$$p(\omega^j) = \frac{1 - \omega^{rj}}{1 - \omega^j} - 1 = -1$$

since $\omega^{rj} = 1$ for $j \ge 1$. □

Theorem 90. *Let* $\mathbf{A}$ *be the adjacency matrix of the complete graph* K_r *on* r *vertices. Then the* r *sequences* $\mathbf{a}_{1j}^{(n)}$ *defined by the first row of* $\mathbf{A}^n$ *satisfy the recurrence*

$$a_n = (r-2)a_{n-1} + (r-1)a_{n-2}$$

with initial conditions

$$\begin{aligned} a_0 = 1,\ a_1 = 0,\ j = 1 \\ a_0 = 0,\ a_1 = 1,\ j = 2 \ldots r. \end{aligned}$$

In addition, we have

$$(r-1)^n = \sum_{j=1}^{r} a_{1j}^{(n)}.$$

Proof. We have

$$\begin{aligned} \begin{pmatrix} a_{11}^{(n)} \\ a_{12}^{(n)} \\ \vdots \\ a_{1r}^{(n)} \end{pmatrix} &= \frac{1}{r}\mathbf{F}_r \begin{pmatrix} \lambda_1^n \\ \lambda_2^n \\ \vdots \\ \lambda_r^n \end{pmatrix} \\ &= \frac{1}{r}\mathbf{F}_r \begin{pmatrix} p(1)^n \\ p(\omega_r^2)^n \\ \vdots \\ p(\omega_r^{r-1})^n \end{pmatrix} \\ &= \frac{1}{r}\mathbf{F}_r \begin{pmatrix} (r-1)^n \\ (-1)^n \\ \vdots \\ (-1)^n \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} a_{11}^{(n)} &= \frac{1}{r}((r-1)^n + (-1)^n + \ldots + (-1)^n) \\ &= \frac{1}{r}((r-1)^n + (r-1)(-1)^n). \end{aligned}$$

It is now easy to show that $a_{11}^{(n)}$ satisfies the recurrence

$$a_n = (r-2)a_{n-1} + (r-1)a_{n-2}$$

with $a_0 = 1$ and $a_1 = 0$.

For $j > 1$, we have

$$\begin{aligned} a_{1j}^{(n)} &= \frac{1}{r}((r-1)^n + (-1)^n\omega^j + \cdots + (-1)^n\omega^{j(k-1)}) \\ &= \frac{1}{r}((r-1)^n + (-1)^n(\omega^j + \ldots + \omega^{j(k-1)}) \\ &= \frac{1}{r}((r-1)^n + (-1)^n(\frac{1-\omega^{jk}}{1-\omega^j} - 1) \\ &= \frac{1}{r}((r-1)^n - (-1)^n). \end{aligned}$$

This is the solution of the recurrence

$$a_n = (r-2)a_{n-1} + (r-1)a_{n-2}$$

with $a_0 = 0$ and $a_1 = 1$ as required. To prove the final assertion, we note that

$$\begin{aligned} \sum_{j=1}^{r} a_{1j}(n) &= a_{11}(n) + (r-1)a_{12}^{(n)} \\ &= \frac{(r-1)^n}{r} + \frac{(-1)^n(r-1)}{r} + (r-1)\left(\frac{(r-1)^n}{r} - \frac{(-1)^n}{r}\right) \\ &= \frac{(r-1)^n}{r}(1+r-1) + \frac{(-1)^n(r-1)}{r} - \frac{(r-1)(-1)^n}{r} \\ &= \frac{(r-1)^n}{r}r = (r-1)^n. \end{aligned}$$

□

Thus the recurrences have solutions

$$a_n = \frac{(r-1)^n}{r} + \frac{(-1)^n(r-1)}{r}$$

when

$$a_0 = 1,\ a_1 = 0,$$

and

$$a_n' = \frac{(r-1)^n}{r} - \frac{(-1)^n}{r}$$

for

$$a_0' = 0,\ a_1' = 1.$$

We recognize in the first expression above the formula for the chromatic polynomial $P(C_n, r)$, divided by the factor r. Hence we have

Corollary 91. $\frac{1}{r}\mathrm{trace}(\mathbf{A}^n) = a_{11}^{(n)} = \frac{1}{r}P(C_n, r)$.

We list below the first few of these sequences, which count walks of length n on the complete graph K_r. Note that we give the sequences in pairs, as for each value of r, there are only two distinct sequences. The first sequence of each pair counts the number of closed walks from a vertex on K_r. In addition, it counts r-colourings on C_n (when multiplied by r).

$$
\begin{aligned}
r &= 3 \\
(2^n + 2(-1)^n)/3 &: \quad 1, 0, 2, 2, 6, 10, 22, \ldots \\
(2^n - (-1)^n)/3 &: \quad 0, 1, 1, 3, 5, 11, 21, \ldots \\
r &= 4 \\
(3^n + 3(-1)^n)/4 &: \quad 1, 0, 3, 6, 21, 60, 183, \ldots \\
(3^n - (-1)^n)/4 &: \quad 0, 1, 2, 7, 20, 61, 182, \ldots \\
r &= 5 \\
(4^n + 4(-1)^n)/5 &: \quad 1, 0, 4, 12, 52, 204, 820, \ldots \\
(4^n - (-1)^n)/5 &: \quad 0, 1, 3, 13, 51, 205, 819, \ldots \\
r &= 6 \\
(5^n + 5(-1)^n)/6 &: \quad 1, 0, 5, 20, 105, 520, 2605, \ldots \\
(5^n - (-1)^n)/6 &: \quad 0, 1, 4, 21, 104, 521, 2604, \ldots
\end{aligned}
$$

We have encountered the first four sequences already. The last four sequences are A109499, A015521, A109500 and A015531.

Chapter 4

Riordan arrays

4.1 The ordinary Riordan group

The *Riordan group* $\mathcal{R}$ [146, 153, 193, 202, 208], is a set of infinite lower-triangular matrices, where each matrix is defined by a pair of ordinary generating functions $g(x) = g_0 + g_1x + g_2x^2 + \ldots$ where $g_0 \neq 0$ and $f(x) = f_1x + f_2x^2 + \ldots$. We sometimes write $f(x) = xh(x)$ where $h(0) \neq 0$. The associated matrix is the matrix whose k-th column is generated by $g(x)f(x)^k$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f), and is often called the *Riordan array* defined by g and f. When $g_0 = 1$, the array is called a *monic* Riordan array. When $f_1 \neq 0$, the array is called a *proper* Riordan array. The group law is given by

$$(g, f) * (u, v) = (g(u \circ f), v \circ f). \tag{4.1}$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.
To each proper Riordan matrix $(g(x), f(x)) = (g(x), xh(x)) = (d_{n,k})_{n,k\geq 0}$ there exist [75] two sequences $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)(\alpha_0 \neq 0)$ and $\xi = (\xi_0, \xi_1, \xi_2, \ldots)$ such that

1. Every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, the coefficients being the elements of the sequence ξ, i.e.

$$d_{n+1,0} = \xi_0 d_{n,0} + \xi_1 d_{n,1} + \xi_2 d_{n,2} + \ldots$$

2. Every element $d_{n+1,k+1}$ not lying in column 0 or row 0 can be expressed as a linear combination of the elements of the preceding row, starting from the preceding column on, the coefficients being the elements of the sequence α, i.e.

$$d_{n+1,k+1} = \alpha_0 d_{n,k} + \alpha_1 d_{n,k+1} + \alpha_2 d_{n,2} + \ldots$$

The sequences α and ξ are called the *α-sequence* and the *ξ-sequence* of the Riordan matrix. It is customary to use the same symbols α and ξ as the names of the corresponding generating functions. The functions $g(x)$, $h(x)$, $\alpha(x)$ and $\xi(x)$ are connected as follows:

$$h(x) = \alpha(xh(x)), \qquad g(x) = \frac{d_{0,0}}{1 - x\xi(xh(x))}.$$

The first relation implies that

$$\alpha(x) = [h(t)|t = xh(t)^{-1}].$$

The α-sequence is sometimes called the *A-sequence* of the array and then we write $A(x) = \alpha(x)$. A matrix equipped with such sequences α and ξ can be shown to be a proper Riordan array. A Riordan array of the form $(g(x), x)$, where $g(x)$ is the ordinary generating function of the sequence a_n, is called the *Appell array* (or sometimes the *sequence array*) of the sequence a_n. Its general term is a_{n-k}.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{u} = (u_0, u_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{U}\ (x)$, then the sequence $\mathbf{Mu}$ has ordinary generating function $g(x)\mathcal{U}(f(x))$. We shall sometimes write

$$(g, f) \cdot \mathcal{U} = (g, f)\mathcal{U} = g(x)\mathcal{U}(f(x)).$$

Example 92. The binomial matrix $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$. For this matrix we have $A(x) = 1 + x$, which translates the usual defining relationship for Pascal's triangle

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

Example 93. We let $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ A000108. The array $(1, xc(x))$ is the inverse of the array $(1, x(1-x))$ while the array $(1, xc(x^2))$ is the inverse of the array $(1, \frac{x}{1+x^2})$.

Example 94. The row sums of the matrix (g, f) have generating function $g(x)/(1-f(x))$ while the diagonal sums of (g, f) have generating function $g(x)/(1-xf(x))$. The row sums of the array $(1, xc(x))$, or A106566, are the Catalan numbers C_n since $\frac{1}{1-xc(x)} = c(x)$. The diagonal sums have g.f. $\frac{1}{1-x^2c(x)}$, A132364.

4.2 A note on the Appell subgroup

We denote by $\mathcal{A}$ the Appell subgroup of $\mathcal{R}$. Let $\mathbf{A} \in \mathcal{R}$ correspond to the sequence $(a_n)_{n\geq 0}$, with o.g.f. $f(x)$. Let $\mathbf{B} \in \mathcal{R}$ correspond to the sequence (b_n), with o.g.f. $g(x)$. Then we have

1. The row sums of $\mathbf{A}$ are the partial sums of (a_n).
2. The inverse of $\mathbf{A}$ is the sequence array for the sequence with o.g.f. $\frac{1}{f(x)}$.
3. The product $\mathbf{AB}$ is the sequence array for the convolution $a * b(n) = \sum_{k=0}^{n} a_k b_{n-k}$ with o.g.f. $f(x)g(x)$.

Example 95. We consider the sequence $a_n = Pell(n+1)$ with o.g.f. $f(x) = \frac{1}{1-2x-x^2}$, and general term

$$a_n = \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{2\sqrt{2}}.$$

We obtain $\frac{1}{f(x)} = 1 - 2x - x^2$ and thus we see that the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 5 & 2 & 1 & 0 & 0 & 0 & \dots \\ 12 & 5 & 2 & 1 & 0 & 0 & \dots \\ 29 & 12 & 5 & 2 & 1 & 0 & \dots \\ 70 & 29 & 12 & 5 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has as inverse the simple matrix

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -2 & 1 & 0 & 0 & \dots \\ 0 & 0 & -1 & -2 & 1 & 0 & \dots \\ 0 & 0 & 0 & -1 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the k-th column has o.g.f. $x^k(1-2x-x^2)$. Of course, this example extends in a natural way to all sequences with o.g.f. of the form $\frac{1}{1-\alpha x-\beta x^2}$, and more generally to sequences with o.g.f. $\frac{1}{P(x)}$ where $P(x) = \sum_{k=0}^{n} \alpha_k x^k$ is a polynomial with $\alpha_0 = 1$.

Another simple example (corresponding to the simple sequence $a_n = 1$ with o.g.f. $\frac{1}{1-x}$) is given by the fact that the 'partial sum' matrix $\left(\frac{1}{1-x}, x\right)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has inverse the ‘first difference’ matrix $(1-x,x)$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

4.3 The subgroup $(g(x), xg(x))$

We let $f(x) = xg(x)$ and hence $(g(x), xg(x)) = (g(x), f(x))$. Then

$$\begin{aligned} (g(x), xg(x))^{-1} &= (g(x), f(x))^{-1} \\ &= (1/(g \circ \bar{f}), \bar{f}) \\ &= \left(\frac{\bar{f}}{x}, \bar{f}\right). \end{aligned}$$

In addition, we have

$$\begin{aligned} (g(x), xg(x)) \cdot (h(x), xh(x)) &= (g(x)h(xg(x)), xg(x)h(xg(x)) \\ &= (\tilde{g}(x), x\tilde{g}(x)) \end{aligned}$$

where $\tilde{g}(x) = g(x)h(xg(x))$. Thus the subset of Riordan arrays of the form $(g(x), xg(x))$ constitutes a sub-group of $\mathcal{R}$.

Example 96. Let $g(x) = c(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the generating function of the Catalan numbers. Then if $f(x) = xg(x)$ we have seen that $\bar{f}(x) = x(1-x)$. Hence

$$(c(x), xc(x))^{-1} = (1-x, x(1-x)).$$

Thus the inverse of the matrix

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 5 & 5 & 3 & 1 & 0 & 0 & \dots \\ 14 & 14 & 9 & 4 & 1 & 0 & \dots \\ 42 & 42 & 28 & 14 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & -4 & 1 & 0 & \dots \\ 0 & 0 & -1 & 6 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term

$$(-1)^{n-k}\binom{k+1}{n-k}.$$

The general term $T_{n,k}$ of the matrix $(c(x), xc(x))$ may be expressed as

$$\begin{aligned} T_{n,k} &= \frac{(k+1)\binom{2n-k+1}{n+1}}{2n-k+1}[k \le n] \\ &= \frac{(k+1+0^n)\binom{2n-k+1}{n+1}}{2n-k+1+0^{2n-k}}. \end{aligned}$$

4.4 The subgroup $(1, xg(x))$

We let $f(x) = xg(x)$ where $g(0) \neq 0$. Then we clearly have

$$(1, f(x))^{-1} = (1, \bar{f}(x))$$

and

$$(1, f_1(x))(1, f_2(x)) = (1, f_2(f_1(x))).$$

Clearly, $f_2(f_1(x)) = x\tilde{g}(x)$ where $\tilde{g}(0) \neq 0$.

Example 97. We consider the Riordan array $(1, c(x))$. This is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 5 & 5 & 3 & 1 & 0 & \dots \\ 0 & 14 & 14 & 9 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term

$$T_{n,k} = \frac{\binom{2n-k-1}{n-k}(k+0^{n+k})}{n+0^{nk}}.$$

We have

$$(1, c(x))^{-1} = (1, x(1-x))$$

which has general term $(-1)^{n-k}\binom{k}{n-k}$.

4.5 The exponential Riordan group

The *exponential Riordan group* [17], [75], [73], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1 x + g_2 x^2 + \dots$ and $f(x) = f_1 x + f_2 x^2 + \dots$ where $f_1 \neq 0$. The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. It is *monic* if $g_0 = 1$. The group law is then given by

$$[g, f] * [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f. We use the notation $e\mathcal{R}$ to denote this group.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n \geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}\ (x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g, f]$ are given by $g(x)e^{f(x)}$ since the sequence $1, 1, 1, \dots$ has exponential generating function e^x.

As an element of the group of exponential Riordan arrays, we have $\mathbf{B} = [e^x, x]$. By the above, the exponential generating function of its row sums is given by $e^x e^x = e^{2x}$, as expected (e^{2x} is the e.g.f. of 2^n).

Example 98. We consider the exponential Riordan array $[\frac{1}{1-x}, x]$. This array has elements

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

and general term $[k \leq n]\frac{n!}{k!}$ with inverse

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which is the array $[1 - x, x]$. In particular, we note that the row sums of the inverse, which begin $1, 0, -1, -2, -3, \dots$ (that is, $1 - n$), have e.g.f. $(1 - x)\exp(x)$. This sequence is thus the binomial transform of the sequence with e.g.f. $(1 - x)$ (which is the sequence starting $1, -1, 0, 0, 0, \dots$).

Example 99. We consider the exponential Riordan array $[1, \frac{x}{1-x}]$. The general term of this matrix may be calculated as follows:

$$\begin{aligned} T_{n,k} &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\ &= \frac{n!}{k!}[x^{n-k}](1-x)^{-k} \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j \\ &= \frac{n!}{k!}\binom{k+n-k-1}{n-k} \\ &= \frac{n!}{k!}\binom{n-1}{n-k}. \end{aligned}$$

Thus its row sums, which have e.g.f. $\exp\left(\frac{x}{1-x}\right)$, have general term $\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{n-k}$. This is A000262, the 'number of "sets of lists": the number of partitions of $\{1,..,n\}$ into any number of lists, where a list means an ordered subset'. Its general term is equal to $(n-1)!L_{n-1}(1,-1)$. The inverse of $\left[1, \frac{x}{1-x}\right]$ is the exponential Riordan array $\left[1, \frac{x}{1+x}\right]$. The row sums of this sequence have e.g.f. $\exp\left(\frac{x}{1+x}\right)$, and start $1, 1, -1, 1, 1, -19, 151, \ldots$. This is A111884. For more information on these matrices, see Chapter 8.

Example 100. The exponential Riordan array $\mathbf{A} = \left[\frac{1}{1-x}, \frac{x}{1-x}\right]$, or

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 6 & 18 & 9 & 1 & 0 & 0 & \ldots \\ 24 & 96 & 72 & 16 & 1 & 0 & \ldots \\ 120 & 600 & 600 & 200 & 25 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has general term

$$T_{n,k} = \frac{n!}{k!}\binom{n}{k}.$$

It is closely related to the Laguerre polynomials. Its inverse is $\left[\frac{1}{1+x}, \frac{x}{1+x}\right]$ with general term $(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$. This is A021009, the triangle of coefficients of the Laguerre polynomials $L_n(x)$. It is noted in [205], that

$$\mathbf{A} = \exp(\mathbf{S}),$$

where

$$
\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 9 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 16 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 25 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
$$

Example 101. The exponential Riordan array $\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]$, or

$$
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 8 & 6 & 1 & 0 & 0 & \dots \\ 1 & 24 & 29 & 10 & 1 & 0 & \dots \\ 1 & 89 & 145 & 75 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

is the coefficient array for the polynomials

$$
{}_2F_0(-n, x; -1)
$$

which are an unsigned version of the Charlier polynomials (of order 0) [99, 195, 218]. This is equal to

$$
[e^x, x]\left[1, \ln\left(\frac{1}{1-x}\right)\right],
$$

or the product of the binomial array $\mathbf{B}$ and the array of (unsigned) Stirling numbers of the first kind.

4.6 A note on the exponential Appell subgroup

By the *exponential Appell subgroup* of $e\mathcal{R}$ we understand the set of arrays of the form $[f(x), x]$.

Let $\mathbf{A} \in e\mathcal{R}$ correspond to the sequence $(a_n)_{n\geq 0}$, with e.g.f. $f(x)$. Let $\mathbf{B} \in e\mathcal{R}$ correspond to the sequence (b_n), with e.g.f. $g(x)$. Then we have

1. The row sums of $\mathbf{A}$ are the partial sums of (a_n).
2. The inverse of $\mathbf{A}$ is the sequence array for the sequence with e.g.f. $\frac{1}{f(x)}$.
3. The product $\mathbf{AB}$ is the sequence array for the exponential convolution $a * b(n) = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$ with e.g.f. $f(x)g(x)$.

Example 102. We consider the matrix $[\cosh(x), x]$ with elements

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & 6 & 0 & 1 & 0 & \ldots \\ 0 & 5 & 0 & 10 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The row sums of this matrix have e.g.f. $\cosh(x)\exp(x)$, which is the e.g.f. of the sequence $1, 1, 2, 4, 8, 16, \ldots$. The inverse matrix is $[\text{sech}(x), x]$ with entries

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & -3 & 0 & 1 & 0 & 0 & \ldots \\ 5 & 0 & -6 & 0 & 1 & 0 & \ldots \\ 0 & 25 & 0 & -10 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

The row sums of this matrix have e.g.f. $\text{sech}(x)\exp(x)$.

Riordan group techniques have been used to provide alternative proofs of many binomial identities that originally appeared in works such as [191, 192]. See for instance [208, 209].

4.7 Conditional Riordan arrays

On occasion, we find it useful to work with arrays that are "almost" of Riordan type. One particular case is where after the first (0-th) column, subsequent columns follow a Riordan type rule. We will call such arrays *conditional* Riordan arrays, and will use the notation

$$(h(x)|(g(x), f(x)))$$

to denote an array whose first column is generated by $h(x)$, and whose k-th column, for $k > 0$, is generated by $g(x)f(x)^k$. We will sometimes also use the notation

$$(h(x)|g(x)f(x)^k)$$

to denote such an array.

Example 103. The conditional Riordan array

$$\left(\frac{1}{1+x}| \left(\frac{1}{1-x}, \frac{x}{1-x}\right)\right)$$

begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ -1 & 3 & 3 & 1 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & 0 & \dots \\ -1 & 5 & 10 & 10 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

On occasion, we shall also find it useful to use the notation

$$(h(x)||(g(x)/x, f(x)))$$

to denote the matrix whose first column is generated by $h(x)$, with subsequent columns generated by $\frac{g(x)}{x} f(x)^k$.

Example 104. The array

$$\left(\frac{1}{1-x}|| \left(\frac{1}{x(1-x)}, x\right)\right)$$

begins

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

4.8 Generalized Riordan arrays

Recently, the authors in [226] have defined the notion of "generalized" Riordan array. Given a sequence $(c_n)_{n\geq 0}$, with $c_n \neq 0 \,\forall n$, a *generalized Riordan array with respect to the sequence* c_n is a pair $(g(t), f(t))_c$ of formal power series, where $g(t) = \sum_{k=0}^{\infty} g_k t^k/c_k$ and $f(t) = \sum_{k=0}^{\infty} f_k t^k/c_k$ with $f_1 \neq 0$. The generalized Riordan array $(g(t), f(t))_c$ defines an infinite, lower triangular $(d_{n,k})_{0\leq k\leq n<\infty}$ according to the rule:

$$d_{n,k} = \left[\frac{t^n}{c_n}\right] g(t) \frac{(f(t))^k}{c_k},$$

where the functions $g(t)(f(t))^k/c_k$ are called the *column generating functions* of the generalized Riordan array. Here, if $f(t) = \sum_{k=0}^{\infty} f_k t^k/c_k$, then $[t^n/c_n]f(t) = f_n$. We have

$$\left[\frac{t^n}{c_n}\right] f(t) = c_n[t^n]f(t).$$

We now note that all the results that are valid for Riordan arrays remain valid for generalized Riordan arrays, whenever we use $[t^n/c^n]$ in place of $[t^n]$ in the ordinary case. Thus to say that $(g(t), f(t))_c$ is a generalized Riordan array for the sequence c_n is equivalent to saying that $g(t) = \sum_{k=0}^{\infty} g_k t^k/c_k$ and $f(t) = \sum_{k=0}^{\infty} f_k t^k/c_k$. We have for instance the following theorem [226] :

Theorem 105. *Let $(g(t), f(t))_c = (d_{n,k})_{n,k\in\mathbb{N}}$ be a generalized Riordan array with respect to c_n, and let $h(t) = \sum_{k=0}^{\infty} h_k t^k/c_k$ be the generalized generating function of the sequence h_n. Then we have*

$$\sum_{k=0}^{n} d_{n,k} h_k = \left[\frac{t^n}{c_n}\right] g(t)h(f(t)),$$

or equivalently,

$$(g(t), f(t))_c \cdot h(t) = g(t)h(f(t)).$$

4.9 Egorychev arrays

Egorychev defined a set matrices, defined to be of type $R^q(\alpha_n, \beta_k; \phi, f, \psi)$ [81, 83, 82] which are more general then Riordan arrays. They in fact include the (ordinary) Riordan arrays, exponential Riordan arrays and implicit Riordan arrays [156, 159].
Specifically, a matrix $C = (c_{nk})_{n,k=0,1,2,\ldots}$ is of type $R^q(\alpha_n, \beta_k; \phi, f, \psi)$ if its general term is defined by the formula

$$c_{nk} = \frac{\beta_k}{\alpha_n} \mathbf{res}_x(\phi(x) f^k(x) \psi^n(x) x^{-n+qk-1})$$

where $\mathbf{res}_x A(x) = a_{-1}$ for a given formal power series $A(x) = \sum_j a_j x^j$ is the formal residue of the series.

For the exponential Riordan arrays, we have $\alpha_n = \frac{1}{n!}$, $\beta_k = \frac{1}{k!}$, and $q = 1$.

4.10 Production arrays

The concept of a *production matrix* [75, 74] is a general one, but for this work we find it convenient to review it in the context of Riordan arrays. Thus let P be an infinite matrix (most often it will have integer entries). Letting r_0 be the row vector

$$r_0 = (1, 0, 0, 0, \ldots),$$

we define $r_i = r_{i-1}P$, $i \geq 1$. Stacking these rows leads to another infinite matrix which we denote by A_P. Then P is said to be the *production matrix* for A_P.
If we let

$$u^T = (1, 0, 0, 0, \ldots, 0, \ldots)$$

then we have

$$A_P = \begin{pmatrix} u^T \\ u^T P \\ u^T P^2 \\ \vdots \end{pmatrix}$$

and

$$DA_P = A_P P$$

where $D = (\delta_{i,j+1})_{i,j\geq 0}$ (where δ is the usual Kronecker symbol).

In [174, 203] P is called the Stieltjes matrix associated to A_P.

The sequence formed by the row sums of A_P often has combinatorial significance and is called the sequence associated to P. Its general term a_n is given by $a_n = u^T P^n e$ where

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

In the context of Riordan arrays, the production matrix associated to a proper Riordan array takes on a special form :

Proposition 106. *[75] Let P be an infinite production matrix and let A_P be the matrix induced by P. Then A_P is an (ordinary) Riordan matrix if and only if P is of the form*

$$P = \begin{pmatrix} \xi_0 & \alpha_0 & 0 & 0 & 0 & 0 & \dots \\ \xi_1 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \dots \\ \xi_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \dots \\ \xi_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots \\ \xi_4 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \dots \\ \xi_5 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Morever, columns 0 and 1 of the matrix P are the ξ- and α-sequences, respectively, of the Riordan array A_P.

Example 107. We consider the Riordan array $\mathbf{L}$ where

$$L^{-1} = \left(\frac{1 - \lambda x - \mu x^2}{1 + ax + bx^2}, \frac{x}{1 + ax + bx^2} \right).$$

The production matrix (Stieltjes matrix) of

$$L = \left(\frac{1 - \lambda x - \mu x^2}{1 + ax + bx^2}, \frac{x}{1 + ax + bx^2} \right)^{-1}$$

is given by

$$P = S_L = \begin{pmatrix} a+\lambda & 1 & 0 & 0 & 0 & 0 & \dots \\ b+\mu & a & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that since

$$\begin{aligned} L &= \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) \\ &= (1-\lambda x-\mu x^2, x)\cdot\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right), \end{aligned}$$

we have

$$L = \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1} = \left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}\cdot\left(\frac{1}{1-\lambda x-\mu x^2}, x\right).$$

If we now let

$$L_1 = \left(\frac{1}{1+ax}, \frac{x}{1+ax}\right)\cdot L,$$

then (see [175]) we obtain that the Stieltjes matrix for L_1 is given by

$$S_{L_1} = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & \cdots \\ b+\mu & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & b & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & b & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & b & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & b & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have in fact the following general result [175]:

Proposition 108. *If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal, then necessarily*

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ b_1 & a & 1 & 0 & 0 & 0 & \cdots \\ 0 & b & a & 1 & 0 & 0 & \cdots \\ 0 & 0 & b & a & 1 & 0 & \cdots \\ 0 & 0 & 0 & b & a & 1 & \cdots \\ 0 & 0 & 0 & 0 & b & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$f(x) = Rev\frac{x}{1+ax+bx^2} \qquad \text{and} \qquad g(x) = \frac{1}{1-a_1x-b_1xf},$$

and vice-versa.

We have the important corollary

Corollary 109. *If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal, with*

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ b_1 & a & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

then L^{-1} is the coefficient array of the family of orthogonal polynomials $p_n(x)$ where $p_0(x) = 1$, $p_1(x) = x - a_1$, and

$$p_{n+1}(x) = (x - a)p_n(x) - b_n p_{n-1}(x), \qquad n \geq 2,$$

where b_n is the sequence $0, b_1, b, b, b, \dots$.

We note that the elements of the rows of L^{-1} can be identified with the coefficients of the characteristic polynomials of the successive principal sub-matrices of P.

Example 110. We consider the Riordan array

$$\left(\frac{1}{1 + ax + bx^2}, \frac{x}{1 + ax + bx^2}\right).$$

Then the production matrix (Stieltjes matrix) of the inverse Riordan array $\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}$ left-multiplied by the k-th binomial array

$$\left(\frac{1}{1 - kx}, \frac{x}{1 - kx}\right) = \left(\frac{1}{1 - x}, \frac{x}{1 - x}\right)^k$$

is given by

$$P = \begin{pmatrix} a+k & 1 & 0 & 0 & 0 & 0 & \dots \\ b & a+k & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a+k & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a+k & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a+k & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a+k & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and vice-versa. This follows since

$$\left(\frac{1}{1 + ax + bx^2}, \frac{x}{1 + ax + bx^2}\right) \cdot \left(\frac{1}{1 + kx}, \frac{x}{1 + kx}\right) = \left(\frac{1}{1 + (a + k)x + bx^2}, \frac{x}{1 + (a + k)x + bx^2}\right).$$

In fact we have the more general result :

$$\left(\frac{1 + \lambda x + \mu x^2}{1 + ax + bx^2}, \frac{x}{1 + ax + bx^2}\right) \cdot \left(\frac{1}{1 + kx}, \frac{x}{1 + kx}\right) = \left(\frac{1 + \lambda x + \mu x^2}{1 + (a + k)x + bx^2}, \frac{x}{1 + (a + k)x + bx^2}\right).$$

The inverse of this last matrix therefore has production array

$$\begin{pmatrix} a+k-\lambda & 1 & 0 & 0 & 0 & 0 & \dots \\ b-\mu & a+k & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a+k & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a+k & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a+k & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a+k & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 111. The series reversion of $\frac{x(1+\gamma x)}{1+\alpha x+\beta x^2}$, which has g.f.

$$\frac{\sqrt{1+(2\gamma-\alpha)x+(\alpha^2-4\beta)x^2}+\alpha x-1}{2(\gamma-\beta x)},$$

is such that the first column of the Riordan array

$$\left(\frac{1-\gamma x}{1+(\alpha-2\gamma)x+(\gamma^2-\alpha\gamma+\beta)x^2}, \frac{x}{1+(\alpha-2\gamma)x+(\gamma^2-\alpha\gamma+\beta)x^2}\right)^{-1}$$

is equal to

$$[x^{n+1}]\mathrm{Rev}\frac{x(1+\gamma x)}{1+\alpha x+\beta x^2}.$$

The production array of this matrix is given by

$$\begin{pmatrix} \alpha-\gamma & 1 & 0 & 0 & 0 & 0 & \dots \\ -\alpha\gamma+\beta+\gamma^2 & \alpha-2\gamma & 1 & 0 & 0 & 0 & \dots \\ 0 & -\alpha\gamma+\beta+\gamma^2 & \alpha-2\gamma & 1 & 0 & 0 & \dots \\ 0 & 0 & -\alpha\gamma+\beta+\gamma^2 & \alpha-2\gamma & 1 & 0 & \dots \\ 0 & 0 & 0 & -\alpha\gamma+\beta+\gamma^2 & \alpha-2\gamma & 1 & \dots \\ 0 & 0 & 0 & 0 & -\alpha\gamma+\beta+\gamma^2 & \alpha-2\gamma & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A consequence of this is that the Hankel transform of $[x^{n+1}]\mathrm{Rev}\frac{x(1+\gamma x)}{1+\alpha x+\beta x^2}$ or

$$[x^n]\frac{\sqrt{1+(2\gamma-\alpha)x+(\alpha^2-4\beta)x^2}+\alpha x-1}{2x(\gamma-\beta x)}$$

is equal to

$$(-\alpha\gamma+\beta+\gamma^2)^{\binom{n+1}{2}}.$$

The sequence $[x^{n+1}]\mathrm{Rev}\frac{x(1+\gamma x)}{1+\alpha x+\beta x^2}$ has g.f. given by the following continued fraction:

$$\cfrac{1}{1-(\alpha-\gamma)-\cfrac{(-\alpha\gamma+\beta+\gamma^2)x^2}{1-(\alpha-2\gamma)-\cfrac{(-\alpha\gamma+\beta+\gamma^2)x^2}{1-(\alpha-2\gamma)-\cfrac{(-\alpha\gamma+\beta+\gamma^2)x^2}{1-\cdots}}}}$$

Equivalently the first column of the matrix

$$\left(\frac{1+\gamma x}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right)^{-1}$$

is equal to the sequence with general term

$$[x^{n+1}]\text{Rev}\frac{x(1-\gamma x)}{1+(\alpha-2\gamma)x+(\gamma^2+\beta-\alpha\gamma)x^2},$$

which has Hankel transform $\beta^{\binom{n+1}{2}}$ since, as we have seen earlier, the inverse matrix above has production array

$$\begin{pmatrix} \alpha-\gamma & 1 & 0 & 0 & 0 & 0 & \ldots \\ \beta & \alpha & 1 & 0 & 0 & 0 & \ldots \\ 0 & \beta & \alpha & 1 & 0 & 0 & \ldots \\ 0 & 0 & \beta & \alpha & 1 & 0 & \ldots \\ 0 & 0 & 0 & \beta & \alpha & 1 & \ldots \\ 0 & 0 & 0 & 0 & \beta & \alpha & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The sequence $[x^{n+1}]\text{Rev}\frac{x(1-\gamma x)}{1+(\alpha-2\gamma)x+(\gamma^2+\beta-\alpha\gamma)x^2}$ has generating function given by the continued fraction

$$\cfrac{1}{1-(\alpha-\gamma)x-\cfrac{\beta x^2}{1-\alpha x-\cfrac{\beta x^2}{1-\alpha x-\cfrac{\beta x^2}{1-\cdots}}}}$$

Example 112. The matrix A_P with production matrix

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 1 & 1 & 0 & \ldots \\ 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or

$$\left(\frac{1}{1-x}||\frac{x^{k-1}}{1-x}\right)$$

is the Catalan matrix $(c(x), xc(x)) = (1-x, x(1-x))^{-1}$ (A033184). Similarly, the matrix with production matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the Riordan array $(1, xc(x))$ (A106566).

Example 113. The matrix A_P with production matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or

$$\left(\frac{1}{1-x^2} || \frac{x^{k-1}}{1-x^2}\right)$$

is the aerated "ternary" matrix $(1-x^2, x(1-x^2))^{-1}$. This begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 3 & 0 & 3 & 0 & 1 & 0 & \cdots \\ 0 & 7 & 0 & 4 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with first column given by $\frac{\binom{3n}{n}}{2n+1}$ aerated, or

$$\frac{\binom{3n/2}{n/2}}{n+1}(1+(-1)^n)/2.$$

The Hankel transforms of these sequences are of combinatorial importance [102].

Letting $\widehat{P}$ denote the production array P augmented by the addition of a first row equal to $r_0 = (1, 0, 0, 0, \ldots)$ we see that

$$\widehat{P} = \left(\frac{1}{1-x^2}, x\right)$$

with general term $\frac{1+(-1)^{n-k}}{2}$. We then have

$$\begin{aligned} A_P \widehat{P}^{-1} A_P^{-1} &= (1-x^2, x(1-x^2))^{-1} \cdot (1-x^2, x) \cdot (1-x^2, x(1-x^2)) \\ &= (1-f^2, x), \end{aligned}$$

where

$$f(1-f^2)=x.$$

Thus

$$A_P \cdot \left(\frac{1}{1-x^2}, x\right) = \left(\frac{1}{1-f^2}, x\right) \cdot A_P,$$

with

$$f(x) = \frac{2}{\sqrt{3}} \cos\left(\frac{\cos^{-1}\left(\frac{-3\sqrt{3}x}{2}\right)}{3}\right).$$

Example 114. The production matrix for the array given by

$$\left(\frac{1+\gamma x}{1-\alpha x-\beta x^2}, \frac{x(1+\gamma x)}{1-\alpha x-\beta x^2}\right)^{-1}$$

is given by removing the first row from

$$\left(\frac{1-\alpha x-\beta x^2}{1+\gamma x}, x\right).$$

For example, the array A154929 defined by

$$\left(\frac{1+x}{1-x-x^2}, \frac{x(1+x)}{1-x-x^2}\right)$$

which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 4 & 1 & 0 & 0 & 0 & \dots \\ 5 & 10 & 6 & 1 & 0 & 0 & \dots \\ 8 & 22 & 21 & 8 & 1 & 0 & \dots \\ 13 & 45 & 59 & 36 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

has inverse

$$\left(\frac{\sqrt{1+6x+5x^2}-x-1}{2x(1+x)}, \frac{\sqrt{1+6x+5x^2}-x-1}{2(1+x)}\right)$$

or equivalently,

$$\left(\frac{1}{1+x}c\left(\frac{-x}{1+x}\right), \frac{x}{1+x}c\left(\frac{-x}{1+x}\right)\right)$$

with production matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & 0 & \ldots \\ -1 & 1 & -2 & 1 & 0 & 0 & \ldots \\ 1 & -1 & 1 & -2 & 1 & 0 & \ldots \\ -1 & 1 & -1 & 1 & -2 & 1 & \ldots \\ 1 & -1 & 1 & -1 & 1 & -2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is

$$\left(\frac{1-x-x^2}{1+x}, x\right)$$

with its first row removed. We note that the general term of

$$\left(\frac{1+x}{1-x-x^2}, \frac{x(1+x)}{1-x-x^2}\right)$$

is given by

$$\sum_{j=0}^{n} \binom{j+1}{n-j}\binom{j}{k}$$

while the general term of the inverse matrix is given by

$$\sum_{j=0}^{n} (-1)^{n-k} \frac{k+1}{j+1}\binom{n}{j}\binom{2j-k}{j-k}.$$

We note further that the inverse of $\left(\frac{1-x-x^2}{1+x}, x\right)$ is $\left(\frac{1+x}{1-x-x^2}, x\right)$ which is the sequence array for $F(n+2)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 3 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 5 & 3 & 2 & 1 & 0 & 0 & \ldots \\ 8 & 5 & 3 & 2 & 1 & 0 & \ldots \\ 13 & 8 & 5 & 3 & 2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Removing the first row of this matrix, we obtain the production matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 3 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 5 & 3 & 2 & 1 & 0 & 0 & \ldots \\ 8 & 5 & 3 & 2 & 1 & 0 & \ldots \\ 13 & 8 & 5 & 3 & 2 & 1 & \ldots \\ 21 & 13 & 8 & 5 & 3 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

of the matrix that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 7 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 31 & 18 & 6 & 1 & 0 & 0 & \ldots \\ 154 & 90 & 33 & 8 & 1 & 0 & \ldots \\ 870 & 481 & 185 & 52 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

where the first column is A007863.
This is the matrix

$$\left(\frac{1-x-x^2}{1+x}, \frac{x(1-x-x^2)}{1+x}\right)^{-1}$$

or

$$(f/x, f)$$

where

$$f(x) = \frac{1}{3}\left(\sqrt{4-3x}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{18x+11}{2(4-3x)^{\frac{3}{2}}}\right)\right)\right) - \frac{1}{3}$$

is the reversion of $\frac{x(1-x-x^2)}{1+x}$.
The matrix

$$\left(\frac{1-x-x^2}{1+x}, \frac{x(1-x-x^2)}{1+x}\right)$$

starts

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -4 & 1 & 0 & 0 & 0 & \ldots \\ -1 & 6 & -6 & 1 & 0 & 0 & \ldots \\ 1 & -6 & 15 & -8 & 1 & 0 & \ldots \\ -1 & 7 & -23 & 28 & -10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We further note that

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 7 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 31 & 18 & 6 & 1 & 0 & 0 & \ldots \\ 154 & 90 & 33 & 8 & 1 & 0 & \ldots \\ 870 & 481 & 185 & 52 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \cdot \mathbf{B}^{-1}$$

is generated by the matrix

$$\mathbf{B}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -4 & 1 & 0 & 0 & 0 & \dots \\ -1 & 6 & -6 & 1 & 0 & 0 & \dots \\ 1 & -6 & 15 & -8 & 1 & 0 & \dots \\ -1 & 7 & -23 & 28 & -10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with its first row removed.

Example 115. The matrix A_P with production matrix

$$P=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & 0 & \dots \\ 1 & 5 & 10 & 10 & 5 & 1 & \dots \\ 1 & 6 & 15 & 20 & 15 & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term

$$\binom{n+1}{k}$$

or equivalently the matrix

$$\left(\frac{1}{1-x}\Big|\Big|\frac{x^{k-1}}{(1-x)^{k+1}}\right)$$

is the (unsigned) Stirling matrix of the first kind

$$\left[\frac{1}{1-x},\ln\left(\frac{1}{1-x}\right)\right].$$

In this case P is equal to $\left(\frac{1}{1-x},\frac{x}{1-x}\right)$ (or $[e^x,x]$) less its top row.

We now define the augmented production array $\widehat{P}$ to be the matrix with general term $\binom{n}{k}$, that is, $\widehat{P}=\mathbf{B}$. This is the matrix P with the row

$$r_0=(1,0,0,0,\dots)$$

added as its first row. We find that

$$A_P\widehat{P}^{-1}A_P^{-1}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or $[1-x,x]$ whose inverse is given by

$$A_P\widehat{P}A_P^{-1}=\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

or $[\frac{1}{1-x},x]$. This translates the identity of Riordan arrays

$$\left[\frac{1}{1-x},\ln\left(\frac{1}{1-x}\right)\right][e^x,x][e^{-x},1-e^{-x}]=\left[\frac{1}{1-x},x\right].$$

Example 116. This example continues the theme of the last example. We start with the array $[1-x,x]$; removing its first row we obtain the matrix

$$\left(\begin{array}{ccccccc} -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & -6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This is the production matrix of the inverse of the Stirling matrix of the first kind $\left[\frac{1}{1-x},\ln\left(\frac{1}{1-x}\right)\right]$, or $[e^{-x},1-e^{-x}]$ which starts

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -3 & 1 & 0 & 0 & 0 & \dots \\ -1 & 7 & -6 & 1 & 0 & 0 & \dots \\ 1 & -15 & 25 & -10 & 1 & 0 & \dots \\ -1 & 31 & -90 & 65 & -15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This is the (signed) Stirling matrix of the second kind. Now taking as production array the 0-column augmented matrix

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -4 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & -5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

we find that this generates the matrix

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -3 & 1 & 0 & 0 & \dots \\ 0 & -1 & 7 & -6 & 1 & 0 & \dots \\ 0 & 1 & -15 & 25 & -10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which is the Stirling matrix of the second kind $[1, 1-e^{-x}]$. We note that if we square this production matrix, and remove the first column, we obtain the matrix

$$\left(\begin{array}{ccccccc} -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -3 & 1 & 0 & 0 & 0 & \dots \\ 0 & 4 & -5 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9 & -7 & 1 & 0 & \dots \\ 0 & 0 & 0 & 16 & -9 & 1 & \dots \\ 0 & 0 & 0 & 0 & 25 & -11 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This matrix generates the exponential Riordan array $\left[\frac{1}{1+x}, \frac{x}{1+x}\right]$ of (signed) Laguerre coefficients. This matrix begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & -4 & 1 & 0 & 0 & 0 & \dots \\ -6 & 18 & -9 & 1 & 0 & 0 & \dots \\ 24 & -96 & 72 & -16 & 1 & 0 & \dots \\ -120 & 600 & -600 & 200 & -25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The inverse of this is the unsigned version of the above matrix, which is the Riordan array

$$\mathbf{Lag} = \left[\frac{1}{1-x}, \frac{x}{1-x}\right]$$

(see Chapter 8). This latter matrix has production matrix given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 0 & 4 & 5 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9 & 7 & 1 & 0 & \dots \\ 0 & 0 & 0 & 16 & 9 & 1 & \dots \\ 0 & 0 & 0 & 0 & 25 & 11 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We note that since the first column of **Lag** is $n!$, the above indicates that the generating function of $n!$ is given by

$$\cfrac{1}{1-x-\cfrac{x^2}{1-3x-\cfrac{4x^2}{1-5x-\cfrac{9x^2}{1-7x-\cfrac{16x^2}{1-\cdots}}}}}.$$

Example 117. The matrix A_P with production matrix P with general term

$$r^{n-k+1}\binom{n+1}{k}$$

or equivalently the matrix

$$\left(\frac{r}{1-rx}||\frac{x^{k-1}}{(1-rx)^{k+1}}\right)$$

is the exponential Riordan array

$$\left[\frac{1}{1-rx}, \frac{1}{r}\ln\left(\frac{1}{1-rx}\right)\right]$$

with general term

$$d_{n,k} = \frac{n!}{r^k k!}[x^n]\frac{1}{1-rx}\left(\ln\left(\frac{1}{1-rx}\right)\right)^k.$$

(See Chapter 4). In this case P is equal to $\left(\frac{1}{1-rx}, \frac{x}{1-rx}\right)$ less its top row (that is, $\widehat{P}$ is equal to $\left(\frac{1}{1-rx}, \frac{x}{1-rx}\right)$).

Example 118. The matrix A_P with production matrix P with general term

$$\binom{n+1}{k} + (r-1)\binom{n}{k}$$

is the exponential Riordan array

$$\mathbf{Lag}^{(r-1)} = \left[\frac{1}{(1-x)^r}, \ln\left(\frac{1}{1-x}\right)\right].$$

(See Chapter 8 for the above notation). The matrix P is the r-Pascal matrix

$$\left(\frac{1+(r-1)x}{1-x}, \frac{x}{1-x}\right),$$

less its top row.

Example 119. The Lah matrix (see Chapter 8) is defined as

$$\mathbf{Lah} = \left[1, \frac{x}{1-x}\right].$$

The production matrix P for this matrix is given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 4 & 1 & 0 & 0 & \dots \\ 0 & 0 & 6 & 6 & 1 & 0 & \dots \\ 0 & 0 & 0 & 12 & 8 & 1 & \dots \\ 0 & 0 & 0 & 0 & 20 & 10 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix obtained from P by removing the leftmost column is the exponential Riordan array $[(1+x)^2, x]$. The row sums of this latter matrix have e.g.f. $(1+x)^2 e^x$. By prepending the row $(1, 0, 0, 0, \dots)$ we obtain a matrix $\widehat{P}$ with row sums equal the central polygonal numbers $n^2 - n + 1$ with e.g.f. $(1 + x^2)e^x$. Thus

$$\widehat{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 4 & 1 & 0 & 0 & \dots \\ 0 & 0 & 6 & 6 & 1 & 0 & \dots \\ 0 & 0 & 0 & 12 & 8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$A_P \widehat{P} A_P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 6 & 4 & 1 & 0 & 0 & \dots \\ 0 & 24 & 18 & 6 & 1 & 0 & \dots \\ 0 & 120 & 96 & 36 & 8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with general term $0^{n+k} + \binom{n-1}{k-1}(n-k+1)!$. We now note that

$$A_P \widehat{P}^{-1} A_P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & -4 & 1 & 0 & 0 & \dots \\ 0 & 0 & 6 & -6 & 1 & 0 & \dots \\ 0 & 0 & 0 & 12 & -8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is a signed version of $\widehat{P}$. The corresponding signed production array generates the signed Lah numbers $\left[1, \frac{x}{1+x}\right]$.

Example 120. We consider the conditional Riordan array

$$\mathbf{P}(\alpha, \beta) = \left(\frac{\beta x}{1-x} || (\alpha - (\alpha-1)0^{k-1})x^{k-1}\right).$$

The row sums of A_P have o.g.f.

$$\frac{1-(\alpha-1)x}{1-\alpha x-\beta x^2}.$$

Furthermore, the row sums of the matrix with production array $\mathbf{I}+\mathbf{P}$ where $\mathbf{I}$ is the (infinite) identity matrix, are the binomial transform of the first sequence. For example, with $\alpha = 2$ and $\beta = 3$, we get

$$\mathbf{P}(2,3) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 2 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 2 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 0 & 2 & 0 & \cdots \\ 3 & 0 & 0 & 0 & 0 & 2 & \cdots \\ 3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the row sums of $A_{\mathbf{P}}$ are given by $1, 1, 5, 13, 41, 121, 365, \ldots$, A046717, with o.g.f. $\frac{1-x}{1-2x-3x^2}$. The row sums of $A_{\tilde{\mathbf{P}}}$ where $\tilde{\mathbf{P}} = \mathbf{P}(2,3) + \mathbf{I}$ are given by $1, 2, 8, 32, 128, 512, \ldots$, A081294, with o.g.f. $\frac{1-2x}{1-4x}$, which is the binomial transform of A046717. In general the row sums of the matrix with production array $\mathbf{P} + k\mathbf{I}$ will be the k-th binomial transform of the matrix with production array $\mathbf{P}$.

Example 121. The production array of the Riordan array

$$\left(\frac{1}{1-x-rx^2}, \frac{x}{1-x}\right)$$

is given by

$$\mathbf{I} + \left(\frac{rx}{1-x} || x^{k-1}\right).$$

Thus in particular, Pascal's triangle $\mathbf{B} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ has production array

$$\mathbf{I} + \left(0 || x^{k-1}\right)$$

with

$$\widehat{\mathbf{P}} = (1+x, x).$$

In [75], we find the following result concerning matrices that are production matrices for exponential Riordan arrays.

Proposition 122. *Let $A = (a_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \dots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \dots \tag{4.2}$$

be two formal power series that that

$$r(f(x)) = f'(x) \tag{4.3}$$

$$c(f(x)) = \frac{g'(x)}{g(x)}. \tag{4.4}$$

Then

$$(i) \qquad a_{n+1,0} = \sum_i i! c_i a_{n,i} \tag{4.5}$$

$$(ii) \qquad a_{n+1,k} = r_0 a_{n,k-1} + \frac{1}{k!}\sum_{i\geq k} i!(c_{i-k} + k r_{i-k+1}) a_{n,i} \tag{4.6}$$

or, defining $c_{-1} = 0$,

$$a_{n+1,k} = \frac{1}{k!}\sum_{i\geq k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}. \tag{4.7}$$

Conversely, starting from the sequences defined by 4.2, the infinite array $(a_{n,k})_{n,k\geq 0}$ defined by 4.7 is an exponential Riordan array.

A consequence of this proposition is that $P = (p_{i,j})_{i,j\geq 0}$ where

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{r-j+1}) \qquad (c_{-1} = 0).$$

Furthermore, the bivariate exponential function

$$\phi_P(t,z) = \sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}$$

of the matrix P is given by

$$\phi_P(t,z) = e^{tz}(c(z) + t r(z)).$$

Example 123. The exponential array $\mathbf{s} = \left[1, \ln\left(\frac{1}{1-x}\right)\right]$ of Stirling numbers of the first kind, which starts

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 3 & 1 & 0 & 0 & \dots \\ 0 & 6 & 11 & 6 & 1 & 0 & \dots \\ 0 & 24 & 50 & 35 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has production array with bivariate generating function

$$te^{(t+1)x} = e^{tx}(0 + e^x t)$$

so that $c(x) = 0$, $r(x) = e^x$. This begins

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 3 & 3 & 1 & 0 & \dots \\ 0 & 1 & 4 & 6 & 4 & 1 & \dots \\ 0 & 1 & 5 & 10 & 10 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The array

$$\mathbf{s} \cdot \mathbf{B}$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 3 & 1 & 0 & 0 & 0 & \dots \\ 6 & 11 & 6 & 1 & 0 & 0 & \dots \\ 24 & 50 & 35 & 10 & 1 & 0 & \dots \\ 120 & 274 & 225 & 85 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has production array with bivariate g.f. equal to $e^{(1+t)x}(1+t) = e^{tx}(1+t)e^x$ so that $c(x) = e^x$ and $r(x) = e^x$. The production array then begins

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & 0 & \dots \\ 1 & 5 & 10 & 10 & 5 & 1 & \dots \\ 1 & 6 & 15 & 20 & 15 & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 124. We consider the exponential array

$$\left[e^{\alpha x}, \log\left(\frac{1}{1-x}\right)\right]$$

which is the product of $\mathbf{B}^\alpha$ and the Stirling matrix of the first kind $\mathbf{s} = \left[1, \ln\left(\frac{1}{1-x}\right)\right]$. This matrix also defines the (unsigned) Charlier polynomials of order α. We now find, using Equations (4.3) and (4.4), that

$$\begin{aligned} r\left(\log\left(\frac{1}{1-x}\right)\right) &= \frac{1}{1-x} \Longrightarrow r(x) = e^x \\ c\left(\log\left(\frac{1}{1-x}\right)\right) &= \frac{(e^{\alpha x})'}{e^{\alpha x}} = \alpha \Longrightarrow c(x) = \alpha. \end{aligned}$$

Thus the production array P for this array has bivariate generating function

$$\phi_P(t,z) = e^{tz}(\alpha + te^z).$$

This implies that P takes the form

$$\begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \alpha+1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & \alpha+2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 3 & \alpha+3 & 1 & 0 & \dots \\ 0 & 1 & 4 & 6 & \alpha+4 & 1 & \dots \\ 0 & 1 & 5 & 10 & 10 & \alpha+5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

or

$$\alpha \mathbf{I} + \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 3 & 3 & 1 & 0 & \dots \\ 0 & 1 & 4 & 6 & 4 & 1 & \dots \\ 0 & 1 & 5 & 10 & 10 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 125. Taking

$$\begin{aligned} c(x) &= 1 + \alpha x \\ r(x) &= 1 + \alpha x \end{aligned}$$

we find that A_P is the array

$$\left[e^{\frac{e^{\alpha x}-1}{\alpha}}, \frac{e^{\alpha x}-1}{\alpha}\right].$$

For $\alpha = 2$, this is the array

$$\left[e^{\sinh(x)e^x}, \sinh(x)e^x\right],$$

whose production array P has generating function

$$e^{tx}(1+t)(1+2x).$$

This array begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 4 & 1 & 0 & 0 & 0 & \dots \\ 11 & 19 & 9 & 1 & 0 & 0 & \dots \\ 49 & 104 & 70 & 16 & 1 & 0 & \dots \\ 257 & 641 & 550 & 190 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A154602. The first column of this array is given by A004211, while the row sums are A055882, equal to $2^n Bell(n)$, with e.g.f.

$$\exp(\exp(2x) - 1).$$

The array with

$$\begin{aligned} c(x) &= 1 + x \\ r(x) &= 1 + x, \end{aligned}$$

and hence with production array with generating function

$$e^{tx}(1+t)(1+x)$$

is equal to

$$\left[e^{e^x-1}, e^x - 1\right]$$

whose inverse is the array of coefficients of the (signed) Charlier polynomials (see A094816). This array begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 5 & 10 & 6 & 1 & 0 & 0 & \ldots \\ 15 & 37 & 31 & 10 & 1 & 0 & \ldots \\ 52 & 151 & 160 & 75 & 15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A049020. It can be expressed as

$$[1, 1, 1, 2, 1, 3, 1, 4, 1, \ldots] \quad \Delta \quad [1, 0, 1, 0, 1, 0, \ldots].$$

It has production matrix which starts

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 4 & 5 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 5 & 6 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

indicating that the generating function of the Bell numbers can be expressed as

$$\cfrac{1}{1 - x - \cfrac{x^2}{1 - 2x - \cfrac{2x^2}{1 - 3x - \cfrac{3x^2}{1 - \cdots}}}}.$$

The product of this array with $\mathbf{B}^{-1}$ is the matrix $\mathbf{S} = [1, e^x - 1]$ of Stirling numbers $\left\{ {n \atop k} \right\}$ of the second kind.

Example 126. We let

$$\begin{aligned} c_n &= -(-\alpha)^n, \qquad c(x) = -\frac{1}{1+\alpha x}, \\ r_n &= (-\alpha)^n, \qquad r(x) = \frac{1}{1+\alpha x}. \end{aligned}$$

Then we find that

$$A_{P_\alpha}^{-1} = \left[e^x, x(1+\frac{\alpha x}{2})\right].$$

For instance, $\alpha = 3$ gives us the array

$$A_{P_3}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 5 & 1 & 0 & 0 & 0 & \dots \\ 1 & 12 & 12 & 1 & 0 & 0 & \dots \\ 1 & 22 & 69 & 22 & 1 & 0 & \dots \\ 1 & 35 & 235 & 235 & 35 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have

$$A_{P_\alpha} = \left[e^{\frac{1-\sqrt{1+2\alpha x}}{\alpha}}, \frac{\sqrt{1+2\alpha x}-1}{\alpha}\right]$$

and the bivariate generating function of P_α is given by

$$\phi_{P_\alpha} = e^{tx}\frac{t-1}{1+\alpha x}.$$

For more about these arrays, see Chapters 11 and 13.

Example 127. The production matrix with bivariate generating function

$$\phi_P = e^{tx}\frac{t-1}{1+x^2}$$

generates the exponential Riordan array

$$\left[e^x, x+\frac{x^3}{3}\right]^{-1}.$$

More generally, the production matrix with generating function

$$\phi_P = e^{tx}\frac{t-1}{1+x^m}$$

generates the array

$$\left[e^x, x+\frac{x^{m+1}}{m+1}\right]^{-1}.$$

Similarly, the production matrix with generating function

$$\phi_P = e^{tx}\frac{t-1}{1+x^m/m}$$

generates the array

$$\left[e^x, x+\frac{x^{m+1}}{m(m+1)}\right]^{-1}.$$

For example, the array

$$\left[e^x, x+\frac{x^3}{2\cdot 3}\right] = \left[e^x, x+\frac{x^3}{3!}\right]$$

begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 8 & 10 & 4 & 1 & 0 & \ldots \\ 1 & 15 & 30 & 20 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that the second column is essentially the so-called "cake numbers", A000125.

Example 128. The production matrix with bivariate generating function

$$\phi_{P_\alpha} = e^{tx}\frac{t-1}{(1+\alpha x)^2}$$

generates the exponential Riordan array

$$\left[e^x, x(1+\alpha x+\frac{\alpha^2}{3}x^2)\right]^{-1}.$$

For instance

$$\phi_{P_\alpha} = e^{tx}\frac{t-1}{(1+x)^2}$$

generates the inverse of the array $[e^x, x(1+x+x^2/3)]$ which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 11 & 9 & 1 & 0 & 0 & \ldots \\ 1 & 24 & 50 & 16 & 1 & 0 & \ldots \\ 1 & 45 & 210 & 150 & 25 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of this array, which begin $1, 2, 6, 22, 92, 432, \ldots$ have e.g.f. $e^{2x+x^2+x^3/3}$.

Example 129. The production matrix with bivariate generating function

$$\phi_P = e^{tx}(1+t)(1+x)^2$$

corresponding to

$$c(x) = r(x) = (1+x)^2$$

generates the array

$$P = \left[e^{\frac{x}{1-x}}, \frac{x}{1-x}\right]$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 4 & 1 & 0 & 0 & 0 & \dots \\ 13 & 21 & 9 & 1 & 0 & 0 & \dots \\ 73 & 136 & 78 & 16 & 1 & 0 & \dots \\ 501 & 1045 & 730 & 210 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is A059110. This array is equal to $\mathbf{Lah} \cdot \mathbf{B}$ (see Chapter 8).

Example 130. The matrix $[e^{\frac{x^2}{2}}, x]$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 0 & 1 & 0 & 0 & \dots \\ 3 & 0 & 6 & 0 & 1 & 0 & \dots \\ 0 & 15 & 0 & 10 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with production matrix given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 4 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 5 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A099174, which is linked to the Hermite and Bessel polynomials. Its general element $T(n,k)$ is the number of involutions of $\{1,2,...,n\}$ having k fixed points. A066325 is a signed version of this array.

Chapter 5

The Deleham construction

5.1 Definition of the Deleham construction

The Deleham construction is a powerful method for constructing number triangles. Based on the theory of continued fractions and orthogonal polynomials, it provides insight into the construction of many important number triangles. Its input is two integer sequences, which we shall denote by r_n and s_n, or r and s (where $r(n) = r_n$ etc). We can then construct a two dimensional integer array, called the *Deleham array determined by* r *and* s, as follows. First, we form the function of n, x and y defined by

$$q(n, x, y) = xr_n + ys_n. \tag{5.1}$$

Then we form a family of polynomials $P(n, m, x, y)$ as follows:

$$P(n, m, x, y) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \text{ and } m = -1 \\ P(n, m-1, x, y) + q(m, x, y)P(n-1, m+1, x, y) & \end{cases} \tag{5.2}$$

Finally, we form the array with general term $\Delta_{n,k}$ determined by

$$\Delta_{n,k} = [x^{n-k}]P(n, 0, x, 1). \tag{5.3}$$

The array so formed will be denoted by $r\Delta s$ or $\Delta(r, s)$. We shall on occasion also use the notation

$$\Delta^{(m)}_{n,k} = [x^{n-k}]P(n, m, x, 1) \tag{5.4}$$

and

$$\Delta^{(m)}_{n,k}(r, s; \alpha, \beta) = [x^{n-k}]P(n, m, x + \alpha, \beta). \tag{5.5}$$

Example 131. We take $r_n = \frac{1-(-1)^n}{2}$ and $s_n = \frac{1+(-1)^n}{2}$. Thus r_n starts $0, 1, 0, 1, 0, 1, \dots$

while s_n starts $1, 0, 1, 0, 1, 0, \ldots$. The array $P(n, m, x, y)$ then starts

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & \cdots \\ y & x+y & x+2y & 2x+2y & \cdots \\ y(x+y) & x^2+3xy+y^2 & x^2+5xy+3y^2 & 3x^2+8xy+3y^2 & \cdots \\ y(x^2+3xy+y^2) & x^3+6x^2y+6xy^2+y^3 & x^3+9x^2y+14xy^2+4y^3 & 4(x^3+5x^2y+5xy^2+y^3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right.$$

Thus $P(n, 0, x, y)$, $n = 0, 1, 2, \ldots$ gives the family of polynomials

$$1, y, y(x+y), y(x^2+3xy+y^2), y(x^3+6x^2y+6xy^2+y^3), \ldots$$

which leads to the following matrix representation of $r_n \, \Delta \, s_n$:

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 6 & 6 & 1 & 0 & \cdots \\ 0 & 1 & 10 & 20 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which is the Narayana triangle A090181 $\tilde{\tilde{N}}$ with generating function

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}$$

or equivalently

$$\cfrac{1}{1-(y-1)x-\cfrac{x}{1-(y-1)x-\cfrac{x}{1-(y-1)x-\cfrac{x}{1-\ldots}}}}.$$

We note further that

$$\Delta^{(1)}_{n,k} = \tilde{N}(n, k)$$

where

$$\tilde{N}(n, k) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k} = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

This is therefore the triangle

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ 1 & 15 & 50 & 50 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which has generating function

$$\cfrac{1}{1-xy-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cdots}}}}}.$$

Example 132. We now take $r_n = 1$ and $s_n = \frac{1+(-1)^n}{2}$. We obtain the family of polynomials $P(n, 0, x, y)$ that begins

$$1, x+y, 2x^2+3xy+y^2, 5x^3+10x^2y+6xy^2+y^3, 14x^4+35x^3y+30x^2y^2+10xy^3+y^4, \dots$$

This gives us the following Deleham array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 3 & 1 & 0 & 0 & 0 & \dots \\ 5 & 10 & 6 & 1 & 0 & 0 & \dots \\ 14 & 35 & 30 & 10 & 1 & 0 & \dots \\ 42 & 126 & 140 & 70 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is A060693 with general term $\binom{2n-k}{k}C_{n-k}$. Note that this array is the product of the previous array and $\mathbf{B}$:

$$[1,1,1,1,1,\dots] \quad \Delta \quad [1,0,1,0,\dots] = ([0,1,0,1,0,\dots] \quad \Delta \quad [1,0,1,0,\dots]) \cdot \mathbf{B}.$$

The generating function of this array can be expressed in continued fraction form as

$$\cfrac{1}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cdots}}}}}$$

or

$$\cfrac{1}{1-\cfrac{x+xy}{1-\cfrac{x}{1-\cfrac{x+xy}{1-\cfrac{x}{1-\cfrac{x+xy}{1-\cdots}}}}}}$$

Example 133. In this example, we reverse the roles of r and s in the previous example. Thus we take $r_n = \frac{1+(-1)^n}{2}$ and $s_n = 1$. This gives us the Deleham array $\Delta(r, s)$ or

$$[1, 0, 1, 0, 1, 0, \ldots] \quad \Delta \quad [1, 1, 1, 1, \ldots]$$

that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 2 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 10 & 5 & 0 & 0 & \ldots \\ 1 & 10 & 30 & 35 & 14 & 0 & \ldots \\ 1 & 15 & 70 & 140 & 126 & 42 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term $\binom{n+k}{2k}C_k = \frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}$. This is A088617. This is the reverse of the previous array. This array is linked to the coefficient array of the series reversion of

$$\frac{x(1-\alpha x)}{1+x}.$$

We can see this as follows. The sequence $[x^{n+1}]\text{Rev}\frac{x(1-\alpha x)}{1+x}$ has generating function

$$\frac{1-x-\sqrt{(1-x)^2-4\alpha x}}{2\alpha x}.$$

Expanding this as a power series, we find the coefficients

$$1, \alpha+1, 2\alpha^2+3\alpha+1, 5\alpha^3+10\alpha^2+6\alpha+1, \ldots$$

with coefficient array A088617. We note that the generating function for this array can be expressed as the continued fraction

$$\cfrac{1}{1-x-\cfrac{xy}{1-x-\cfrac{xy}{1-x-\cfrac{xy}{1-x-\cfrac{xy}{1-x-\cdots}}}}}$$

or

$$\cfrac{1}{1-\cfrac{x(1+y)}{1-\cfrac{xy}{1-\cfrac{x(1+y)}{1-\cfrac{xy}{1-\cdots}}}}}$$

or indeed as

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cdots}}}}}.$$

This last expression follows from

$$[1,0,1,0,\ldots] \quad \Delta \quad [1,1,1,\ldots] = [1,0,1,0,\ldots] \quad \Delta^{(1)} \quad [0,1,1,1,\ldots].$$

As an example, we take the sequence A103210, or

$$a_n = [x^{n+1}]\mathrm{Rev}\frac{x(1-2x)}{1+x}.$$

We can express the g.f. of this sequence, given by

$$g(x) = \frac{1-x-\sqrt{1-10x+x^2}}{4x}$$

in continued fraction form as

$$\begin{aligned} g(x) &= 1/(1-3x/(1-2x/(1-3x/(1-2x/(1-3x/(1-\ldots, \\ &= 1/(1-x-2x/(1-x-2x/(1-x-2x/(1-\ldots, \\ &= 1/(1-3x-6x^2/(1-5x-6x^2/(1-5x-6x^2/(1-\ldots \end{aligned}$$

Correspondingly we have the following expressions for a_n :

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k-1}3^k2^{n-k}, n>0, a_0=1, \\ &= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k+1}2^k3^{n-k}, n>0, a_0=1, \\ &= \sum_{k=0}^{n}\binom{n+k}{2k}2^kC_k, \\ &= \sum_{k=0}^{n}\binom{2n-k}{k}2^{n-k}C_{n-k}. \end{aligned}$$

As noted by Deleham in A103210, we also have

$$a_n = a_{n-1} + 2\sum_{k=0}^{n-1} a_k a_{n-k-1}, \quad n > 0, \quad a_0 = 1.$$

In fact, we have the following general result.

Proposition 134. *Let*

$$a_n = [x^{n+1}]Rev\frac{x(1-\alpha x)}{1-\beta x}.$$

Then the g.f. of a_n can be expressed in continued fraction form as

$$\begin{aligned} g(x) &= 1/(1-(\alpha-\beta)x/(1-\alpha x/(1-(\alpha-\beta)x/(1-\alpha x/(1-\ldots, \\ &= 1/(1+\beta x-\alpha x/(1+\beta x-\alpha x/(1+\beta x-\alpha x/(1-\ldots, \\ &= \cfrac{1}{1-(\alpha-\beta)x-\cfrac{\alpha(\alpha-\beta)x^2}{1-(2\alpha-\beta)x-\cfrac{\alpha(\alpha-\beta)x^2}{1-(2\alpha-\beta)x-\ldots}}}. \end{aligned}$$

We then have

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k-1}(\alpha-\beta)^k\alpha^{n-k}, n > 0, a_0 = 1, \\ &= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k+1}\alpha^k(\alpha-\beta)^{n-k}, n > 0, a_0 = 1, \\ &= \sum_{k=0}^{n}\binom{n+k}{2k}(-\beta)^{n-k}\alpha^k C_k, \\ &= \sum_{k=0}^{n}\binom{2n-k}{k}(-\beta)^k\alpha^{n-k}C_{n-k}. \end{aligned}$$

In addition, a_n satisfies the recurrence

$$a_n = (-\beta)a_{n-1} + \alpha\sum_{k=0}^{n-1} a_k a_{n-k-1}, \quad n > 0, \quad a_0 = 1.$$

Corollary 135. *We have*

$$\begin{aligned} a_n &= \sum_{k=0}^{n}\tilde{\tilde{N}}_{n,k}(\alpha-\beta)^k\alpha^{n-k} \\ &= \sum_{k=0}^{n} N_{n,k}\alpha^k(\alpha-\beta)^{n-k}. \end{aligned}$$

Corollary 136. *a_n is the moment sequence for the family of orthogonal polynomials defined by*

$$P_0(x) = 1, \qquad P_2(x) = x - (\alpha - \beta), \qquad P_{n+2}(x) = (x - (2\alpha - \beta))P_{n+1}(x) - \alpha(\alpha - \beta)P_n(x).$$

We now note that the product of the array

$$[1, 0, 1, 0, \ldots] \quad \Delta \quad [1, 1, 1, \ldots]$$

with $\mathbf{B}$, that is, $\Delta(r, s) \cdot \mathbf{B}$, has generating function

$$\cfrac{1}{1 - x - \cfrac{x(y+1)}{1 - x - \cfrac{x(y+1)}{1 - x - \cfrac{x(y+1)}{1 - x - \cfrac{x(y+1)}{1 - x - \cdots}}}}}$$

or

$$\cfrac{1}{1 - \cfrac{x(2+y)}{1 - \cfrac{x(1+y)}{1 - \cfrac{x(2+y)}{1 - \cfrac{x(1+y)}{1 - \cdots}}}}}$$

This is

$$[2, 1, 2, 1, 2, \ldots] \quad \Delta \quad [1, 1, 1, 1, \ldots].$$

A related matrix is A107131, which has general term

$$[k \le n]\binom{n}{2n - 2k} C_{n-k}.$$

This matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 6 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 10 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with generating function

$$\cfrac{1}{1-\cfrac{xy(1+x)}{1-\cfrac{x^2y}{1-\cfrac{xy(1+x)}{1-\cfrac{x^2y}{1-\cdots}}}}}.$$

Other forms of its generating function are

$$\cfrac{1}{1-xy-\cfrac{x^2y}{1-xy-\cfrac{x^2y}{1-\ldots}}}$$

and

$$\cfrac{1}{1-\cfrac{xy+x^2y}{1-\cfrac{x^2y}{1-\cfrac{xy+x^2y}{1-\cfrac{x^2y}{1-\ldots}}}}}.$$

This matrix is the coefficient array for the polynomials $x^n\,{}_2F_1(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;\frac{4}{x})$. The product of $\mathbf{B}$ and this array gives the Narayana triangle $\tilde{N}$, which therefore has generating function of the following form:

$$\cfrac{1}{1-x-xy-\cfrac{x^2y}{1-x-xy-\cfrac{x^2y}{1-x-xy-\cfrac{x^2y}{1-\ldots}}}}.$$

Example 137. We can define a "q-Catalan triangle" to be the following:

$$[1,q,q^2,q^3,q^4,\ldots] \quad \Delta \quad [1,0,1,0,1,\ldots].$$

For instance, when $q=2$, we get the triangle that begins

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 3 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 17 & 25 & 9 & 1 & 0 & 0 & \ldots \\ 171 & 258 & 102 & 16 & 1 & 0 & \ldots \\ 3113 & 4635 & 1788 & 290 & 25 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with g.f. given by

$$\cfrac{1}{1-\cfrac{x+xy}{1-\cfrac{2x}{1-\cfrac{4x+xy}{1-\cfrac{8x}{1-\cfrac{16x+xy}{1-\cdots}}}}}}$$

The first column, which has g.f.

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{4x}{1-\cfrac{8x}{1-\cfrac{16x}{1-\cdots}}}}}}$$

is the sequence A015083 of q-Catalan numbers for $q=2$. The row sums of this matrix are A154828. They may be considered as q-Schröder numbers for $q=2$.

Example 138. We let r be the sequence $0,1,0,2,0,3,\ldots$ and $s_n = \lfloor \frac{n+2}{2} \rfloor$. Then

$$[0,1,0,2,0,3,0,\ldots] \quad \Delta \quad [1,1,2,2,3,3,\ldots]$$

is the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 2 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 0 & 0 & \ldots \\ 0 & 1 & 14 & 36 & 24 & 0 & \ldots \\ 0 & 1 & 30 & 150 & 240 & 120 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which has general term $k!\left\{ {n \atop k} \right\} = \sum_{j=0}^{k}(-1)^{k-j} j^n \binom{k}{j}$ where $\left\{ {n \atop k} \right\}$ denotes the Stirling numbers of the second kind. This matrix has bi-variate generating function

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cfrac{2x}{1-\cfrac{2(x+xy)}{1-\cfrac{3xy}{1-\cfrac{3(x+xy)}{1-\cdots}}}}}}}$$

This is A019538.

Example 139. We let $r_n = 0^n$ and $s_n = 1 - 0^n$. Then $\Delta(r,s)$ is the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 0 & 0 & 0 & \dots \\ 1 & 3 & 5 & 5 & 0 & 0 & \dots \\ 1 & 4 & 9 & 14 & 14 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\Delta^{(1)}(r,s)$ is the Catalan array A009766 which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 0 & 0 & 0 & \dots \\ 1 & 3 & 5 & 5 & 0 & 0 & \dots \\ 1 & 4 & 9 & 14 & 14 & 0 & \dots \\ 1 & 5 & 14 & 28 & 42 & 42 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term $\binom{n+k}{n}\frac{n-k+1}{n+1}$ and generating function

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{xy}{1-\cfrac{xy}{1-\cfrac{xy}{1-\cdots}}}}}$$

Example 140. Similarly, if $r_n = 0^n$ and s_n is the sequence

$$0, 2, 1, 1, 1, \dots$$

then $\Delta^{(1)}(r,s)$ is the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 1 & 4 & 6 & 0 & 0 & 0 & \dots \\ 1 & 6 & 16 & 20 & 0 & 0 & \dots \\ 1 & 8 & 30 & 64 & 70 & 0 & \dots \\ 1 & 10 & 48 & 140 & 256 & 352 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with generating function

$$\cfrac{1}{1-x-\cfrac{2xy}{1-\cfrac{xy}{1-\cfrac{xy}{1-\cfrac{xy}{1-\cdots}}}}}$$

The row sums of this triangle are A026671 with g.f. given by

$$\frac{1}{1-x-2c(x)}=\cfrac{1}{1-x-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}}$$

We note the following proposition :

Proposition 141. *We have*

$$\Delta(r+s,s)=\Delta(r,s)\cdot\mathbf{B}.$$

Proof. We let $\Delta^*=\Delta(r+s,s)$ and $\Delta=\Delta(r,s)$. We note first that since

$$x(r_n+s_n)+ys_n=xr_n+(x+y)s_n,$$

Δ^* is defined by $P(n,m,x,x+y)$ where Δ is defined by $P(n,m,x,y)$. $\square$

Example 142. We take $r_n=1$ and $s_n=\frac{1-(-1)^n}{2}$. The $\Delta(r,s)$ or

$$[1,1,1,\ldots]\quad\Delta\quad[0,1,0,1,0,\ldots]$$

is the array that starts

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 5 & 5 & 1 & 0 & 0 & 0 & \ldots \\ 14 & 21 & 9 & 1 & 0 & 0 & \ldots \\ 42 & 84 & 56 & 14 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term

$$[k<=n]\frac{1}{n+1}\binom{n-1}{k}\binom{2n-k}{n}$$

and generating function

$$\cfrac{x}{1-\cfrac{x+xy}{1-\cfrac{x}{1-\cfrac{x+xy}{1-\cdots}}}}.$$

This array has as row sums the little Schröder numbers A001003. The generating function of this array can also be expressed as the continued fraction

$$\cfrac{1}{1-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-\cdots}}}}}$$

The array $\Delta^{(1)}(r,s)$ is the array A126216 which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 5 & 5 & 1 & 0 & 0 & 0 & \ldots \\ 14 & 21 & 9 & 1 & 0 & 0 & \ldots \\ 42 & 84 & 56 & 14 & 1 & 0 & \ldots \\ 132 & 330 & 300 & 120 & 20 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

This array has general term $\frac{1}{n+2}\binom{n}{k}\binom{2n-k+2}{n+1} = \frac{1}{n+1}\binom{n+1}{k}\binom{2n-k+2}{n+2}$ and generating function

$$\cfrac{1}{1-x-\cfrac{x+xy}{1-\cfrac{x}{1-\cfrac{x+xy}{1-\cdots}}}}.$$

Now

$$\Delta^{(1)}(r,s)\cdot \mathbf{B}^{-1} = \tilde{\mathbf{N}}$$

where $\tilde{N}(n,k) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k}$.

5.2 The fundamental theorem

For the purposes of this work, we shall take as definition of the the Deleham array

$$[r_0, r_1, r_2, r_3, \ldots] \quad \Delta \quad [s_0, s_1, s_2, s_3, \ldots]$$

the number triangle with generating function

$$\cfrac{1}{1-\cfrac{(r_0x+s_0xy)}{1-\cfrac{(r_1x+s_1xy)}{1-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}}.$$

Then we have

Theorem 143. *The first column of the Deleham array*

$$[r_0, r_1, r_2, r_3, \ldots] \quad \Delta \quad [s_0, s_1, s_2, s_3, \ldots]$$

has g.f.

$$\cfrac{1}{1-\cfrac{r_0x}{1-\cfrac{r_1x}{1-\cfrac{r_2x}{1-\cdots}}}}.$$

The main diagonal of the array has g.f.

$$\cfrac{1}{1-\cfrac{s_0x}{1-\cfrac{s_1x}{1-\cfrac{s_2x}{1-\cdots}}}}.$$

The row sums of the array have g.f.

$$\cfrac{1}{1-\cfrac{(r_0+s_0)x}{1-\cfrac{(r_1+s_1)x}{1-\cfrac{(r_2+s_2)x}{1-\cdots}}}}.$$

The diagonal sums of the array have g.f.

$$\cfrac{1}{1-\cfrac{(r_0x+s_0x^2)}{1-\cfrac{(r_1x+s_1x^2)}{1-\cfrac{(r_2x+s_2x^2)}{1-\cdots}}}}.$$

The product of the array with $\mathbf{B}$ *has generating function*

$$\cfrac{1}{1-\cfrac{((r_0+s_0)x+s_0xy)}{1-\cfrac{((r_1+s_1)x+s_1xy)}{1-\cfrac{((r_2+s_2)x+s_2xy)}{1-\cdots}}}}=\cfrac{1}{1-\cfrac{r_0x+s_0x(1+y)}{1-\cfrac{r_1x+s_1x(1+y)}{1-\cfrac{r_2x+s_2x(1+y)}{1-\cdots}}}}.$$

The product of $\mathbf{B}$ *and the array has generating function*

$$\cfrac{1}{1-x-\cfrac{(r_0x+s_0xy)}{1-\cfrac{(r_1x+s_1xy)}{1-x-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}}.$$

Proof. The g.f. of the first column is obtained by setting $y=0$ in the bivariate g.f. Similarly, the g.f. of the row sums is obtained by setting $y=1$, while that of the diagonal sums is found by setting $y=x$. The g.f. of the binomial transform of the array will be given by

$$\frac{1}{1-x}\cfrac{1}{1-\cfrac{(r_0+s_0y)\frac{x}{1-x}}{1-\cfrac{(r_1+s_1y)\frac{x}{1-x}}{1-\cfrac{(r_2+s_2y)\frac{x}{1-x}}{1-\cdots}}}},$$

which simplifies to

$$\cfrac{1}{1-x-\cfrac{(r_0x+s_0xy)}{1-\cfrac{(r_1x+s_1xy)}{1-x-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}}.$$

□

The array

$$[r_0,r_1,r_2,r_3,\ldots]\quad\Delta^{(1)}\quad[s_0,s_1,s_2,s_3,\ldots]$$

has generating function

$$\cfrac{1}{1-(r_0x+s_0xy)-\cfrac{r_1x+s_1xy}{1-\cfrac{r_2x+s_2xy}{1-\cdots}}}.$$

5.3 The Deleham construction and Riordan arrays

In the particular case of r_n being the sequence

$$\alpha, \beta, \gamma, 0, 0, 0, 0, \ldots$$

and $s_n = 0^n$, we have the following result:

Proposition 144. *The Deleham array*

$$[\alpha, \beta, \gamma, 0, 0, 0, \ldots] \quad \Delta \quad [1, 0, 0, 0, \ldots]$$

is given by the Riordan array

$$\left(\frac{1-(\beta+\gamma)x}{1-(\alpha+\beta+\gamma)x+\alpha\gamma x^2}, \frac{x(1-\gamma x)}{1-(\alpha+\beta+\gamma)x+\alpha\gamma x^2}\right).$$

This array has row sums with g.f. given by

$$\frac{1-(\beta+\gamma)x}{1-(1+\alpha+\beta+\gamma)x+(1+\alpha)\gamma x^2}.$$

Proof. The generating function of this array is given by

$$\cfrac{1}{1-\cfrac{\alpha x+xy}{1-\cfrac{\beta}{1-\gamma x}}}$$

This is equal to

$$\frac{1-(\beta+\gamma)x}{1-(\alpha+\beta+\gamma+y)x+\gamma(\alpha+y)x^2}.$$

$\square$

A trivial consequence of this is that

$$\mathbf{B} = [1, 0, 0, 0, \ldots] \quad \Delta \quad [1, 0, 0, 0, \ldots]$$

while

$$\mathbf{B}^{-1} = [-1, 0, 0, 0, \ldots] \quad \Delta \quad [1, 0, 0, 0, \ldots].$$

Example 145. We take for r_n the sequence $1, -1, 1, 0, 0, 0, \ldots$. Then we obtain the array $\left(\frac{1}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2}\right)$ with row sums with g.f. $\frac{1}{1-2x+2x^2}$ (and e.g.f. given by $(\exp(x)\sin(x))'$). The row sums of the inverse of this matrix are the so-called "Motzkin sums" A005043.

Corollary 146.

$$[\alpha, \beta, 0, 0, 0, \ldots] \quad \Delta \quad [1, 0, 0, 0, \ldots]$$

is given by the Riordan array

$$\left(\frac{1-\beta x}{1-(\alpha+\beta)x}, \frac{x}{1-(\alpha+\beta)x}\right).$$

For instance,

$$[k, 0, 0, 0, \ldots] \quad \Delta \quad [1, 0, 0, 0, \ldots] = \mathbf{B}^k.$$

5.4 The Deleham construction and associahedra

The Deleham construction leads to many interesting triangular arrays of numbers. The field of *associahedra* [44, 49, 94, 180] is rich in such triangles, including the Narayana triangle. We give two examples from this area.

Example 147. The triangle with general term

$$\frac{1}{k+1}\binom{n}{k}\binom{n+k+2}{k}$$

is given by

$$[1, 0, 1, 0, 1, \ldots] \quad \Delta^{(1)} \quad [1, 1, 1, 1, \ldots].$$

This is the coefficient array for the f-vector for A_n [44, 49]. We recall that

$$[1, 0, 1, 0, 1, \ldots] \quad \Delta \quad [1, 1, 1, 1, \ldots]$$

has generating function

$$\cfrac{1}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cdots}}}}$$

and thus

$$[1, 0, 1, 0, 1, \ldots] \quad \Delta^{(1)} \quad [1, 1, 1, 1, \ldots]$$

has generating function

$$\cfrac{1}{1-(x+xy)-\cfrac{xy}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cdots}}}}.$$

This is the array A033282 that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 5 & 5 & 0 & 0 & 0 & \ldots \\ 1 & 9 & 21 & 14 & 0 & 0 & \ldots \\ 1 & 14 & 56 & 84 & 42 & 0 & \ldots \\ 1 & 20 & 120 & 1300 & 330 & 132 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We note that we have

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n+k+2}{k}r^k = \sum_{k=0}^{n}\binom{n}{2k}(2r+1)^{n-2k}\left(2\binom{r+1}{2}\right)^k C_k \tag{5.6}$$

$$= [x^{n+1}]\text{Rev}\frac{x}{1+(2r+1)x+2\binom{r+1}{2}x^2}. \tag{5.7}$$

We deduce that the row sums (case $r=1$) are given by s_{n+1}, where s_n are the little Schröder numbers A001003. They have generating function

$$\cfrac{1}{1-2x-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cdots}}}}}$$

In addition, the diagonal sums of this array, given by A005043$(n+2)$ where A005043 are the so-called "Motzkin sums", have generating function

$$\cfrac{1}{1-x(1+x)-\cfrac{x^2}{1-\cfrac{x(1+x)}{1-\cfrac{x^2}{1-\cfrac{x(1+x)}{1-\cdots}}}}}$$

We note that this generating function may also be represented as

$$\cfrac{1}{1-x(1+x)-\cfrac{x^2}{1-x-\cfrac{x^2}{1-x-\cfrac{x^2}{1-\cdots}}}}$$

The Hankel transform of this sequence begins

$$1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, \ldots$$

The corresponding h-vector array is given by the Narayana numbers

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & \ldots \\ 1 & 15 & 50 & 50 & 15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with generating function

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}$$

Thus the transition from f-vector array to h-vector array is symbolized by

$$\cfrac{1}{1-(x+xy)-\cfrac{xy}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cdots}}}}} \Rightarrow \cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}$$

There is also a well-defined transition from

$$[1,0,1,0,1,\ldots] \quad \Delta \quad [1,1,1,1,\ldots]$$

to the Narayana numbers. $[1,0,1,0,1,\ldots] \quad \Delta \quad [1,1,1,1,\ldots]$ is the array with general element $T_{n,k}=\frac{1}{k+1}\binom{n+k}{k}\binom{n}{k}$. Forming the array with general element $T_{k,n-k}$ or

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 6 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 10 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we find that the product of $\mathbf{B}$ and this matrix is the triangle of Narayana numbers. Alternatively, reversing $[1,0,1,0,1,\ldots] \quad \Delta^{(1)} \quad [1,1,1,1,\ldots]$ to give

$$[1,1,1,1,\ldots] \quad \Delta^{(1)} \quad [1,0,1,0,1,\ldots]$$

or

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 5 & 5 & 1 & 0 & 0 & 0 & \ldots \\ 14 & 21 & 9 & 1 & 0 & 0 & \ldots \\ 42 & 84 & 56 & 14 & 1 & 0 & \ldots \\ 132 & 330 & 300 & 120 & 20 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we find that the product of this matrix and $\mathbf{B}^{-1}$ gives the Narayana numbers $\tilde{\mathbf{N}}$.

Example 148. The triangle with general term

$$\binom{n}{k}\binom{n+k}{k} = \binom{n+k}{2k}\binom{2k}{k}$$

A063007 is the coefficient array for the f-vector for B_n [49]. This array is given by

$$[1, 0, 1, 0, 1, 0, \ldots] \quad \Delta^{(1)} \quad [0, 2, 1, 1, 1, 1, \ldots]$$

and thus has generating function

$$\cfrac{1}{1 - x - \cfrac{2xy}{1 - \cfrac{x+xy}{1 - \cfrac{xy}{1 - \cfrac{x+xy}{1 - \cdots}}}}}$$

which can also be expressed as

$$\cfrac{1}{1 - x - \cfrac{2xy}{1 - x - \cfrac{xy}{1 - x - \cfrac{xy}{1 - x - \cfrac{xy}{1 - \cdots}}}}}$$

or as

$$\cfrac{1}{1 + x - \cfrac{2(x+xy)}{1 - \cfrac{xy}{1 - \cfrac{x+xy}{1 - \cfrac{xy}{1 - \cfrac{x+xy}{1 - \cdots}}}}}}$$

The array begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 0 & 0 & 0 & \ldots \\ 1 & 12 & 30 & 20 & 0 & 0 & \ldots \\ 1 & 20 & 90 & 140 & 70 & 0 & \ldots \\ 1 & 30 & 210 & 560 & 630 & 252 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Its row sums are A001850, the central Delannoy numbers. We deduce the following generating functions for A001850:

$$\cfrac{1}{1-x-\cfrac{2x}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cdots}}}}}}$$

and

$$\cfrac{1}{1-x-\cfrac{2x}{1-x-\cfrac{x}{1-x-\cfrac{x}{1-x-\cfrac{x}{1-\cdots}}}}}$$

Reversing this array to get the array

$$[0,2,1,1,1,1,\ldots] \quad \Delta^{(1)} \quad [1,0,1,0,1,0,\ldots]$$

with general term $\binom{n}{k}\binom{2n-k}{n}$ we find that the product of this matrix with $\mathbf{B}^{-1}$ is the matrix with general term $\binom{n}{k}^2$. This is the h-vector array for B_n. We note that in the case of $T_{n,k}=\binom{n}{k}\binom{n+k}{k}$ the product of $\mathbf{B}$ and the matrix with general term $T_{k,n-k}$ is the matrix with general term $\binom{n}{k}^2$. This matrix has generating function

$$\cfrac{1}{1+x-xy-\cfrac{2x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}}$$

We thus have the transition

$$\cfrac{1}{1+x-\cfrac{2(x+xy)}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cdots}}}}}} \Rightarrow \cfrac{1}{1+x-xy-\cfrac{2x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}}$$

We note that we obtain the identities

$$\binom{n}{k}^2 = \sum_{j=0}^{n} \binom{n}{j}\binom{j}{k}\binom{2n-j}{n}(-1)^{j-k}$$

and

$$\binom{n}{k} = \sum_{j=0}^{n} \binom{n-k}{j-k}\binom{2n-j}{n}(-1)^{j-k}.$$

Chapter 6

Riordan arrays and a Catalan transform [1]

6.1 Introduction

In this chapter, we report on a transformation of integer sequences that might reasonably be called the Catalan transformation. It is easy to describe both by formula (in relation to the general term of a sequence) and in terms of its action on the ordinary generating function of a sequence. It and its inverse can also be described succinctly in terms of the Riordan group.
Many classical "core" sequences can be paired through this transformation. It is also linked to several other known transformations, most notably the binomial transformation.
Unless otherwise stated, the integer sequences we shall study will be indexed by $\mathbb{N}_0$, the nonnegative integers. Thus the Catalan numbers, with general term C_n, are described by

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

with ordinary generating function given by

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

In the following, all sequences a_n will have offset 0, that is, they begin $a_0, a_1, a_2, \ldots$. We use the notation 1^n to denote the all $1's$ sequence $1, 1, 1, \ldots$ with ordinary generating function $1/(1-x)$ and 0^n to denote the sequence $1, 0, 0, 0, \ldots$ with ordinary generating function 1. This is A000007. We have $0^n = \delta_{n,0} = \binom{0}{n}$ as an integer sequence. This notation allows us to regard $\ldots(-2)^n, (-1)^n, 0^n, 1^n, 2^n, \ldots$ as a sequence of successive binomial transforms (see next section).
In order to characterize the effect of the so-called Catalan transformation, we shall look at its effect on some common sequences, including the Fibonacci and Jacobsthal numbers. The

[1] This chapter reproduces the content of the published article "P. Barry, A Catalan Transform and Related Transformations on Integer Sequences, J. Integer Seq., **8** (2005), Art. 05.4.5" [15].

Fibonacci numbers [237] are amongst the most studied of mathematical objects. They are easy to define, and are known to have a rich set of properties. Closely associated to the Fibonacci numbers are the Jacobsthal numbers [239]. In a sense that will be made exact below, they represent the next element after the Fibonacci numbers in a one-parameter family of linear recurrences. These and many of the integer sequences that will be encountered are to be found in The On-Line Encylopedia of Integer Sequences [205], [206]. Sequences in this database will be referred to by their *Annnnnn* number. For instance, the Catalan numbers are A000108.
The Fibonacci numbers $F(n)$ A000045 are the solutions of the recurrence

$$a_n = a_{n-1} + a_{n-2}, \ a_0 = 0, \ a_1 = 1$$

with

n	0	1	2	3	4	5	6	...
$F(n)$	0	1	1	2	3	5	8	...

The Jacobsthal numbers $J(n)$ A001045 are the solutions of the recurrence

$$a_n = a_{n-1} + 2a_{n-2}, \ a_0 = 0, \ a_1 = 1$$

with

n	0	1	2	3	4	5	6	...
$J(n)$	0	1	1	3	5	11	21	...

$$J(n) = \frac{2^n}{3} - \frac{(-1)^n}{3}.$$

When we change the initial conditions to $a_0 = 1, \ a_1 = 0$, we get a sequence which we will denote by $J_1(n)$ A078008, given by

n	0	1	2	3	4	5	6	...
$J_1(n)$	1	0	2	2	6	10	22	...

We see that

$$2^n = 2J(n) + J_1(n).$$

The Jacobsthal numbers are the case $k = 2$ for the one-parameter family of recurrences

$$a_n = a_{n-1} + ka_{n-2}, \ a_0 = 0, \ a_1 = 1$$

where the Fibonacci numbers correspond to the case $k = 1$. The Pell numbers $Pell(n)$ A000129 are the solutions of the recurrence

$$a_n = 2a_{n-1} + a_{n-2}, \ a_0 = 0, \ a_1 = 1$$

with

n	0	1	2	3	4	5	6	...
$Pell(n)$	0	1	2	5	12	29	70	...

The Pell numbers correspond to the case $k = 2$ of the one-parameter family of recurrences

$$a_n = ka_{n-1} + a_{n-2}, \ a_0 = 0, \ a_1 = 1$$

where again the Fibonacci numbers correspond to the case $k = 1$.

6.2 Transformations and the Riordan Group

In this chapter we use transformations that operate on integer sequences. An example of such a transformation that is widely used in the study of integer sequences is the so-called binomial transform [230], which associates to the sequence with general term a_n the sequence with general term b_n where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

If we consider the sequence to be the vector $(a_0, a_1, \ldots)$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector with the lower-triangle matrix **Bin** whose (i,j)-th element is equal to $\binom{i}{j}$:

$$\mathbf{Bin} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & 0 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that we index matrices starting at $(0,0)$. This transformation is invertible, with

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k.$$

We note that **Bin** corresponds to Pascal's triangle. Its row sums are 2^n, while its diagonal sums are the Fibonacci numbers $F(n+1)$. The inverse matrix $\mathbf{Bin}^{-1}$ has form

$$\mathbf{Bin}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & 0 & \ldots \\ -1 & 3 & -3 & 1 & 0 & 0 & \ldots \\ 1 & -4 & 6 & -4 & 1 & 0 & \ldots \\ -1 & 5 & -10 & 10 & -5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

If $\mathcal{A}(x)$ is the ordinary generating function of the sequence a_n, then the generating function of the transformed sequence b_n is $(1/(1-x))\mathcal{A}(x/(1-x))$.

Thus the transformation represented by the Binomial matrix **Bin** is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group, while its inverse is the element $(\frac{1}{1+x}, \frac{x}{1+x})$. It can be shown more generally [209] that the matrix with general term $\binom{n+ak}{m+bk}$ is the element $(x^m/(1-x)^{m+1}, x^{b-a}/(1-x)^b)$ of the Riordan group. This result will be used in a later section, along with characterizations of terms of the form $\binom{2n+ak}{n+bk}$. As an example, we cite the result that the

lower triangular matrix with general term $\binom{2n}{n-k} = \binom{2n}{n+k}$ is given by the Riordan array $(\frac{1}{\sqrt{1-4x}}, \frac{1-2x-\sqrt{1-4x}}{2x}) = (\frac{1}{\sqrt{1-4x}}, xc(x)^2)$ where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers.
A lower-triangular matrix that is related to $(\frac{1}{\sqrt{1-4x}}, xc(x)^2)$ is the matrix $(\frac{1}{\sqrt{1-4x}}, x^2c(x)^2)$. This is no longer a *proper* Riordan array: it is a *stretched* Riordan array, as described in [59]. The row sums of this array are then the diagonal sums of $(\frac{1}{\sqrt{1-4x}}, xc(x)^2)$, and hence have expression $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2(n-k)}{n}$.

6.3 The Catalan transform

We initially define the Catalan transformation by its action on ordinary generating functions. For this, we let $\mathcal{A}(x)$ be the generating function of a sequence. The *Catalan transform* of that sequence is defined to be the sequence whose generating function is $\mathcal{A}(xc(x))$. The Catalan transform thus corresponds to the element of the Riordan group given by $(1, xc(x))$. This has bivariate generating function $\frac{1}{1-xyc(x)}$. That this transformation is invertible is demonstrated by

Proposition 149. *The inverse of the Catalan transformation is given by*

$$\mathcal{A}(x) \to \mathcal{A}(x(1-x)).$$

Proof. We prove a more general result. Consider the Riordan matrix $(1, x(1-kx))$. Let $(g^*, \bar{f})$ denote its Riordan inverse. We then have

$$(g^*, \bar{f})(1, x(1-kx)) = (1, x).$$

Hence

$$\begin{aligned} \bar{f}(1-k\bar{f}) = x \quad &\Rightarrow \quad k\bar{f}^2 - \bar{f} + x = 0 \\ &\Rightarrow \quad \bar{f} = \frac{1-\sqrt{1-4kx}}{2k}. \end{aligned}$$

Since $g = 1$, $g^* = 1/(g \circ \bar{f}) = 1$ also, and thus

$$(1, x(1-kx))^{-1} = \left(1, \frac{1-\sqrt{1-4kx}}{2k}\right).$$

Setting $k = 1$, we obtain

$$(1, x(1-x))^{-1} = (1, xc(x))$$

Taking inverses, we obtain

$$(1, xc(x))^{-1} = (1, x(1-x))$$

as required. □

We note that in the sequel, the following identities will be useful: $c(x(1-x)) = \frac{1}{1-x}$ and $c(x) = \frac{1}{1-xc(x)}$.
In terms of the Riordan group, the Catalan transform and its inverse are thus given by the elements $(1, xc(x))$ and $(1, x(1-x))$. The lower-triangular matrix representing the Catalan transformation has the form

$$\mathbf{Cat} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 5 & 5 & 3 & 1 & 0 & \dots \\ 0 & 14 & 14 & 9 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Where convenient, we shall denote this transformation by **Cat**. We note the the row sums of this matrix have generating function given by $(1, xc(x))\frac{1}{1-x} = \frac{1}{1-xc(x)} = c(x)$. That is, the matrix **Cat** has the Catalan numbers as row sums. The bivariate generating function of this matrix may be expressed as

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}}} \tag{6.1}$$

or as

$$\cfrac{1}{1-\cfrac{xy}{1-x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}} \tag{6.2}$$

The inverse Catalan transformation $\mathbf{Cat}^{-1}$ has the form

$$\mathbf{Cat}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 3 & -4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The general term of the matrix $(1, x(1-x))$ is given by $\binom{k}{n-k}(-1)^{n-k}$. This can be shown by observing that the $k-$th column of $(1, x(1-x))$ has generating function $(x(1-x))^k$.

But $[x^n](x(1-x))^k = \binom{k}{n-k}(-1)^{n-k}$. We note also that the bivariate generating function of $(1, x(1-x))$ is $\frac{1}{1-xy(1-x)}$.
We now characterize the general term of the matrix for the Catalan transform.

Proposition 150. *The general term $T(n,k)$ of the Riordan matrix $(1, c(x))$ is given by*

$$T(n,k) = \sum_{j=0}^{k} \binom{k}{j}\binom{j/2}{n}(-1)^{n+j}2^{2n-k}.$$

Proof. We seek $[x^n](xc(x))^k$. To this end, we develop the term $(xc(x))^k$ as follows:

$$\begin{aligned}
x^k c(x)^k &= x^k\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k \\
&= \frac{1}{2^k}(1-\sqrt{1-4x})^k \\
&= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(-\sqrt{1-4x})^j \\
&= \frac{1}{2^k}\sum_{j}\binom{k}{j}(-1)^j(1-4x)^{j/2} \\
&= \frac{1}{2^k}\sum_{j}\binom{k}{j}(-1)^j\sum_{i}\binom{j/2}{i}(-4x)^i \\
&= \frac{1}{2^k}\sum_{j}\binom{k}{j}(-1)^j\sum_{i}\binom{j/2}{i}(-4)^i x^i \\
&= \sum_{j}\binom{k}{j}\sum_{i}\binom{j/2}{i}(-1)^{i+j}2^{2i-k}x^i
\end{aligned}$$

Thus $[x^n](xc(x))^k = \sum_{j=0}^{k}\binom{k}{j}\binom{j/2}{n}(-1)^{n+j}2^{2n-k}$.

□

The above proposition shows that the Catalan transform of a sequence a_n has general term b_n given by

$$b_n = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{\frac{j}{2}}{n}(-1)^{n+j}2^{2n-k}a_k.$$

The following proposition gives alternative versions for this expression.

Proposition 151. *Given a sequence a_n, its Catalan transform b_n is given by*

$$\begin{aligned}
b_n &= \sum_{k=0}^{n}\frac{k}{2n-k}\binom{2n-k}{n-k}a_k \\
&= \sum_{k=0}^{n}\frac{k}{n}\binom{2n-k-1}{n-k}a_k
\end{aligned}$$

or

$$b_n = \sum_{j=0}^{n} \sum_{k=0}^{n} \frac{2k+1}{n+k+1} (-1)^{k-j} \binom{2n}{n-k} \binom{k}{j} a_j.$$

The inverse transformation is given by

$$\begin{aligned} a_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k b_{n-k} \\ &= \sum_{k=0}^{n} \binom{k}{n-k} (-1)^{n-k} b_k. \end{aligned}$$

Proof. Using [209] (3.164) we have

$$\begin{aligned} T(n,k) &= 2^{2n-k}(-1)^n \sum_{j=0}^{k} \binom{k}{j} \binom{\frac{j}{2}}{n} (-1)^j \\ &= 2^{2n-k}(-1)^n \left\{ (-1)^n 2^{k-2n} \left(\binom{2n-k-1}{n-1} - \binom{2n-k-1}{n} \right) \right\} \\ &= \binom{2n-k-1}{n-1} - \binom{2n-k-1}{n} \\ &= \frac{k}{n} \binom{2n-k-1}{n-1} = \frac{k}{n} \binom{2n-k-1}{n-k}. \end{aligned}$$

But

$$\begin{aligned} \frac{k}{n} \binom{2n-k-1}{n-k} &= \frac{k}{n} \frac{2n-k-(n-k)}{2n-k} \binom{2n-k}{n-k} \\ &= \frac{k}{n} \frac{2n-k-n+k}{2n-k} \binom{2n-k}{n-k} \\ &= \frac{k}{2n-k} \binom{2n-k}{n-k} \end{aligned}$$

This proves the first two assertions of the proposition. Note that we could have used Lagrange inversion to prove these results, as in Example 13.

The last assertion follows from the expression for the general term of the matrix $(1, x(1-x))$ obtained above. The equivalence of this and the accompanying expression is easily obtained. The remaining assertion will be a consequence of results in Section 5.

$\square$

Using the last proposition, and [205], it is possible to draw up the following representative list of Catalan pairs, that is, sequences and their Catalan transforms.

Table Catalan pairs

a_n	b_n	a_n	b_n
0^n	0^n	A000007	A000007
1^n	C_n	1^n	A000108
2^n	$\binom{2n}{n}$	A000079	A000984
2^n-1	$\binom{2n}{n-1}$	A000225	A001791
$2^n-\sum_{j=0}^{k-1}\binom{n}{j}$	$\binom{2n}{n-k}$	various	various
n	$3nC_n/(n+2)$	A001477	A000245
$n+1$	C_{n+1}	A000027	A000108$(n+1)$
$\binom{n}{2}$	$5\binom{2n}{n-2}/(n+3)$	A000217$(n-1)$	A000344
$\binom{n+1}{2}$	$4\binom{2n+1}{n-1}/(n+3)$	A000217	A002057
$\binom{n+2}{2}$	$3\binom{2n+2}{n}/(n+3)$	A000217$(n+1)$	A000245
$0^n-(-1)^n$	Fine's sequence	-	A000957
$(1+(-1)^n)/2$	Fine's sequence	A000035$(n+1)$	A000957$(n+1)$
$2-0^n$	$(2-0^n)C_n$	A040000	A068875
$F(n)$	$[x^n]\frac{xc(x)}{x+\sqrt{1-4x}}$	A000045	-
$F(n+1)$	$[x^n]\frac{1}{x+\sqrt{1-4x}}$	A000045$(n+1)$	A081696
$J(n)$	$\sum_{j=0}^{\lfloor(n-1)/2\rfloor}\binom{2n-2j-2}{n-1}$	A001045	A014300
$J(n+1)$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{2n-2j-1}{n-1}+0^n$	A001045$(n+1)$	A026641$(n)+0^n$
$J_1(n)$	$\sum_{k=0}^{n}\binom{n+k-1}{k}(-1)^{n-k}$	A078008	A072547
$2\sin(\frac{\pi n}{3}+\frac{\pi}{3})/\sqrt{3}$	1^n	A010892	1^n
$(-1)^n\mathrm{F}(n+1)$	$(-1)^n$	$(-1)^n$A000045$(n+1)$	$(-1)^n$
$2^{\frac{n}{2}}(\cos(\frac{\pi n}{4})+\sin(\frac{\pi n}{4}))$	2^n	A009545	A000079

We note that the result above concerning Fine's sequence A000957 is implicit in the work [71]. We deduce immediately that the generating function for Fine's sequence can be written as

$$\frac{x}{1-(xc(x))^2}=\frac{2x}{1+2x+\sqrt{1-4x}}=\frac{x}{1+x-xc(x)}.$$

See also [173].

6.4 Transforms of a Jacobsthal family

From the above, we see that the Jacobsthal numbers $J(n)$ transform to give the sequence with general term

$$\sum_{j=0}^{\lfloor(n-1)/2\rfloor}\binom{2n-2j-2}{n-1}$$

and generating function

$$\frac{xc(x)}{(1+xc(x))(1-2xc(x))}.$$

This prompts us to characterize the family of sequences with general term

$$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n-2j-2}{n-1} k^j$$

where the transform of the Jacobsthal numbers corresponds to the case $k = 1$. Re-casting the ordinary generating function of the Jacobsthal numbers as

$$\frac{x}{(1+x)(1-2x)} = \frac{x(1-x)}{(1-x^2)(1-2x)}$$

we see that the Jacobsthal numbers are the case $k = 1$ of the one-parameter family of sequences with generating functions

$$\frac{x(1-x)}{(1-kx^2)(1-2x)}.$$

For instance, the sequence for $k = 0$ has g.f. $\frac{x(1-x)}{1-2x}$ which is $0, 1, 1, 2, 4, 8, 16, \ldots$. For $k = 2$, we obtain $0, 1, 1, 4, 6, 16, \ldots$, or A007179. The general term for these sequences is given by

$$\frac{(\sqrt{k})^{n-1}(1-\sqrt{k})}{2(2-\sqrt{k})} + \frac{(-\sqrt{k})^{n-1}(1+\sqrt{k})}{2(2+\sqrt{k})} + \frac{2^n}{4-k}$$

for $k \neq 4$. They are solutions of the family of recurrences

$$a_n = 2a_{n-1} + ka_{n-2} - 2ka_{n-3}$$

where $a_0 = 0$, $a_1 = 1$ and $a_2 = 1$.

Proposition 152. *The Catalan transform of the generalized Jacobsthal sequence with ordinary generating function*

$$\frac{x(1-x)}{(1-kx^2)(1-2x)}$$

has ordinary generating function given by

$$\frac{x}{\sqrt{1-4x}(1-k(xc(x))^2)}$$

and general term

$$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n-2j-2}{n-1} k^j.$$

Proof. By definition, the Catalan transform of $\frac{x(1-x)}{(1-kx^2)(1-2x)}$ is

$$\frac{xc(x)(1-xc(x))}{(1-kx^2c(x)^2)(1-2xc(x))}.$$

But $c(x)(1-xc(x))=1$ and $1-2xc(x)=\sqrt{1-4x}$. Hence we obtain the first assertion. We now recognize that

$$\frac{1}{\sqrt{1-4x}(1-k(xc(x))^2)}=\left(\frac{1}{\sqrt{1-4x}}, x^2c(x)^2\right)\frac{1}{1-kx}.$$

But this is $\sum_{j=0}^{\lfloor n/2\rfloor}\binom{2n-2j}{n}k^j$. The second assertion follows from this. □

For example, the transform of the sequence $0,1,1,2,4,8,\ldots$ can be recognized as $\binom{2n-2}{n-1}=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-2}{n-1}0^j$.

As noted in A014300, the Catalan transform of the Jacobsthal numbers corresponds to the convolution of the central binomial numbers (with generating function $\frac{1}{\sqrt{1-4x}}$) and Fine's sequence A000957 (with generating function $\frac{x}{1-(xc(x))^2}$). The above proposition shows that the Catalan transform of the generalized Jacobsthal numbers corresponds to a convolution of the central binomial numbers and the "generalized" Fine numbers with generating function $\frac{x}{1-k(xc(x))^2}$.

Using the inverse Catalan transform, we can express the general term of this Jacobsthal family as

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}(-1)^i\sum_{j=0}^{\lfloor (n-i-1)/2\rfloor}\binom{2n-2i-2j-2}{n-i-1}k^j.$$

This provides us with a closed form for the case $k=4$ in particular.
We now wish to find an expression for the transform of $J_1(n)$. To this end, we note that

$$J(n+1)\to\sum_{j=0}^{\lfloor n/2\rfloor}\binom{2n-2j-1}{n-1}+0^n=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-1}{n-1}+\frac{1+(-1)^n}{2}.$$

Then $J_1(n)=J(n+1)-J(n)$ is transformed to

$$\sum_{j=0}^{\lfloor\frac{n-1)}{2}\rfloor}\binom{2n-2j-1}{n-1}+\frac{1+(-1)^n}{2}-\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{2n-2j-2}{n-1}$$

or

$$\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-2}{n-2}+\frac{1+(-1)^n}{2}.$$

The first term of the last expression deserves comment. Working with generating functions, it is easy to show that under the Catalan transform, we have

$$(-1)^nF(n+1)/2+\cos\left(\frac{\pi n}{3}\right)/2+\sqrt{3}\sin\left(\frac{\pi n}{3}\right)/6\to\frac{1+(-1)^n}{2}.$$

Hence $\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-2}{n-2}$ is the Catalan transform of

$$J_1(n)-(-1)^nF(n+1)/2-\cos\left(\frac{\pi n}{3}\right)/2-\sqrt{3}\sin\left(\frac{\pi n}{3}\right)/6.$$

6.5 The Generalized Ballot Transform

In this section, we introduce and study a transformation that we will link to the generalized ballot numbers studied in [165]. For this, we define a new transformation **Bal** as the composition of the Catalan transform and the Binomial transform:

$$\mathbf{Bal} = \mathbf{Cat} \circ \mathbf{Bin}.$$

The Riordan matrix formulation of this transformation is thus given by

$$\begin{aligned}\mathbf{Bal} &= \mathbf{Cat} \circ \mathbf{Bin}\\ &= (1, c(x))\left(\frac{1}{1-x}, \frac{x}{1-x}\right)\\ &= \left(\frac{1}{1-xc(x)}, \frac{xc(x)}{1-xc(x)}\right)\\ &= (c(x), c(x)-1) = (c(x), xc(x)^2).\end{aligned}$$

This has generating function

$$\cfrac{1}{1-\cfrac{x+xy}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}}.$$

The row sums of this array are $\binom{2n}{n}$ since

$$\frac{c(x)}{1-xc(x)^2} = \frac{1}{\sqrt{1-4x}}.$$

In similar fashion, we can find the Riordan description of the inverse of this transformation by

$$\begin{aligned}\mathbf{Bal}^{-1} &= \mathbf{Bin}^{-1} \circ \mathbf{Cat}^{-1}\\ &= \left(\frac{1}{1-x}, \frac{x}{1-x}\right)^{-1}(1, c(x))^{-1}\\ &= \left(\frac{1}{1+x}, \frac{x}{1+x}\right)(1, x(1-x))\\ &= \left(\frac{1}{1+x}.1, \frac{x}{1+x}\left(1-\frac{x}{1+x}\right)\right)\\ &= \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right).\end{aligned}$$

The general term of $\mathbf{Bal}^{-1}$ is easily derived from the last expression: it is $[x^n](x^k(1+x)^{-2k+1}) = \binom{n+k}{n-k}(-1)^{n-k} = \binom{n+k}{2k}(-1)^{n-k}$. Alternatively we can find the general term in the

matrix product of $\mathbf{Bin}^{-1}$, or $\binom{n}{k}(-1)^{n-k}$, with $\mathbf{Cat}^{-1}$, or $\binom{k}{n-k}(-1)^{n-k}$ to get the equivalent expression $\sum_{j=0}^{k+n}\binom{n}{j}\binom{k}{j-k}(-1)^{n-k}$.
We now examine the general term of the transformation $\mathbf{Bal} = (c(x), c(x)-1) = (c(x), xc(x)^2)$. An initial result is given by

Proposition 153. *The general term $T(n,k)$ of the Riordan matrix $(c(x), c(x)-1)$ is given by*

$$T(n,k) = \sum_{j=0}^{k}\sum_{i=0}^{j+1}\binom{k}{j}\binom{j+1}{i}\binom{i/2}{n+j+1}(-1)^{k+n+i+1}2^{2n+j+1}$$

and

$$T(n,k) = 2.4^n\sum_{j=0}^{2k+1}\binom{2k+1}{j}\binom{j/2}{n+k+1}(-1)^{n+k+j+1}.$$

Proof. The first assertion follows by observing that the $k-$th column of $(c(x), c(x)-1)$ has generating function $c(x)(c(x)-1)^k$. We are thus looking for $[x^n]c(x)(c(x)-1)^k$. Expanding and substituting for c(x) yields the result. The second assertion follows by taking $[x^n]c(x)(xc(x)^2)^k$. □

We now show that this transformation has in fact a much easier formulation, corresponding to the generalized Ballot numbers of [165]. We recall that the generalized Ballot numbers or generalized Catalan numbers [165] are defined by

$$B(n,k) = \binom{2n}{n+k}\frac{2k+1}{n+k+1}.$$

$B(n,k)$ can be written as

$$B(n,k) = \binom{2n}{n+k}\frac{2k+1}{n+k+1} = \binom{2n}{n-k} - \binom{2n}{n-k-1}$$

where the matrix with general term $\binom{2n}{n-k}$ is the element

$$\left(\frac{1}{\sqrt{1-4x}}, \frac{1-2x-\sqrt{1-4x}}{2x}\right) = \left(\frac{1}{\sqrt{1-4x}}, x\frac{c(x)-1}{x}\right) = \left(\frac{1}{\sqrt{1-4x}}, xc(x)^2\right)$$

of the Riordan group [11]. The inverse of this matrix is $(\frac{1-x}{1+x}, \frac{x}{(1+x)^2})$ with general term $(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{2k}$.

Proposition 154. *The general term of the Riordan matrix $(c(x), c(x)-1)$ is given by*

$$B(n,k) = \binom{2n}{n+k}\frac{2k+1}{n+k+1}.$$

Proof. We provide two proofs - one indirect, the other direct. The first, indirect proof is instructive as it uses properties of Riordan arrays.
We have $B(n,k) = \binom{2n}{n-k} - \binom{2n}{n-k-1}$. Hence the generating function of the $k-$th column of the matrix with general term $B(n,k)$ is given by

$$\frac{1}{\sqrt{1-4x}}(c(x)-1)^k - \frac{1}{\sqrt{1-4x}}(c(x)-1)^{k+1}$$

Then

$$\begin{aligned}\frac{1}{\sqrt{1-4x}}(c(x)-1)^k - \frac{1}{\sqrt{1-4x}}(c(x)-1)^{k+1} &= (c(x)-1)^k\left(\frac{1}{\sqrt{1-4x}} - \frac{c(x)-1}{\sqrt{1-4x}}\right) \\ &= (c(x)-1)^k\left(\frac{1}{\sqrt{1-4x}}(1-(c(x)-1))\right) \\ &= (c(x)-1)^k\left(\frac{-c(x)+2}{\sqrt{1-4x}}\right) \\ &= (c(x)-1)^k c(x)\end{aligned}$$

But this is the generating function of the $k-$th column of $(c(x), c(x)-1)$.

The second, direct proof follows from the last proposition. We first seek to express the term $\sum_{j=0}^{2k+1}\binom{2k+1}{j}\binom{j/2}{n+k+1}(-1)^j$ in simpler terms. Using [209] (3.164), this is equivalent to

$$\left\{(-1)^{n+k+1}2^{2k+1-2(n+k+1)}\left(\binom{2n+2k+2-2k-1-1}{n+k} - \binom{2n+2k+2-2k-1-1}{n+k+1}\right)\right\}$$

or $(-1)^{n+k+1}\frac{1}{2.4^n}\left\{\binom{2n}{n+k} - \binom{2n}{n+k+1}\right\}$. Hence

$$\begin{aligned}B(n,k) &= 2.4^n(-1)^{n+k+1}\sum_{j=0}^{2k+1}\binom{2k+1}{j}\binom{j/2}{n+k+1}(-1)^j \\ &= \left\{\binom{2n}{n+k} - \binom{2n}{n+k+1}\right\} \\ &= \left\{\binom{2n}{n-k} - \binom{2n}{n-k-1}\right\} \\ &= \binom{2n}{n+k}\frac{2k+1}{n+k+1}.\end{aligned}$$

□

The numbers $B(n,k)$ have many combinatorial uses. For instance,

$$B(n,k) = D(n+k+1, 2k+1)$$

where $D(n,k)$ is the number of Dyck paths of semi-length n having height of the first peak equal to k [173]. $B(n,k)$ also counts the number of paths from $(0,-2k)$ to $(n-k, n-k)$

with permissible steps $(0,1)$ and $(1,0)$ that don't cross the diagonal $y=x$ [165].

We recall that the classical ballot numbers are given by $\frac{k}{2n+k}\binom{2n+k}{n}=\frac{k}{2n+k}\binom{2n+k}{n+k}$ [117].

We now define the *generalized Ballot transform* to be the transformation corresponding to the Riordan array $\mathbf{Bal}=\mathbf{Cat}\circ\mathbf{Bin}=(c(x),c(x)-1)=(c(x),xc(x)^2)$. By the above, the generalized Ballot transform of the sequence a_n is the sequence b_n where

$$b_n=\sum_{k=0}^{n}\binom{2n}{n+k}\frac{2k+1}{n+k+1}a_k.$$

In terms of generating functions, the generalized Ballot transform maps the sequence with ordinary generating function $g(x)$ to the sequence with generating function $\mathrm{c}(x)g(\mathrm{c}(x)-1)=\mathrm{c}(x)\mathrm{g}(x\mathrm{c}(x)^2)$ where $\mathrm{c}(x)$ is the generating function of the Catalan numbers. We then have

Proposition 155. *Given a sequence a_n, its inverse generalized Ballot transform is given by*

$$b_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{2k}a_k.$$

If a_n has generating function $g(x)$ then b_n has generating function $\frac{1}{1+x}g\left(\frac{x}{(1+x)^2}\right)$.

Proof. We have seen that $\mathbf{Bal}^{-1}=\mathbf{Bin}^{-1}\circ\mathbf{Cat}^{-1}=(\frac{1}{1+x},\frac{x}{(1+x)^2})$. The second statement follows from this. We have also seen that the general term of $\mathbf{Bal}^{-1}$ is $(-1)^{n-k}\binom{n+k}{2k}$. Hence the transform of the sequence a_n is as asserted. $\square$

We can now characterize the Catalan transformation as

$$\mathbf{Cat}=\mathbf{Bal}\circ\mathbf{Bin}^{-1}.$$

Since the general term of $\mathbf{Bin}^{-1}$ is $(-1)^{n-k}\binom{n}{k}$ we immediately obtain the expression

$\sum_{j=0}^{n}\sum_{k=0}^{n}\frac{2k+1}{n+k+1}(-1)^{k-j}\binom{2n}{n-k}\binom{k}{j}$ for the general term of $\mathbf{Cat}$.

The following table identifies some Ballot transform pairings [205].

Table. Generalized Ballot transform pairs

a_n	b_n	a_n	b_n
$(-1)^n$	0^n	$(-1)^n$	A000007
0^n	C_n	A000007	A000108
1^n	$\binom{2n}{n}$	1^n	A000984
2^n	$[x^n]\frac{1}{1-3xc(x)}$	A000079	A007854
n	$[x^n]\frac{xc(x)}{1-4c(x)}$	A001477	A000346$(n-1)+0^n/2$
$n+1$	$\sum_{k=0}^{n}\binom{2n}{k}$	A000027	A032443
$2n+1$	4^n	A005408	A000302
$(1+(-1)^n)/2$	$\binom{2n+1}{n}$	A059841	A088218
$2-0^n$	$\binom{2n+1}{n+1}$	A040000	A001700
$(2^n+0^n)/2$	-	A011782	A090317
$\cos(\frac{2\pi n}{3})+\sin(\frac{2\pi n}{3})/\sqrt{3}$	1^n	A057078	1^n
$\cos(\frac{\pi n}{2})+\sin(\frac{\pi n}{2})$	2^n	-	A000079
$\cos(\frac{\pi n}{3})+\sqrt{3}\sin(\frac{\pi n}{3})$	3^n	A057079	A000244
$F(n)$	–	A000045	A026674
$F(n+1)$	–	A000045$(n+1)$	A026726

In terms of the Riordan group, the above implies that the generalized Ballot transform and its inverse are given by the elements $(c(x), x(c(x)-1)/x)$ and $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$. The lower-triangular matrix representing the Ballot transformation thus has the form

$$\mathbf{Bal} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 3 & 1 & 0 & 0 & 0 & \dots \\ 5 & 9 & 5 & 1 & 0 & 0 & \dots \\ 14 & 28 & 20 & 7 & 1 & 0 & \dots \\ 42 & 90 & 75 & 35 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while the inverse Ballot transformation is represented by the matrix

$$\mathbf{Bal}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -3 & 1 & 0 & 0 & 0 & \dots \\ -1 & 6 & -5 & 1 & 0 & 0 & \dots \\ 1 & -10 & 15 & -7 & 1 & 0 & \dots \\ -1 & 15 & -35 & 28 & -9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The first matrix is A039599, while the absolute value of the second matrix is A085478.

6.6 The Signed Generalized Ballot transform

For completeness, we consider a transformation that may be described as the signed generalized Ballot transform. As a member of the Riordan group, this is the element $(c(-x), 1-$

$c(-x)) = (c(-x), xc(-x)^2)$. For a given sequence a_n, it yields the sequence with general term

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{n+k} \frac{2k+1}{n+k+1} a_k.$$

Looking at generating functions, we get the mapping

$$\mathcal{A}(x) \to c(-x)\mathcal{A}(1 - c(-x)) = c(-x)\mathcal{A}(xc(-x)^2).$$

The inverse of this map is given by

$$b_n = \sum_{k=0} \binom{n+k}{2k} a_k$$

or, in terms of generating functions

$$\mathcal{A}(x) \to \frac{1}{1-x}\mathcal{A}\left(\frac{x}{(1-x)^2}\right).$$

Example mappings under this transform are $0^n \to (-1)^n C_n$, $1^n \to 0^n$, $F(2n+1) \to 1^n$. We note that if the sequence a_n has generating function of the form

$$\cfrac{1}{1 - \cfrac{\alpha_1 x}{1 - \cfrac{\alpha_2 x}{1 - \cdots}}}$$

then the matrix with general term

$$\binom{n+k}{2k} a_k$$

has generating function

$$\cfrac{1}{1 - x - \cfrac{\alpha_1 xy}{1 - x - \cfrac{\alpha_2 xy}{1 - x - \cdots}}}.$$

Hence the inverse signed generalized Ballot transform of the sequence a_n will in this case have the generating function

$$\cfrac{1}{1 - x - \cfrac{\alpha_1 x}{1 - x - \cfrac{\alpha_2 x}{1 - x - \cdots}}}.$$

We can characterize the effect of this inverse on the power sequences $n \to k^n$ as follows: the image of k^n under the inverse signed generalized Ballot transform is the solution to the recurrence

$$a_n = (k+2)a_{n-1} - a_{n-2}$$

with $a_0 = 1$, $a_1 = F(2k+1)$.
The matrices that represent this inverse pair of transformations are, respectively, the alternating sign versions of A039599 and A085478.
The latter matrix has a growing literature in which it is known as the DFF triangle [87], [88], [216]. As an element of the Riordan group, it is given by $\left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right)$.
It has a "companion" matrix with general element

$$b_{n,k} = \binom{n+k+1}{2k+1} = \frac{n+k+1}{2k+1}\binom{n+k}{2k}$$

called the DFFz triangle. This is the element

$$\left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right)$$

of the Riordan group with inverse

$$\left(\frac{1-c(-x)}{x}, 1-c(-x)\right) = (c(-x)^2, xc(-x)^2).$$

Another number triangle that is related to the above [216] has general term

$$a_{n,k} = \frac{2n}{n+k}\binom{n+k}{2k}.$$

Taking $a_{0,0} = 1$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 9 & 6 & 1 & 0 & 0 & \ldots \\ 2 & 16 & 20 & 8 & 1 & 0 & \ldots \\ 2 & 25 & 50 & 35 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the element $\left(\frac{1+x}{1-x}, \frac{x}{(1-x)^2}\right)$ of the Riordan group. Its inverse is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 6 & -4 & 1 & 0 & 0 & 0 & \ldots \\ -20 & 15 & -6 & 1 & 0 & 0 & \ldots \\ 70 & -56 & 28 & -8 & 1 & 0 & \ldots \\ -252 & 210 & -120 & 45 & -10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general element $(-1)^{n-k}\binom{2n}{n-k}$ which is the element

$$\left(\frac{1}{\sqrt{1+4x}}, \frac{1+2x-\sqrt{1+4x}}{2x}\right) = \left(\frac{1}{\sqrt{1+4x}}, 1-c(-x)\right)$$

of the Riordan group. We note that $\frac{1+2x-\sqrt{1+4x}}{2x^2}$ is the generating function of $(-1)^n C_{n+1}$. Applying $\mathbf{Bin}^2$ to this matrix yields the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 1 & 0 & 0 & \ldots \\ 6 & 0 & 4 & 0 & 1 & 0 & \ldots \\ 0 & 10 & 0 & 5 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the element $\left(\frac{1}{\sqrt{1-4x^2}}, xc(x^2)\right)$ of the Riordan group.

6.7 An Associated Transformation

We briefly examine a transformation associated to the Ballot transformation. Unlike other transformations in this study, this is not invertible. However, it transforms some "core" sequences to other "core" sequences, and hence deserves study. An example of this transformation is given by

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

where M_n is the nth Motzkin number A001006. In general, if a_n is the general term of a sequence with generating function $\mathcal{A}(x)$ then we define its transform to be

$$b_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} a_k$$

which has generating function

$$\frac{1}{1-x}\mathcal{A}\left(\frac{x^2}{(1-x)^2}\right).$$

As this sequence represents the diagonal sums of the array with general term

$$\binom{n+k}{2k} a_k,$$

it will have generating function

$$\cfrac{1}{1-x-\cfrac{\alpha_1 x^2}{1-x-\cfrac{\alpha_2 x^2}{1-x-\cfrac{\alpha_3 x^2}{1-x-\cdots}}}}$$

in the event that a_n has generating function

$$\cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cdots}}}.$$

The opening assertion concerning the Motzkin numbers follows from the fact that

$$\frac{1}{1-x}c\left(\frac{x^2}{(1-x)^2}\right)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$$

which is the generating function of the Motzkin numbers. We also deduce that the generating function of the Motzkin numbers may be expressed as

$$\cfrac{1}{1-x-\cfrac{x^2}{1-x-\cfrac{x^2}{1-x-\cdots}}}.$$

The effect of this transform on the power sequences $n \to k^n$ is easy to describe. We have

$$\frac{1}{1-kx} \to \frac{1-x}{1-2x-(k-1)x^2}.$$

In other words, the sequences $n \to k^n$ are mapped to the solutions of the one parameter family of recurrences

$$a_n = 2a_{n-1} + (k-1)a_{n-2}$$

satisfying $a_0 = 1$, $a_1 = 1$.
For instance,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^k = ((1+\sqrt{2})^n + (1-\sqrt{2})^n)/2$$

is the general term of the sequence A001333 which begins $1, 1, 3, 7, 17, \ldots$. Related to this is the following formula for the Pell numbers A000129

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k+1} 2^k = Pell(n).$$

Under this mapping, the central binomial numbers $\binom{2n}{n}$ are mapped to the central trinomial numbers A002426 since

$$\frac{1}{\sqrt{1-4x}} \to \frac{1}{1-x}\frac{1}{\sqrt{1-4x^2/(1-x)^2}} = \frac{1}{\sqrt{1-2x-3x^2}}.$$

This is an interesting result, as the central trinomial numbers are also the inverse binomial transform of the central binomial numbers :

$$\frac{1}{\sqrt{1-4x}} \to \frac{1}{1+x}\frac{1}{\sqrt{1-4x/(1+x)}} = \frac{1}{\sqrt{1-2x-3x^2}}.$$

We also deduce the following form of the generating function for the central trinomial numbers

$$\cfrac{1}{1-x-\cfrac{2x^2}{1-x-\cfrac{x^2}{1-x-\cdots}}}.$$

This transformation can be represented by the "generalized" Riordan array $(\frac{1}{1-x}, \frac{x^2}{(1-x)^2})$. As such, it possesses two interesting factorizations. Firstly, we have

$$\left(\frac{1}{1-x}, \frac{x^2}{(1-x)^2}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)(1, x^2) = \mathbf{Bin} \circ (1, x^2).$$

Thus the effect of this transform is to "aerate" a sequence with interpolated zeros and then follow this with a binomial transform. This is obvious from the following identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} a_k = \sum_{k=0}^{n} \binom{n}{k} \frac{1+(-1)^k}{2} a_{k/2}$$

where we use the usual convention that $a_{k/2}$ is to be interpreted as 0 when k is odd, that is when $k/2$ is not an integer. Secondly, we have

$$\left(\frac{1}{1-x}, \frac{x^2}{(1-x)^2}\right) = \left(\frac{1-x}{1-2x+2x^2}, \frac{x^2}{1-2x+2x^2}\right)\left(\frac{1}{1-x}, \frac{x}{1-x}\right).$$

As pointed out in [59], this transformation possesses a left inverse. Using the methods of [59] or otherwise, it is easy to see that the stretched Riordan array $(1, x^2)$ has left inverse $(1, \sqrt{x})$. Hence the first factorization yields

$$\begin{aligned}
\left(\frac{1}{1-x}, \frac{x^2}{(1-x)^2}\right)^{\sim 1} &= (\mathbf{Bin} \circ (1, x^2))^{\sim 1} \\
&= (1, \sqrt{x}) \circ \mathbf{Bin}^{-1} \\
&= (1, \sqrt{x}) \left(\frac{1}{1+x}, \frac{x}{1+x}\right) \\
&= = \left(\frac{1}{1+\sqrt{x}}, \frac{\sqrt{x}}{1+\sqrt{x}}\right)
\end{aligned}$$

where we have used $(.)^{\sim 1}$ to denote left-inverse.

It is instructive to represent these transformations by their general terms. We look at $(1, x^2)$ first. We have

$$[x^n](x^2)^k = [x^n]x^{2k} = [x^n]\sum_{j=0}^{\infty}\binom{0}{j}x^{2k+j} = \binom{0}{n-2k}$$

Hence

$$b_n = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}a_k = \sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}\binom{0}{j-2k}a_j.$$

We now wish to express b_n in terms of a_n. We have

$$[x^n](\sqrt{x})^k = [t^{2n}]t^k = [t^{2n}]\sum_{j=0}^{\infty}\binom{0}{j}t^{k+j} = \binom{0}{2n-k}.$$

Hence

$$a_n = \sum_{k=0}^{2n}\sum_{j=0}^{k}\binom{0}{2n-k}\binom{k}{j}(-1)^{k-j}b_j.$$

We can use the methods of [59] to further elucidate the relationship between $(1, x^2)$ and $(1, \sqrt{x})$. Letting $b(x)$ be the generating function of the image of the sequence a_n under $(1, x^2)$, we see that $b(x) = a(x^2)$ where $a(x) = \sum_{k=0}^{\infty} a_k x^k$. We wish to find the general term a_n in terms of the b_n. We have $a(x) = [b(t)|x = t^2]$ and so

$$\begin{aligned}
a_n &= [x^n]a(x) = [x^n](b(t)|t = \sqrt{x}) \\
&= [w^{2n}](b(t)|t = w) \\
&= \frac{1}{2n}[t^{2n-1}](b'(t))^{2n} = \frac{1}{2n}2n.b_{2n} \\
&= b_{2n}.
\end{aligned}$$

Table 3 displays a list of sequences and their transforms under this transformation. Note that by the above, we can recover the original sequence a_n by taking every second element of the inverse binomial transform of the transformed sequence b_n.

Table 3. Transform pairs

a_n	b_n	a_n	b_n
0^n	1^n	A000007	1^n
1^n	$(2^n + 0^n)/2$	1^n	A011782
2^n	$\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n)}{2}$	A000079	A001333
n	$n2^{n-3} - \frac{\binom{1}{n}-\binom{0}{n}}{4}$	A001477	-
$n+1$	$[x^n]\frac{(1-x)^3}{(1-2x)^2}$	A000027	A045891
$2n+1$	$2^n(n+2)/2,\ n > 1$	A005408	A087447
$(1+(-1)^n)/2$	$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n}{4k}$	A059841	A038503
$\binom{2n}{n}$	Central trinomial	A000984	A002426
C_n	Motzkin	A000108	A001006

6.8 Combining transformations

We finish this chapter by briefly looking at the effect of combining transformations. For this, we will take the Fibonacci numbers as an example. We look at two combinations: the Catalan transform followed by the binomial transform, and the Catalan transform followed by the inverse binomial transform. For the former, we have

$$\mathbf{Bin} \circ \mathbf{Cat} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)(1, xc(x)) = \left(\frac{1}{1-x}, \frac{x}{1-x}c\left(\frac{x}{1-x}\right)\right).$$

while for the latter we have

$$\mathbf{Bin}^{-1} \circ \mathbf{Cat} = \left(\frac{1}{1+x}, \frac{x}{1+x}\right)(1, xc(x)) = \left(\frac{1}{1+x}, \frac{x}{1+x}c\left(\frac{x}{1+x}\right)\right).$$

Applying the first combined transformation to the Fibonacci numbers yields the sequence $0, 1, 4, 15, 59, 243, \ldots$ with generating function

$$\frac{(\sqrt{5x-1} - \sqrt{x-1})}{2((x-1)\sqrt{5x-1} - x\sqrt{x-1}))}$$

or

$$\frac{\sqrt{1-6x+5x^2} - (1-5x+4x^2)}{2(1-x)(1-6x+4x^2)}.$$

Applying the second combined transformation (Catalan transform followed by the inverse binomial transform) to the Fibonacci numbers we obtain the sequence $0, 1, 0, 3, 3, 13, 26, \ldots$ with generating function

$$\frac{(1+2x)\sqrt{1-2x-3x^2} - (1-x-2x^2)}{2(1+x)(1-2x-4x^2)}.$$

It is instructive to reverse these transformations. Denoting the first by $\mathbf{Bin} \circ \mathbf{Cat}$ we wish to look at $(\mathbf{Bin} \circ \mathbf{Cat})^{-1} = \mathbf{Cat}^{-1} \circ \mathbf{Bin}^{-1}$. As elements of the Riordan group, we obtain

$$(1, x(1-x))\left(\frac{1}{1+x}, \frac{x}{1+x}\right) = \left(\frac{1}{1+x-x^2}, \frac{x(1-x)}{1+x-x^2}\right).$$

Applying the inverse transformation to the family of functions k^n with generating functions $\frac{1}{1-kx}$, for instance, we obtain

$$\frac{1}{1+x-x^2}\frac{1}{1-\frac{kx(1-x)}{1+x-x^2}} = \frac{1}{1-(k-1)x+(k-1)x^2}.$$

In other words, the transformation $(\mathbf{Bin} \circ \mathbf{Cat})^{-1}$ takes a power k^n and maps it to the solution of the recurrence

$$a_n = (k-1)a_{n-1} - (k-1)a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = k-1$. In particular, it takes the constant sequence 1^n to 0^n.

The Jacobsthal numbers $J(n)$, for instance, are transformed into the sequence with ordinary generating function $\frac{x(1-x)}{1+x-3x^2+4x^3-2x^4}$ with general term

$$\begin{aligned} b_n &= \sum_{k=0}^{n} \binom{k}{n-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{n-j} J(j) \\ &= 2\sqrt{3}\sin(\pi n/3+\pi/3)/9 - \frac{\sqrt{3}}{18}\left\{(\sqrt{3}-1)^{n+1} - (-1)^n(\sqrt{3}+1)^{n+1}\right\}. \end{aligned}$$

We now look at $(\mathbf{Bin}^{-1} \circ \mathbf{Cat})^{-1} = \mathbf{Cat}^{-1} \circ \mathbf{Bin}$. As elements of the Riordan group, we obtain

$$(1, x(1-x))(\frac{1}{1-x}, \frac{x}{1-x}) = \left(\frac{1}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2}\right).$$

Applying this inverse transformation to the family of functions k^n with generating functions $\frac{1}{1-kx}$, for instance, we obtain

$$\frac{1}{1-x+x^2}\frac{1}{1-\frac{kx(1-x)}{1-x+x^2}} = \frac{1}{1-(k+1)x+(k+1)x^2}.$$

Thus the transformation $(\mathbf{Bin}^{-1} \circ \mathbf{Cat})^{-1}$ takes a power k^n and maps it to the solution of the recurrence

$$a_n = (k+1)a_{n-1} - (k+1)a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = k+1$. In particular, it takes the constant sequence 1^n to $2^{\frac{n}{2}}(\cos(\frac{\pi n}{4}) + \sin(\frac{\pi n}{4}))$ (the inverse Catalan transform of 2^n).
As a final example, we apply the combined transformation $\mathbf{Cat}^{-1} \circ \mathbf{Bin}$ to the Fibonacci numbers. We obtain the sequence

$$0, 1, 2, 2, 0, -5, -13, -21, -21, 0, 55, 144, 233, 233, 0, -610, -1597, \ldots$$

whose elements would appear to be Fibonacci numbers. This sequence has generating function

$$\frac{x(1-x)}{1-3x+4x^2-2x^3+x^4}.$$

In closed form, the general term of the sequence is

$$b_n = \phi^n\sqrt{\frac{2}{5}+\frac{2\sqrt{5}}{25}}\sin\left(\frac{\pi n}{5}+\frac{\pi}{5}\right) - (\frac{1}{\phi})^n\sqrt{\frac{2}{5}-\frac{2\sqrt{5}}{25}}\sin\left(\frac{2\pi n}{5}+\frac{2\pi}{5}\right)$$

where $\phi = \frac{1+\sqrt{5}}{2}$. We note that

$$b_n = \frac{2}{5}\left\{\phi^n\sqrt{\sqrt{5}\phi}\sin\left(\frac{\pi n}{5}+\frac{\pi}{5}\right) - (\frac{1}{\phi})^n\sqrt{\sqrt{5}/\phi}\sin\left(\frac{2\pi n}{5}+\frac{2\pi}{5}\right)\right\}.$$

Chapter 7

An application of Riordan arrays to coding theory [1]

7.1 Introduction

In this chapter, we report on a one-parameter family of transformation matrices which can be related to the weight distribution of maximum distance separable (MDS) codes. Regarded as transformations on integer sequences, they are easy to describe both by formula (in relation to the general term of a sequence) and in terms of their action on the ordinary generating function of a sequence. To achieve this, we use the language of the Riordan group of infinite lower-triangular integer matrices. They are also linked to several other known transformations, most notably the binomial transformation.

7.2 Error-correcting codes

Maximum separable codes are a special case of error-correcting code. By *error-correcting code*, we shall mean a linear code over $F_q = GF(q)$, that is, a vector subspace C of F_q^n for some $n > 0$. If C is a k-dimensional vector subspace of F_q^n, then the code is described as a *q-ary* $[n, k]$-code. The elements of C are called the codewords of the code. The weight $w(c)$ of a codeword c is the number of non-zero elements in the vector representation of c. An $[n, k]$ code with minimum weight d is called an $[n, k, d]$ code. A code is called a maximum separable code if the minimum weight of a non-zero codeword in the code is $n - k + 1$. The Reed-Solomon family of linear codes is a well-known family of MDS codes.

An important characteristic of a code is its *weight distribution*. This is defined to be the set of coefficients $A_0, A_1, \ldots, A_n$ where A_i is the number of codewords of weight i in C. The weight distribution of a code plays a significant role in calculating probabilities of error. Except for trivial or 'small' codes, the determination of the weight distribution is normally not easy. The MacWilliams identity for general linear codes is often used to simplify this

[1] This chapter reproduces the content of the published article "P. Barry and P. Fitzpatrick, On a One-parameter Family of Riordan arrays and the Weight Distribution of MDS Codes, J. Integer Seq., **10** (2007), Art. 07.9.8." [20].

task. The special case of MDS codes proves to be tractable. Using the MacWilliams identity [147] or otherwise [179], [222], we obtain the following equivalent results.

Proposition 156. *The number of codewords of weight i, where $n-k+1 \le i \le n$, in a q-ary $[n,k]$ MDS code is given by*

$$\begin{aligned} A_i &= \binom{n}{i}(q-1)\sum_{j=0}^{i-d_{min}}(-1)^j\binom{i-1}{j}q^{i-d_{min}-j} \\ &= \binom{n}{i}\sum_{j=0}(-1)^j\binom{i}{j}(q^{i-d_{min}+1-j}-1) \\ &= \binom{n}{i}\sum_{j=d_{min}}^{i}(-1)^{i-j}\binom{i}{j}(q^{j-d_{min}+1}-1) \end{aligned}$$

where $d_{min}=n-k+1$.

We note that the last expression can be written as

$$A_i=\binom{n}{i}\sum_{j=0}^{i-d_{min}}(-1)^{i-d_{min}-j}\binom{i}{j+d_{min}}(q^{j+1}-1)$$

by a simple change of variable.

We have $A_0=1$, and $A_i=0$ for $1 \le i \le n-k$. The term $\binom{n}{i}$ is a scaling term, which also ensures that $A_i=0$ for $i>n$. In the sequel, we shall study a one-parameter family of Riordan arrays associated to the equivalent summation expressions above.

7.3 Introducing the one-parameter family of 'MDS' transforms

In this section, we shall frequently use n and k to address elements of infinite arrays. Thus the n,k-th element of an infinite array T refers to the element in the n-th row and the k-th column. Row and column indices will start at 0. This customary use of n, k, should not cause any confusion with the use of n, k above to describe $[n,k]$ codes.

We define $\mathbf{T}_m$ to be the transformation represented by the matrix

$$\mathbf{T}_m=\left(\frac{1+x}{1-mx},\frac{x}{1+x}\right)$$

where $m \in \mathbf{N}$. For instance, we have

$$\mathbf{T}_1=\left(\frac{1+x}{1-x},\frac{x}{1+x}\right)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 1 & 0 & 0 & \dots \\ 2 & 1 & 1 & -1 & 1 & 0 & \dots \\ 2 & 1 & 0 & 2 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This triangle is A113310, which has row sums $1, 3, 4, 4, 4, \ldots$ with generating function $\frac{(1+x)^2}{1-x}$. This is A113311. In general, the row sums of $\mathbf{T}_m$ have generating function $\frac{(1+x)^2}{1-mx}$. Note also that

$$\mathbf{T}_0 = \left(1+x, \frac{x}{1+x}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & -2 & 1 & 0 & \ldots \\ 0 & 0 & -1 & 3 & -3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term $\binom{n-2}{n-k}(-1)^{n-k}$.

The A-sequence of each matrix $\mathbf{T}_m$ is clearly $1-x$ (since $f(x) = \frac{x}{1+x}$), and hence we have

$$T_m(n+1, k+1) = T_m(n, k) - T_m(n, k+1).$$

Proposition 157. *For each m, $\mathbf{T}_m$ is invertible with*

$$\mathbf{T}_m^{-1} = \left(1-(m+1)x, \frac{x}{1-x}\right).$$

Proof. Let $\mathbf{T}_m^{-1} = (g^*, \bar{f})$. This exists since $\mathbf{T}_m$ is an element of the Riordan group. Then

$$(g^*, \bar{f})\left(\frac{1+x}{1-mx}, \frac{x}{1+x}\right) = (1, x).$$

Hence

$$\frac{\bar{f}}{1+\bar{f}} = x \Rightarrow \bar{f} = \frac{x}{1-x}$$

and

$$g^* = \frac{1}{g \circ \bar{f}} \Rightarrow g^* = \frac{1-m\bar{f}}{1+\bar{f}} = 1-(m+1)x.$$

□

Corollary 158. *The general term of $\mathbf{T}_m^{-1}$ is given by*

$$\mathbf{T}_m^{-1}(n,k) = \binom{n-1}{n-k} - (m+1)\binom{n-2}{n-k-1}.$$

Proof. We have

$$\begin{aligned}
\mathbf{T}_m^{-1}(n,k) &= [x^n](1-(m+1)x)(\frac{x}{1-x})^k \\
&= [x^{n-k}](1-(m+1)x)\sum_{j\ge 0}\binom{-k}{j}(-1)^j x^j \\
&= [x^{n-k}](1-(m+1)x)\sum_{j\ge 0}\binom{k+j-1}{j}x^j \\
&= [x^{n-k}]\sum_{j\ge 0}\binom{k+j-1}{j}x^j-(m+1)[x^{n-k-1}]\sum_{j\ge 0}\binom{k+j-1}{j}x^j \\
&= \binom{k+n-k+1}{n-k}-(m+1)\binom{k+n-k-1-1}{n-k-1} \\
&= \binom{n-1}{n-k}-(m+1)\binom{n-2}{n-k-1}.
\end{aligned}$$

□

Our main goal in this section is to find expressions for the general term $\mathbf{T}_m(n,k)$ of $\mathbf{T}_m$. To this end, we exhibit certain useful factorizations of $\mathbf{T}_m$.

Proposition 159. *We have the following factorizations of the Riordan array* $\mathbf{T}_m$*:*

$$\begin{aligned}
\mathbf{T}_m &= \left(\frac{1+x}{1-mx},\frac{x}{1+x}\right) \\
&= (1+x,x)\left(\frac{1}{1-mx},\frac{x}{1+x}\right) \\
&= \left(1,\frac{1}{1+x}\right)\left(\frac{x}{1-(m+1)x},x\right) \\
&= \left(\frac{1}{1-mx},x\right)\left(1+x,\frac{x}{1+x}\right) \\
&= \left(\frac{1}{1+x},\frac{x}{1+x}\right)\left(\frac{1}{1-x}\frac{1}{1-(m+1)x},x\right).
\end{aligned}$$

Proof. Each of the assertions is a simple consequence of the product rule for Riordan arrays. For instance,

$$\begin{aligned}
\left(1,\frac{x}{1+x}\right)\left(\frac{1}{1-(m+1)x},x\right) &= \left(1.\frac{1}{1-(m+1)\frac{x}{1-x}},\frac{x}{1+x}\right) \\
&= \left(1.\frac{1+x}{1+x-(m+1)x},\frac{x}{1+x}\right) \\
&= \left(\frac{1+x}{1-mx},\frac{x}{1+x}\right)=\mathbf{T}_m.
\end{aligned}$$

The other assertions follow in a similar manner. □

The last assertion, which can be written

$$\mathbf{T}_m = \mathbf{B}^{-1}\left(\frac{1}{1-x}\frac{1}{1-(m+1)x}, x\right),$$

is a consequence of the fact that the product $\mathbf{BT}_m$ takes on a simple form. We have

$$\begin{aligned}
\mathbf{BT}_m &= \left(\frac{1}{1-x}, \frac{x}{1-x}\right)\left(\frac{1+x}{1-mx}, \frac{x}{1+x}\right) \\
&= \left(\frac{1}{1-x}\frac{1+\frac{x}{1-x}}{1-m\frac{x}{1-x}}, \frac{\frac{x}{1+x}}{1+\frac{x}{1-x}}\right) \\
&= \left(\frac{1}{1-x}\frac{1}{1-(m+1)x}, x\right).
\end{aligned}$$

We can interpret this as the sequence array for the partial sums of the sequence $(m+1)^n$, that is, the sequence array of $\frac{(m+1)^{n+1}-1}{(m+1)-1}$. Thus $\mathbf{T}_m$ is obtained by applying $\mathbf{B}^{-1}$ to this sequence array. We note that the inverse matrix $(\mathbf{BT}_m)^{-1}$ takes the special form

$$((1-x)(1-(m+1)x), x) = (1-(m+2)x+(m+1)^2x^2, x).$$

Thus this matrix is tri-diagonal, of the form

$$(\mathbf{BT}_m)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \dots \\
-(m+2) & 1 & 0 & 0 & 0 & 0 & \dots \\
m+1 & -(m+2) & 1 & 0 & 0 & 0 & \dots \\
0 & m+1 & -(m+2) & 1 & 0 & 0 & \dots \\
0 & 0 & m+1 & -(m+2) & 1 & 0 & \dots \\
0 & 0 & 0 & m+1 & -(m+2) & 1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Corollary 160. *The general term of the array* $\mathbf{T}_m$ *is*

$$\mathbf{T}_m(n,k) = \sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j}((m+1)^{j-k+1}-1)/m, \qquad m \neq 0.$$

Proof. By the last proposition, we have

$$\mathbf{T}_m = \mathbf{B}^{-1}\left(\frac{1}{1-x}\frac{1}{1-(m+1)x}, x\right).$$

The general term of $\mathbf{B}^{-1} = (\frac{1}{1+x}, \frac{x}{1+x})$ is $(-1)^{n-k}\binom{n}{k}$ while that of the second Riordan array is $\frac{(m+1)^{n-k+1}-1}{(m+1)-1}$. The result follows from the product formula for matrices. $\square$

Corollary 161.

$$\mathbf{T}_{m-1}(n,k)=\sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j}\frac{m^{j-k+1}-1}{m-1}=\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n}{j+k}\frac{m^{j+1}-1}{m-1}.$$

Equivalently,

$$(m-1)\binom{n}{k}\mathbf{T}_{m-1}(n,k)=\binom{n}{k}\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n}{j+k}(m^{j+1}-1).$$

This last result makes evident the link between the Riordan array $\mathbf{T}_{m-1}$ and the weight distribution of MDS codes. We now find a number of alternative expressions for the general term of $\mathbf{T}_m$ which will give us a choice of expressions for the weight distribution of an MDS code.

Proposition 162.

$$\begin{aligned}\mathbf{T}_m(n,k) &= \sum_{j=0}^{n-k}(-1)^j\binom{j+k-2}{j}m^{n-k-j}\\ &= \sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-j-2}{n-j-k}m^j\\ &= \sum_{j=k}^{n}\binom{n-1}{n-j}(-1)^{n-j}(m+1)^{j-k}.\end{aligned}$$

Proof. The first two equations result from

$$\begin{aligned}\mathbf{T}_m(n,k) &= [x^n]\frac{1+x}{1-mx}\left(\frac{x}{1+x}\right)^k\\ &= [x^{n-k}](1-mx)^{-1}(1+x)^{-(k-1)}\\ &= [x^{n-k}]\sum_{i\geq 0}m^ix^i\sum_{j\geq 0}\binom{-(k-1)}{j}x^j\\ &= [x^{n-k}]\sum_{i\geq 0}\sum_{j\geq 0}\binom{k+j-2}{j}(-1)^jm^ix^{i+j}.\end{aligned}$$

The third equation is a consequence of the factorization

$$\mathbf{T}_m=\left(1,\frac{1}{1+x}\right)\left(\frac{1}{1-(m+1)x},x\right)$$

since $\left(1,\frac{1}{1+x}\right)$ has general term $\binom{n-1}{n-k}(-1)^{n-k}$. $\square$

Thus we have, for instance,

$$(m-1)\binom{n}{k}\mathbf{T}_{m-1}(n,k) = (m-1)\binom{n}{k}\sum_{j=k}^{n}\binom{n-1}{n-j}(-1)^{n-j}m^{j-k}.$$

Using the notation from the second section, we obtain

$$A_i = (q-1)\binom{n}{i}\sum_{j=d_{min}}^{i}\binom{i-1}{i-j}(-1)^{i-j}q^{j-d_{min}}.$$

7.4 Applications to MDS codes

We begin this section with an example.

Example 163. The dual of the $[7,2,6]$ Reed Solomon code over $GF(2^3)$ is an MDS $[7,5,3]$ code, also over $GF(2^3)$. Thus the code parameters of interest to us are $q=8$, $n=7$, $k=5$ and $d_{min} = n-k+1 = 3$. Let $\mathbf{D} = \mathrm{diag}(\binom{7}{0}, \binom{7}{1}, \ldots, \binom{7}{7}, 0, 0, \ldots)$ denote the infinite square matrix all of whose entries are zero except for those indicated. We form the matrix product $(q-1)\mathbf{D}\mathbf{T}_{q-1}$, with $q=8$, to get

$$7\mathrm{diag}\left\{\binom{7}{j}\right\}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 56 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 392 & 49 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2744 & 343 & 43 & 5 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 19208 & 2401 & 300 & 38 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 134456 & 16807 & 2101 & 262 & 34 & 3 & 1 & 0 & 0 & \ldots \\ 941192 & 117649 & 14706 & 1839 & 228 & 31 & 2 & 1 & 0 & \ldots \\ 6588344 & 823543 & 102943 & 12867 & 1611 & 197 & 29 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 392 & 49 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 8232 & 1029 & 147 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 96040 & 12005 & 1470 & \mathbf{245} & 0 & 0 & 0 & 0 & 0 & \ldots \\ 672280 & 84035 & 10535 & \mathbf{1225} & 245 & 0 & 0 & 0 & 0 & \ldots \\ 2823576 & 352947 & 44100 & \mathbf{5586} & 588 & 147 & 0 & 0 & 0 & \ldots \\ 6588344 & 823543 & 102949 & \mathbf{12838} & 1666 & 147 & 49 & 0 & 0 & \ldots \\ 6588344 & 823543 & 102942 & \mathbf{12873} & 1596 & 217 & 14 & 7 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Column 3 (starting from column 0) of this matrix then yields the weight distribution of the $[7,5,3]$ code. That is, we obtain the vector $(1,0,0,245,1225,5586,12838,12873)$, where we have made the adjustment $A_0 = 1$. We moreover notice that the numbers $(0, 0, 0, 1, 5, 8, 38, 262)$, which correspond to the ratios $A_i/((q-1)\binom{n}{i})$, are elements of the sequence with ordinary generating function $\frac{1+x}{1-7x}\left(\frac{x}{1+x}\right)^3$. Hence they satisfy the recurrence

$$a_n = 4a_{n-1} + 18a_{n-2} + 20a_{n-3} + 7a_{n-4}.$$

This last result leads us to define the *weight ratios* of a q-ary $[n, k, d]$ MDS code to be the ratios $A_i/((q-1)\binom{n}{i})$.

We are now in a position to summarize the results of this paper.

Theorem 164. *Let C be a q-ary $[n, k, d]$ MDS code. The weight distribution of C, adjusted for A_0=1, is obtained from the d-th column of the matrix*

$$(q-1)Diag\left\{\binom{n}{j}\right\}\left(\frac{1+x}{1-(q-1)x}, \frac{x}{1+x}\right).$$

Moreover, the weight ratios of the code satisfy a recurrence defined by the ordinary generating function $\frac{1+x}{1-(q-1)x}(\frac{x}{1+x})^d$.

Proof. Inspection of the expressions for the general term $\mathbf{T}_{q-1}$ and the formulas for A_i yield the first statement. The second statement is a standard property of the columns of a Riordan array. $\square$

Thus the weight ratios satisfy the recurrence

$$a_n = \left((q-1)\binom{d}{0} - \binom{d}{1}\right)a_{n-1} + \left((q-1)\binom{d}{1} - \binom{d}{2}\right)a_{n-2} + \cdots + \left((q-1)\binom{d}{d-1} - \binom{d}{d}\right)a_{n-d} + (q-1)a_{n-d-1}.$$

Letting $R_i = A_i/((q-1)\binom{n}{i})$, we therefore have

$$R_l = \sum_{j=0}^{d}\left((q-1)\binom{d}{j} - \binom{d}{j+1}\right)R_{l-j-1}$$

where $d = d_{min} = n - k + 1$.

Chapter 8

Lah and Laguerre transforms of integer sequences [1]

In this chapter, we show how the simple application of exponential Riordan arrays can bring a unity to the discussion of a number of related topics. Continuing a theme already established, we show that there is a close link between certain simple Riordan arrays and families of orthogonal polynomials (in this case, the Laguerre polynomials). By looking at judicious factorizations and parameterizations, we define interesting transformations and families of polynomials.

Example 165. The Permutation matrix P and its inverse. We consider the matrix

$$\mathbf{P} = \left[\frac{1}{1-x}, x\right].$$

The general term $P(n,k)$ of this matrix is easily found:

$$\begin{aligned} P(n,k) &= \frac{n!}{k!}[x^n]\frac{x^k}{1-x} \\ &= \frac{n!}{k!}[x^{n-k}]\frac{1}{1-x} \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty} x^j \\ &= \frac{n!}{k!} \qquad \text{if} \qquad n-k \ge 0, \qquad =0, \qquad \text{otherwise}, \\ &= [k \le n]\frac{n!}{k!}. \end{aligned}$$

Here, we have used the Iverson bracket notation [106], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false. For instance, $\delta_{ij} = [i = j]$, while $\delta_n = [n = 0]$.

[1]This chapter reproduces and extends the content of the published article "P. Barry, Some observations on the Lah and Laguerre transforms of integer sequences, J. Integer Seq., **10** (2007), Art. 07.4.6." [18].

Clearly, the inverse of this matrix is $\mathbf{P}^{-1} = [1-x, x]$. The general term of this matrix is given by

$$\begin{aligned} P^{-1}(n,k) &= \frac{n!}{k!}[x^n](1-x)x^k \\ &= \frac{n!}{k!}[x^{n-k}](1-x) \\ &= \frac{n!}{k!}(\delta_{n-k} - \delta_{n-k-1}) \\ &= \delta_{n-k} - (k+1)\delta_{n-k-1}. \end{aligned}$$

Thus

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can generalize the construction of $\mathbf{P}$ as follows:

$$\mathbf{P}^{(\alpha)} = \left[\frac{1}{(1-x)^\alpha}, \frac{x}{1-x}\right].$$

The general term of $\mathbf{P}^{(\alpha)}$ is equal to

$$\frac{n!}{k!}\binom{n-k+\alpha-1}{n-k} = \frac{n!}{k!}\binom{-\alpha}{n-k}(-1)^{n-k} = \frac{n!}{k!}\frac{(\alpha)_{n-k}}{(n-k)!}.$$

Clearly, $\mathbf{P} = \mathbf{P}^{(1)}$ and in general, $\mathbf{P}^\alpha = \mathbf{P}^{(\alpha)}$.

8.1 The Lah transform

Introduced by Jovovic (see, for instance, A103194), the Lah transform is the transformation on integer sequences whose matrix is given by

$$\mathbf{Lah} = \left[1, \frac{x}{1-x}\right].$$

Properties of the matrix obtained from the $n \times n$ principal sub-matrix of **Lah**, and related matrices have been studied in [162]. From the above definition, we see that the matrix **Lah** has general term $\text{Lah}(n,k)$ given by

$$\begin{aligned}
\text{Lah}(n,k) &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\
&= \frac{n!}{k!}[x^{n-k}]\sum j\binom{k+j-1}{j}x^j \\
&= \frac{n!}{k!}\binom{n-1}{n-k}
\end{aligned}$$

Thus if b_n is the Lah transform of the sequence a_n, we have

$$b_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n-1}{n-k} a_k.$$

It is clear that the inverse of this matrix $\mathbf{Lah}^{-1}$ is given by $\left[1, \frac{x}{1+x}\right]$ with general term $\text{Lah}(n,k)(-1)^{n-k}$. Thus

$$a_n = \sum_{k=0}^{n} (-1)^{n-k}\frac{n!}{k!}\binom{n-1}{n-k} b_k.$$

Numerically, we have

$$\mathbf{Lah} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 6 & 6 & 1 & 0 & 0 & \dots \\ 0 & 24 & 36 & 12 & 1 & 0 & \dots \\ 0 & 120 & 240 & 120 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Operating on the sequence with e.g.f. $f(x)$, it produces the sequence with e.g.f. $f(\frac{x}{1-x})$.

Example 166. The row sums of the matrix **Lah**, obtained by operating on the sequence $1, 1, 1 \dots$ with e.g.f. e^x, is the sequence $1, 1, 3, 13, 73, 501, \dots$ (A000262) with e.g.f. $e^{\frac{x}{1-x}}$. We observe that this is $n!L(n,-1,-1) = n!L_n^{(-1)}(-1)$ (see Appendix to this chapter for notation). This sequence counts the number of partitions of $\{1,..,n\}$ into any number of lists, where a list means an ordered subset.

8.2 The generalized Lah transform

Extending the definition in [162], we can define, for the parameter t, the generalized Lah matrix

$$\mathbf{Lah}[t] = \left[1, \frac{x}{1-tx}\right].$$

It is immediate that $\mathbf{Lah}[0] = [1, x] = \mathbf{I}$, and $\mathbf{Lah}[1] = \mathbf{Lah}$. The general term of the matrix $\mathbf{Lah}[t]$ is easily computed:

$$\begin{aligned}
\mathrm{Lah}[t](n,k) &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-tx)^k} \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j t^j x^j \\
&= \frac{n!}{k!}[x^{n-k}]\sum j \binom{k+j-1}{j} t^j x^j \\
&= \frac{n!}{k!}\binom{n-1}{n-k} t^{n-k}.
\end{aligned}$$

We can easily establish that

$$\mathbf{Lah}[t]^{-1} = \left[1, \frac{x}{1+tx}\right] = \mathbf{Lah}[-t]$$

with general term $\frac{n!}{k!}\binom{n-1}{n-k}(-t)^{n-k}$. We also have

$$\mathbf{Lah}[u+v] = \mathbf{Lah}[u] \cdot \mathbf{Lah}[v].$$

This follows since

$$\begin{aligned}
\mathbf{Lah}[u] \cdot \mathbf{Lah}[v] &= \left[1, \frac{x}{1-ux}\right]\left[1, \frac{x}{1-vx}\right] \\
&= \left[1, \frac{\frac{x}{1-vx}}{1-\frac{ux}{1-vx}}\right] \\
&= \left[1, \frac{\frac{x}{1-vx}}{\frac{1-vx-ux}{1-vx}}\right] \\
&= \left[1, \frac{x}{1-(u+v)x}\right] \\
&= \mathbf{Lah}[u+v].
\end{aligned}$$

For integer m, it follows that

$$\mathbf{Lah}[mt] = (\mathbf{Lah}[t])^m.$$

8.3 Laguerre related transforms

In this section, we will define the Laguerre transform on integer sequences to be the transform with matrix given by

$$\mathbf{Lag} = \left[\frac{1}{1-x}, \frac{x}{1-x}\right].$$

We favour this denomination through analogy with the Binomial transform, whose matrix is given by

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right).$$

Numerically, we have

$$\mathbf{Lag} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 1 & 0 & 0 & 0 & \dots \\ 6 & 18 & 9 & 1 & 0 & 0 & \dots \\ 24 & 96 & 72 & 16 & 1 & 0 & \dots \\ 120 & 600 & 600 & 200 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The inverse of the Laguerre transform, as we understand it in this section, is given by

$$\mathbf{Lag}^{-1} = \left[\frac{1}{1+x}, \frac{x}{1+x}\right].$$

Clearly, the general term $\text{Lag}(n,k)$ of the matrix $\mathbf{Lag}$ is given by

$$\text{Lag}(n,k) = \frac{n!}{k!}\binom{n}{k}.$$

Thus if b_n is the Laguerre transform of the sequence a_n, we have

$$b_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k} a_k.$$

The e.g.f. of b_n is given by $\frac{1}{1-x} f(\frac{x}{1-x})$ where $f(x)$ is the e.g.f. of a_n. The inverse matrix of $\mathbf{Lag}$ has general term given by $(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$. Thus

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!}\binom{n}{k} b_k.$$

The relationship between the Lah transform with matrix $\mathbf{Lah}$ and the Laguerre transform with matrix $\mathbf{Lag}$ is now clear:

$$\begin{aligned} \mathbf{Lag} &= \left[\frac{1}{1-x}, \frac{x}{1-x}\right] \\ &= \left[\frac{1}{1-x}, x\right]\left[1, \frac{x}{1-x}\right] \\ &= \mathbf{P} \cdot \mathbf{Lah}. \end{aligned}$$

We note that this implies that

$$\begin{aligned}
\mathrm{Lag}(n,k) &= \frac{n!}{k!}\binom{n}{k} \\
&= \sum_{i=0}^{n}[i \le n]\frac{n!}{i!}\frac{i!}{k!}\binom{i-1}{i-k} \\
&= \sum_{i=0}^{n}[k \le n]\frac{n!}{k!}\binom{i-1}{i-k} \\
&= \frac{n!}{k!}\sum_{i=0}^{n}\binom{i-1}{i-k}
\end{aligned}$$

which indeed is true since

$$\binom{n}{k} = \sum_{i=0}^{n}\binom{i-1}{i-k}.$$

Numerically, we have

$$\begin{aligned}
\mathbf{P}\cdot\mathbf{Lah} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 6 & 6 & 1 & 0 & 0 & \dots \\ 0 & 24 & 36 & 12 & 1 & 0 & \dots \\ 0 & 120 & 240 & 120 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 1 & 0 & 0 & 0 & \dots \\ 6 & 18 & 9 & 1 & 0 & 0 & \dots \\ 24 & 96 & 72 & 16 & 1 & 0 & \dots \\ 120 & 600 & 600 & 200 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \mathbf{Lag}.
\end{aligned}$$

Similarly we have

$$\mathbf{Lag}^{-1} = \mathbf{Lah}^{-1}\cdot\mathbf{P}^{-1},$$

which implies that

$$\begin{aligned}
\mathrm{Lag}^{-1}(n,k) &= (-1)^{n-k}\frac{n!}{k!}\binom{n}{k} \\
&= \sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!}\binom{n-1}{n-i}(\delta_{i-k}-(k+1)\delta_{i-k-1}).
\end{aligned}$$

It is of course possible to pass from **Lag** to **Lah** by:

$$\mathbf{Lah} = \mathbf{P}^{-1} \cdot \mathbf{Lag}.$$

Thus

$$\begin{aligned}
\mathrm{Lah}(n,k) &= \frac{n!}{k!}\binom{n-1}{n-k} \\
&= \sum_{i=0}^{n} (\delta_{n-i} - (i+1)\delta_{n-i-1})\frac{i!}{k!}\binom{i}{k}
\end{aligned}$$

We note in passing that this gives us the identity

$$n!\binom{n-1}{n-k} = \sum_{i=0}^{n} (\delta_{n-i} - (i+1)\delta_{n-i-1}) i! \binom{i}{k}.$$

Numerically, we have

$$\begin{aligned}
\mathbf{P}^{-1} \cdot \mathbf{Lag} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 1 & 0 & 0 & 0 & \dots \\ 6 & 18 & 9 & 1 & 0 & 0 & \dots \\ 24 & 96 & 72 & 16 & 1 & 0 & \dots \\ 120 & 600 & 600 & 200 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 6 & 6 & 1 & 0 & 0 & \dots \\ 0 & 24 & 36 & 12 & 1 & 0 & \dots \\ 0 & 120 & 240 & 120 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \mathbf{Lah}.
\end{aligned}$$

Thus if the Laguerre transform of a_n has general term b_n, then the general term of the Lah transform of a_n will be given by

$$b_n - nb_{n-1}$$

(for $n > 0$).

Example 167. A simple consequence of the formula for the general term of the Laguerre array is the following:

Proposition 168. *The Laguerre transform of* $r^n n!$ *is* $(r+1)^n n!$

Proof. We have

$$\begin{aligned}\sum_{k=0}^{n}\frac{n!}{k!}\binom{n}{k}r^k k! &= \sum_{k=0}^{n} n!\binom{n}{k}r^k \\ &= n!\sum_{k=0}^{n}\binom{n}{k}r^k \\ &= n!(r+1)^n.\end{aligned}$$

where we have used the fact that the binomial transform of r^n is r^{n+1}. □

Example 169. The row sums of the matrix **Lag**, that is, the transform of the sequence $1, 1, 1, \ldots$ with e.g.f. e^x, is the sequence $1, 2, 7, 34, 209, 1546, 13327, \ldots$ with e.g.f. $\frac{1}{1-x}e^{\frac{x}{1-x}}$. This is A002720. Among other things, it counts the number of matchings in the bipartite graph $K(n, n)$. Its general term is $\sum_{k=0}^{n}\frac{n!}{k!}\binom{n}{k}$. This is equal to $L_n(-1)$ where $L_n(x)$ is the n-th Laguerre polynomial.

Example 170. The row sums of the matrix $\mathbf{Lag}^{-1}$ yield the sequence

$$1, 0, -1, 4, -15, 56, -185, 204, \ldots$$

with e.g.f. $\frac{1}{1+x}e^{\frac{x}{1+x}}$. It has general term

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$$

which is equal to $(-1)^n L_n(1)$.

8.4 The Associated Laguerre transforms

The Lah and Laguerre transforms, as defined above, are elements of a one-parameter family of transforms, whose general element is given by

$$\mathbf{Lag}^{(\alpha)} = \left[\frac{1}{(1-x)^{\alpha+1}}, \frac{x}{1-x}\right].$$

We can calculate the general term of this matrix in the usual way:

$$\begin{aligned}\mathrm{Lag}^{(\alpha)}(n,k) &= \frac{n!}{k!}[x^n](1-x)^{-(\alpha+1)}x^k(1-x)^{-k} \\ &= \frac{n!}{k!}[x^{n-k}](1-x)^{-(\alpha+k+1)} \\ &= \frac{n!}{k!}\sum_{j=0}^{\infty}\binom{\alpha+k+j}{j}x^j \\ &= \frac{n!}{k!}\binom{n+\alpha}{n-k}.\end{aligned}$$

We note that $\mathbf{Lah} = \mathbf{Lag}^{(-1)}$ while $\mathbf{Lag} = \mathbf{Lag}^{(0)}$. We can factorize $\mathbf{Lag}^{(\alpha)}$ as follows:

$$\begin{aligned}\mathbf{Lag}^{(\alpha)} &= \left[\frac{1}{(1-x)^{\alpha}}, x\right]\left[\frac{1}{1-x}, \frac{x}{1-x}\right] \\ &= \mathbf{P}^{(\alpha)} \cdot \mathbf{Lag}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{Lag}^{(\alpha)} &= \left[\frac{1}{(1-x)^{\alpha+1}}, x\right]\left[1, \frac{x}{1-x}\right] \\ &= \mathbf{P}^{(\alpha+1)} \cdot \mathbf{Lah}.\end{aligned}$$

Now $\mathbf{P}^{(\alpha+1)}$ has general term $\frac{n!}{k!}\binom{n-k+\alpha}{n-k}$ and $\mathbf{Lah}$ has general term $\frac{n!}{k!}\binom{n-1}{n-k}$. We deduce the following identity

$$\binom{n+\alpha}{n-k} = \sum_{j=0}^{n} \binom{n-j+\alpha}{n-j}\binom{j-1}{j-k}.$$

The transform of the sequence a_n by the associated Laguerre transform for α is the sequence b_n with general term $b_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n+\alpha}{n-k} a_k$, which has e.g.f. $\frac{1}{(1-x)^{\alpha+1}} f(\frac{x}{1-x})$.

The exponential Riordan array $\mathbf{Lag}^{(\alpha)}$ has production matrix

$$\left(\begin{array}{ccccccc} 1+\alpha & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1+\alpha & 3+\alpha & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2(2+\alpha) & 5+\alpha & 1 & 0 & 0 & \ldots \\ 0 & 0 & 3(3+\alpha) & 7+\alpha & 1 & 0 & \ldots \\ 0 & 0 & 0 & 4(4+\alpha) & 9+\alpha & 1 & \ldots \\ 0 & 0 & 0 & 0 & 5(5+\alpha) & 11+\alpha & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

or

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 5 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9 & 7 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 16 & 9 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 25 & 11 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) + \alpha \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We note that in the literature of Riordan arrays, the subset of matrices of the form $(1, f(x))$ forms a sub-group, called *the associated group*. We trust that this double use of the term "associated" does not cause confusion.

8.5 The Generalized Laguerre transform

We define, for the parameter t, the generalized Laguerre matrix $\mathbf{Lag}[t]$ to be

$$\mathbf{Lag}[t] = \left[\frac{1}{1-tx}, \frac{x}{1-tx}\right].$$

We immediately have

$$\begin{aligned}\mathbf{Lag}[t] &= \left[\frac{1}{1-tx}, \frac{x}{1-tx}\right] \\ &= \left[\frac{1}{1-tx}, x\right]\left[1, \frac{x}{1-tx}\right] \\ &= \mathbf{P}[t] \cdot \mathbf{Lah}[t].\end{aligned}$$

where the generalized permutation matrix $\mathbf{P}[t]$ has general term $[k \leq n]\frac{n!}{k!}t^{n-k}$. It is clear that

$$\mathbf{Lag}[t]^{-1} = \mathbf{Lag}[-t] = \left[\frac{1}{1+tx}, \frac{x}{1+tx}\right].$$

It is possible to generalize the associated Laguerre transform matrices in similar fashion.

8.6 Transforming the expansion of $\frac{x}{1-\mu x-\nu x^2}$

The e.g.f. of the expansion of $\frac{x}{1-\mu x-\nu x^2}$ takes the form

$$f(x) = A(\mu,\nu)e^{r_1 x} + B(\mu,\nu)e^{r_2 x}$$

which follows immediately from the Binet form of the general term. Thus the transform of this sequence by $\mathbf{Lag}^{(\alpha)}$ will have e.g.f.

$$\frac{A(\mu,\nu)}{(1-x)^{\alpha+1}}e^{\frac{r_1 x}{1-x}} + \frac{B(\mu,\nu)}{(1-x)^{\alpha+1}}e^{\frac{r_2 x}{1-x}}.$$

In the case of the Lah transform ($\alpha = -1$), we get the simple form

$$Ae^{\frac{r_1 x}{1-x}} + Be^{\frac{r_2 x}{1-x}}$$

while in the Laguerre case ($\alpha = 0$) we get

$$A\frac{e^{\frac{r_1 x}{1-x}}}{1-x} + B\frac{e^{\frac{r_2 x}{1-x}}}{1-x}.$$

Now $\frac{e^{\frac{rx}{1-x}}}{1-x}$ is the e.g.f. of the sequence $n!L_n(-r)$. Thus in this case, the transformed sequence has general term

$$An!L_n(-r_1) + Bn!L_n(-r_2).$$

Example 171. The Laguerre transform of the Fibonacci numbers

$$F(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

is given by

$$\frac{1}{\sqrt{5}}n!L_n\left(-\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}n!L_n\left(-\frac{1-\sqrt{5}}{2}\right).$$

This is A105277. It begins $0, 1, 5, 29, 203, 1680, 16058, \ldots$.

Example 172. The Laguerre transform of the Jacobsthal numbers [11, 239] (expansions of $\frac{x}{1-x-2x^2}$)

$$J(n) = \frac{2^n}{3} - \frac{(-1)^n}{3}$$

is given by

$$\frac{1}{3}n!L_n(-2) - \frac{1}{3}n!L_n(1).$$

This is A129695. It begins $0, 1, 5, 30, 221, 1936, 19587, \ldots$. We can use this result to express the Lah transform of the Jacobsthal numbers, since this is equal to $b_n - nb_{n-1}$ where b_n is the Laguerre transform of $J(n)$. We find

$$\frac{n!}{3}(L_n(-2) - L_{n-1}(-2) - (L_n(1) - L_{n-1}(1))).$$

Example 173. We calculate the $\mathbf{Lag}^{(1)}$ transform of the Jacobsthal numbers $J(n)$. Since $\mathbf{Lag}^{(1)} = \mathbf{P} \cdot \mathbf{Lag}$, we apply $\mathbf{P}$ to the Laguerre transform of $J(n)$. This gives us

$$\sum_{k=0}^{n} \frac{n!}{k!}(k!L_k(-2) - k!L_k(1))/3 = \frac{n!}{3}\sum_{k=0}^{n}(L_k(-2) - L_k(1)).$$

This sequence has e.g.f. $\frac{1}{(1-x)^2}\frac{e^{\frac{2x}{(1-x)}} - e^{\frac{-x}{1-x}}}{3}$.

8.7 The Lah and Laguerre transforms and Stirling numbers

In this section, we follow the notation of [106]. Thus the Stirling numbers of the first kind, denoted by $\left[{n \atop k}\right]$, are the elements of the matrix

$$\mathbf{s} = \left[1, \ln\left(\frac{1}{1-x}\right)\right].$$

$\left[{n \atop k}\right]$ counts the number of ways to arrange n objects into k cycles. We have

$$\mathbf{s}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 6 & 11 & 6 & 1 & 0 & \ldots \\ 0 & 24 & 50 & 35 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of this array give $n!$.
The Stirling numbers of the second kind, denoted by $\left\{{n \atop k}\right\}$, count the number of ways to partition a set of n things into k nonempty subsets. $\left\{{n \atop k}\right\}$ are the elements of the matrix

$$\mathbf{S}=[1, e^x-1].$$

This is the matrix

$$[0,1,0,2,0,3,0,\ldots] \quad \Delta \quad [1,0,1,0,1,\ldots],$$

with production matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 3 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 4 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have

$$\begin{aligned} \left\{{n \atop k}\right\} &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n \\ &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \end{aligned}$$

We note that the matrix $[1, e^x-1]$ of Stirling numbers of the second kind is the inverse of the matrix with elements $(-1)^{n-k}\left[{n \atop k}\right]$, which is the matrix $[1, \ln(1+x)]$.
Related matrices include

$$\left[\frac{1}{1-x}, \ln\left(\frac{1}{1-x}\right)\right]$$

whose elements are given by $\left[{n+1 \atop k+1}\right]$ and its signed version,

$$\left[\frac{1}{1+x}, \ln\left(\frac{1}{1+x}\right)\right]$$

whose elements are given by $(-1)^{n-k}\left[{n+1 \atop k+1}\right]$, along with their inverses, given respectively by $[e^{-x}, 1-e^{-x}]$, with general element $(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}$, and $[e^x, e^x-1]$ with general element $\left\{{n+1 \atop k+1}\right\}$.

We can generalize a result linking the Lah matrix to the Stirling numbers [162] to the infinite matrix case as follows:

$$\mathbf{Lah} = \mathbf{s} \cdot \mathbf{S}.$$

This is because we have

$$\begin{aligned}
\mathbf{s} \cdot \mathbf{S} &= \left[1, \ln\left(\frac{1}{1-x}\right)\right][1, e^x - 1] \\
&= [1, e^{\ln\left(\frac{1}{1-x}\right)} - 1] \\
&= \left[1, \frac{1}{1-x} - 1\right] \\
&= \left[1, \frac{x}{1-x}\right] = \mathbf{Lah}.
\end{aligned}$$

Thus we have

$$\mathbf{S} = \mathbf{s}^{-1} \cdot \mathbf{Lah}, \qquad \mathbf{s} = \mathbf{Lah} \cdot \mathbf{S}^{-1}.$$

We now observe that

$$\begin{aligned}
\left[\frac{1}{1-x}, \frac{x}{1-x}\right][1, \ln(1+x)] &= \left[\frac{1}{1-x}, \ln\left(1 + \frac{x}{1-x}\right)\right] \\
&= \left[\frac{1}{1-x}, \ln\left(\frac{1}{1-x}\right)\right]
\end{aligned}$$

or

$$\mathbf{Lag} \cdot \left((-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}\right) = \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}\right).$$

We deduce the identity

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \sum_{j=0}^{n} \frac{n!}{j!}\binom{n}{j}(-1)^{j-k}\begin{bmatrix} j \\ k \end{bmatrix}.$$

Taking the inverse of the matrix identity above, we obtain

$$\left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}\right)^{-1} = \left((-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}\right)^{-1} \cdot \mathbf{Lag}^{-1}$$

which can be established alternatively by noting that

$$\begin{aligned}
[1, e^x - 1]\left[\frac{1}{1+x}, \frac{x}{1+x}\right] &= \left[1.\frac{1}{1+e^x-1}, \frac{e^x-1}{1+e^x-1}\right] \\
&= [e^{-x}, 1-e^{-x}].
\end{aligned}$$

This establishes the identity

$$(-1)^{n-k}\left\{\begin{matrix} n+1 \\ k+1 \end{matrix}\right\} = \sum_{j=0}^{n}(-1)^{j-k}\binom{j}{k}\frac{j!}{k!}\left\{\begin{matrix} n \\ j \end{matrix}\right\}.$$

Finally, from the matrix identity

$$\mathbf{Lag} \cdot \left((-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \right) = \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \right).$$

we deduce that

$$\begin{aligned} \mathbf{Lag} &= \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \right) \cdot \left((-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \right)^{-1} \\ &= \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \right) \cdot \left(\begin{Bmatrix} n \\ k \end{Bmatrix} \right). \end{aligned}$$

Thus

$$\mathrm{Lag}(n,k) = \sum_{j=0}^{n} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix}.$$

This is equivalent to the factorization

$$\mathbf{Lag} = \left[\frac{1}{1-x}, \frac{x}{1-x} \right] = \left[\frac{1}{1-x}, \ln\left(\frac{1}{1-x} \right) \right] [1, e^x - 1].$$

This implies (see Appendix A) that

$$L_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \sum_{j=0}^{n} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix} (-x)^k.$$

It is natural in this context to define as *associated Stirling numbers of the first kind* the elements $\begin{bmatrix} n \\ k \end{bmatrix}_\alpha$ of the matrices

$$\left[\frac{1}{(1-x)^\alpha}, \ln\left(\frac{1}{1-x} \right) \right].$$

For instance, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = \begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$. We note that signed versions of these numbers have been documented by Lang (see for instance A049444 and A049458). To calculate $\begin{bmatrix} n \\ k \end{bmatrix}_2$, we proceed as follows:

$$\begin{aligned} \left(\begin{bmatrix} n \\ k \end{bmatrix}_2 \right) &= \left[\frac{1}{(1-x)^2}, \ln\left(\frac{1}{1-x} \right) \right] \\ &= \mathbf{P} \cdot \left[\frac{1}{1-x}, \ln\left(\frac{1}{1-x} \right) \right] \\ &= \left([k \le n] \frac{n!}{k!} \right) \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \right). \end{aligned}$$

Thus

$$\begin{bmatrix} n \\ k \end{bmatrix}_2 = \sum_{j=0}^{n} \frac{n!}{j!} \begin{bmatrix} j+1 \\ k+1 \end{bmatrix} = \sum_{j=0}^{n} \frac{n!}{j!} \begin{bmatrix} j \\ k \end{bmatrix}_1.$$

More generally, since $[\frac{1}{(1-x)^\alpha}, \ln(\frac{1}{1-x})] = \mathbf{P}[\frac{1}{(1-x)^{\alpha-1}}, \ln(\frac{1}{1-x})]$, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = \sum_{j=0}^{n} \frac{n!}{j!} \begin{bmatrix} j \\ k \end{bmatrix}_{\alpha-1}.$$

$$\mathrm{Lag}^{(\alpha)}(n,k) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{\alpha+1} \left\{ \begin{matrix} j \\ k \end{matrix} \right\}.$$

For example,

$$\begin{aligned} \mathbf{Lag}^{(1)} &= \left[\frac{1}{(1-x)^2}, \ln\left(\frac{1}{1-x}\right)\right][1, e^x - 1] \\ &= \left(\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_2 \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \right) \\ &= \left(\sum_{j=0}^{n} \sum_{i=0}^{n} \frac{n!}{i!} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \right). \end{aligned}$$

In general, we have

$$\mathbf{Lag}^{(\alpha)} = \left[\frac{1}{(1-x)^{\alpha+1}}, \frac{x}{1-x}\right] = \left[\frac{1}{(1-x)^{\alpha+1}}, \ln\left(\frac{1}{1-x}\right)\right][1, e^x - 1].$$

This implies that

$$\mathrm{Lag}^{(\alpha)}(n,k) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{\alpha+1} \left\{ \begin{matrix} j \\ k \end{matrix} \right\}.$$

8.8 The generalized Lah, Laguerre and Stirling matrices

Given a parameter t we define the *generalized Stirling numbers of the first kind* to be the elements of the matrix

$$\mathbf{s}[t] = \left[1, \frac{1}{t} \ln\left(\frac{1}{1-tx}\right)\right].$$

Similarly, we define the *generalized Stirling numbers of the second kind* to be the elements of the matrix

$$\mathbf{S}[t] = \left[1, \frac{e^{tx}-1}{t}\right].$$

Then

$$\mathbf{S}[t]^{-1} = \mathbf{s}[-t].$$

For instance,

$$\begin{aligned}
\mathbf{s}[-t]\cdot\mathbf{S}[t] &= \left[1, -\frac{1}{t}\ln\left(\frac{1}{1+tx}\right)\right]\left[1, \frac{e^{tx}-1}{t}\right] \\
&= \left[1, -\frac{1}{t}\ln\left(\frac{1}{1+t\frac{e^{tx}-1}{t}}\right)\right] \\
&= \left[1, -\frac{1}{t}\ln(\frac{1}{e^{tx}})\right] \\
&= \left[1, \frac{1}{t}\ln(e^{tx})\right] \\
&= [1, x] = \mathbf{I}.
\end{aligned}$$

The general term of $\mathbf{s}[t]$ is given by $t^{n-k}\left[{n \atop j}\right]$ and that of $\mathbf{S}[t]$ is given by $t^{n-k}\left\{{n \atop j}\right\}$. An easy calculation establishes that

$$\mathbf{Lah}[t] = \mathbf{s}[t]\mathbf{S}[t].$$

From this we immediately deduce that

$$\mathbf{Lag}[t] = \mathbf{P}[t]\mathbf{s}[t]\mathbf{S}[t].$$

Similarly results for the generalized associated Laguerre transform matrices can be derived.

8.9 Stirling numbers and Charlier polynomials

We finish this chapter by noting a close relationship between the Stirling numbers of the first kind $\mathbf{s} = \left[1, \ln\left(\frac{1}{1-x}\right)\right]$ and the coefficient array of the (unsigned) Charlier polynomials. In effect, we have

$$\left[e^x, \ln\left(\frac{1}{1-x}\right)\right] = [e^x, x]\cdot\left[1, \ln\left(\frac{1}{1-x}\right)\right] = \mathbf{B}\cdot\mathbf{s}.$$

where the array on the LHS is the coefficient array of the unsigned Charlier polynomials. These polynomials are equal to ${}_2F_0(-n, x; -1)$. The Charlier polynomials are normally defined to be ${}_2F_0(-n, -x; -1/\mu)$. Here, we define the unsigned Charlier polynomials by

$$P_n(x) = \sum_{k=0}^{n}\binom{n}{k}(x)_k,$$

where $(x)_k = \prod_{j=0}^{k-1}(x+j)$. Letting $\mathbf{Ch} = \mathbf{B}\cdot\mathbf{s} = \left[e^x, \ln\left(\frac{1}{1-x}\right)\right]$, we have for example

$$\mathbf{Lah} = \mathbf{B}^{-1}\cdot\mathbf{Ch}\cdot\mathbf{S}$$

and

$$\mathbf{Lag} = \mathbf{P}\cdot\mathbf{B}^{-1}\cdot\mathbf{Ch}\cdot\mathbf{S}.$$

8.10 Appendix A - the Laguerre and associated Laguerre functions

The associated Laguerre polynomials [241] are defined by

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!}\binom{n+\alpha}{n-k}(-x)^k \\ &= \frac{1}{n!}e^x x^{-\alpha}\frac{d^n}{dx^n}[x^{n+\alpha}e^{-x}] \\ &= \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\alpha+1;x). \end{aligned}$$

Their generating function is

$$\frac{e^{\frac{-xz}{1-z}}}{(1-z)^{\alpha+1}}.$$

The Laguerre polynomials are given by $L_n(x) = L_n^{(0)}(x)$. The associated Laguerre polynomials are orthogonal on the interval $[0,\infty)$ for the weight $e^{-x}x^{\alpha}$.
Using the notation developed above, we have

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{n!}\sum_{k=0}^{n} \mathrm{Lag}^{(\alpha)}(n,k)(-x)^k \\ &= \frac{1}{n!}\sum_{k=0}^{n}\sum_{i=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{\alpha+1} \begin{Bmatrix} j \\ k \end{Bmatrix}(-x)^k. \end{aligned}$$

In particular,

$$L_n(x) = \frac{1}{n!}\sum_{k=0}^{n}\sum_{i=0}^{n} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix}(-x)^k.$$

We finish this appendix by illustrating the use of Riordan arrays to establish a well known identity for Laguerre polynomials [5], namely

$$L_n^{(\beta)}(x) = \sum_{k=0}^{n} \frac{(\beta-\alpha)_{n-k}}{(n-k)!}L_k^{(\alpha)}(x).$$

To do this, we first note that

$$\begin{aligned} \mathbf{Lag}^{(\beta)} &= \left[\frac{1}{(1-x)^{\beta+1}}, \frac{x}{1-x}\right] \\ &= \left[\frac{1}{(1-x)^{\beta-\alpha}}, x\right] \cdot \left[\frac{1}{(1-x)^{\alpha+1}}, \frac{x}{1-x}\right] \\ &= \left[\frac{1}{(1-x)^{\beta-\alpha}}, x\right] \cdot \mathbf{Lag}^{(\alpha)}. \end{aligned}$$

Now the general term of the Riordan array $\left[\frac{1}{(1-x)^{\beta-\alpha}}, x\right]$ is given by

$$\frac{n!}{k!}\binom{\alpha-\beta}{n-k}(-1)^{n-k} = \frac{n!}{k!}\frac{(\beta-\alpha)_{n-k}}{(n-k)!}.$$

It follows that

$$\begin{aligned}
L_n^{(\beta)}(x) &= \frac{1}{n!}\sum_{k=0}^{n} Lag^{(\beta)}(n,k)(-x)^k \\
&= \frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{n}\frac{n!}{j!}\binom{\alpha-\beta}{n-j}(-1)^{n-j}Lag^{(\alpha)}(j,k)(-x)^k \\
&= \frac{1}{n!}\sum_{j=0}^{n} n!\binom{\alpha-\beta}{n-j}(-1)^{n-j}\frac{1}{j!}\sum_{k=0}^{j}Lag^{(\alpha)}(j,k)(-x)^k \\
&= \sum_{j=0}^{n}\binom{\alpha-\beta}{n-j}(-1)^{n-j}L_j^{(\alpha)}(x) \\
&= \sum_{j=0}^{n}\frac{(\beta-\alpha)_{n-j}}{(n-j)!}L_j^{(\alpha)}(x).
\end{aligned}$$

8.11 Appendix B - Lah and Laguerre transforms in the OEIS

Table 1. Table of Lah transforms

a_n	Lah transform b_n
$A000012$	$A000262$
$A000027$	$A052852$
$A000027(n+1)$	$A002720$
$A000079$	$A052897$
$A000085$	$A049376$
$A000110$	$A084357$
$A000262$	$A025168$
$A000290$	$A103194$
$A000670$	$A084358$
$A104600$	$A121020$
$\frac{1+(-1)^n}{2}$ (e.g.f. $\cosh(x)$)	$A088312$
$\frac{1-(-1)^n}{2}$ (e.g.f. $\sinh(x)$)	$A088313$

Table 2. Table of Laguerre transforms

a_n	Laguerre transform b_n
$A000007$	$A000142$
$A000012$	$A002720$
$A000027(n+1)$	$A000262(n+1)$
$A000045$	$A105277$
$A000079$	$A087912$
$A000142$	$A000165$
$A000165$	$A032031$
$A001045$	$A129695$
$A032031$	$A047053$
$A005442$	$A052574$
$(-1)^n \cdot A052554$	$A005442$
$(-1)^n \cdot A052598$	$A052585$

Chapter 9

Riordan arrays and Krawtchouk polynomials[1]

9.1 Introduction

The Krawtchouk polynomials play an important role in various areas of mathematics. Notable applications occur in coding theory [147, 144], association schemes [47], and in the theory of group representations [224, 225].

In this chapter, we explore links between the Krawtchouk polynomials and Riordan arrays, of both ordinary and exponential type, and we study integer sequences defined by evaluating the Krawtchouk polynomials at different values of their parameters.

The link between Krawtchouk polynomials and exponential Riordan arrays is implicitly contained in the umbral calculus approach to certain families of orthogonal polynomials. We shall look at these links explicitly in the following.

We define the Krawtchouk polynomials, using exponential Riordan arrays, and look at some general properties of these polynomials from this perspective. We then show that for different values of the parameters used in the definition of the Krawtchouk polynomials, there exist interesting families of (ordinary) Riordan arrays.

9.2 Krawtchouk polynomials

We follow [195] in defining the Krawtchouk polynomials. They form an important family of orthogonal polynomials [53, 218, 240]. Thus the Krawtchouk polynomials will be considered to be the special case $\beta = -N$, $c = \frac{p}{p-1}$, $p + q = 1$ of the Meixner polynomials of the first kind, which form the Sheffer sequence for

$$\begin{aligned} g(t) &= \left(\frac{1-c}{1-ce^t}\right)^{\beta}, \\ f(t) &= \frac{1-e^t}{c^{-1}-e^t}. \end{aligned}$$

[1] This chapter reproduces the content of the published article "P. Barry, A Note on Krawtchouk Polynomials and Riordan Arrays, J. Integer Seq., **11** (2008), Art. 08.2.2." [22].

Essentially, this says that the Meixner polynomials of the first kind are obtained by operating on the vector $(1, x, x^2, x^3, \ldots)'$ by the exponential Riordan array $[g(t), f(t)]^{-1}$, since

$$[g, f]^{-1} = \left[\frac{1}{g \circ \bar{f}}, \bar{f}\right]$$

and

$$\left[\frac{1}{g \circ \bar{f}}, \bar{f}\right] e^{xt} = \frac{1}{g \circ \bar{f}} e^{x\bar{f}(t)}$$

which is the defining expression for the Sheffer sequence associated to g and f. In order to work with this expression, we calculate $[g, f]^{-1}$ as follows. Firstly,

$$\bar{f} = \log\left(\frac{t-c}{c(t-1)}\right)$$

since

$$\frac{1-e^u}{c^{-1}-u} = x \implies e^u = \frac{x-c}{c(x-1)}$$
$$u = \log\left(\frac{x-c}{c(x-1)}\right) \implies \bar{f}(t) = \log\left(\frac{t-c}{c(t-1)}\right)$$

Then we have

$$g(\bar{f}(t)) = \left(\frac{1-c}{1-ce^{\bar{f}(t)}}\right)^{\beta} = \left(\frac{1-c}{1-\frac{t-c}{t-1}}\right)^{\beta} = (1-t)^{\beta}.$$

and

$$e^{x\bar{f}(t)} = e^{x\log\left(\frac{t-c}{c(t-1)}\right)} = \left(\frac{t-c}{c(t-1)}\right)^{x}.$$

Thus we arrive at

$$[g, f]^{-1} = \left[\frac{1}{(1-t)^{\beta}}, \log\left(\frac{t-c}{c(t-1)}\right)\right]$$

and

$$\begin{aligned}\frac{e^{x\bar{f}(t)}}{g(\bar{f}(t))} &= \frac{1}{(1-t)^{\beta}}\left(\frac{t-c}{c(t-1)}\right)^{x} \\ &= \frac{1}{(1-t)^{\beta+x}}\left(\frac{c-t}{c}\right)^{x} \\ &= (1-t)^{-\beta-x}\left(1-\frac{t}{c}\right)^{x}.\end{aligned}$$

Specializing to the values $\beta = -N$ and $c = \frac{p}{p-1} = -\frac{p}{q}$, we get

$$\frac{e^{x\bar{f}(t)}}{g(\bar{f}(t))} = (1-t)^{N-x}(1+\frac{q}{p}t)^{x}.$$

Extracting the coefficient of t^k in this expression, we obtain

$$\begin{aligned}
[t^k]\frac{e^{x\bar{f}(t)}}{g(\bar{f}(t))} &= [t^k]\sum_{i=0}\binom{N-x}{i}(-1)^i t^i \sum_{j=0}\binom{x}{j}\left(\frac{q}{p}\right)^j t^j \\
&= \sum_{j=0}^{k}\binom{N-x}{k-j}\binom{x}{j}(-1)^{k-j}q^j p^{-j}.
\end{aligned}$$

Scaling by p^k, we thus obtain

$$p^k[x^k]\frac{e^{x\bar{f}(t)}}{g(\bar{f}(t))} = \sum_{j=0}^{k}\binom{N-x}{k-j}\binom{x}{j}(-1)^{k-j}q^j p^{k-j}.$$

We use the notation

$$\kappa_n^{(p)}(x,N) = \sum_{j=0}^{n}\binom{N-x}{n-j}\binom{x}{j}(-1)^{n-j}q^j p^{n-j}$$

for the Krawtchouk polynomial with parameters N and p. This can be expressed in hypergeometric form as

$$\kappa_n^{(p)}(x,N) = (-1)^n\binom{N}{n}p^n\,{}_2F_1(-n,-x;-N;1/p).$$

The form of $[g,f]^{-1}$ allows us to make some interesting deductions. For instance, if we write

$$[g(t),f(t)]^{-1} = \left[\frac{1}{(1-t)^\beta}, \log\left(\frac{1-\frac{t}{c}}{1-t}\right)\right]$$

then setting $\beta = -N$ and $c = \frac{p}{p-1}$, we get

$$[g(t),f(t)]^{-1} = \left[\frac{1}{(1-t)^{-N}}, \log\left(\frac{1-\frac{p-1}{p}t}{1-t}\right)\right].$$

Now we let $t = ps$, giving

$$\begin{aligned}
[g(t),f(t)]^{-1} &= \mathrm{Diag}(1/p^n) * \left[(1-ps)^N, \log\left(\frac{1-(p-1)s}{1-ps}\right)\right] \\
&= \mathrm{Diag}(1/p^n) * [(1-ps)^N, s] * \left[1, \frac{s}{1-(p-1)s}\right] * \left[1, \log\left(\frac{1}{1-s}\right)\right] \\
&= \mathrm{Diag}(1/p^n) * \mathbf{P}[p]^{-N} * \mathbf{Lah}[p-1] * \mathbf{s}.
\end{aligned}$$

where we have used the notation of [18] and where for instance $\mathbf{s} = \left[1, \log\left(\frac{1}{1-s}\right)\right]$ is the Stirling array of the first kind.

The matrix $\mathbf{P}[p]^{-N} * \mathbf{Lah}[p-1] * \mathbf{s} = \left[(1-pt)^N, \log\left(\frac{1-(p-1)t}{1-pt}\right)\right]$ is of course a monic exponential Riordan array. If its general term is $T(n,k)$, then that of the corresponding array $[g,f]^{-1}$ is given by $T(n,k)/p^n$.

The above matrix factorization indicates that the Krawtchouk polynomials can be expressed as combinations of the Stirling polynomials of the first kind $1, x, x(x+1), x(x^2+3x+2), x(x^3+6x^2+11x+6), \ldots$.

Example 174. Taking $N=-1$ and $p=2$ we exhibit an interesting property of the matrix $\left[(1-pt)^N, \log\left(\frac{1-(p-1)t}{1-pt}\right)\right]$, which in this case is the matrix $\left[\frac{1}{1-2t}, \log\left(\frac{1-t}{1-2t}\right)\right]$. An easy calculation shows that

$$\left[\frac{1}{1-2t}, \log\left(\frac{1-t}{1-2t}\right)\right]^{-1} = \left[\frac{1}{2e^t-1}, \frac{e^t-1}{2e^t-1}\right].$$

We recall that the Binomial matrix with general term $\binom{n}{k}$ is the Riordan array $[e^t, t]$. Now

$$\left[\frac{1}{1-2t}, \log\left(\frac{1-t}{1-2t}\right)\right][e^t, t]\left[\frac{1}{2e^t-1}, \frac{e^t-1}{2e^t-1}\right] = \left[\frac{1-t}{1-2t}, t\right].$$

Hence the matrices $[e^t, t]$ and $\left[\frac{1-t}{1-2t}, t\right]$ are similar, with $\left[\frac{1}{1-2t}, \log\left(\frac{1-t}{1-2t}\right)\right]$ serving as matrix of change of basis for the similarity.

9.3 Krawtchouk polynomials and Riordan arrays

In this section, we shall use the following notation, where we define a variant on the polynomial family $\kappa_n^{(p)}(x,N)$. Thus we let

$$K(n,k,x,q) = \sum_{j=0}^{k} (-1)^j \binom{x}{j}\binom{n-x}{k-j}(q-1)^{k-j}.$$

We then have

$$K(n,k,x,q) = [t^k](1-t)^x(1+(q-1)t)^{n-x},$$

which implies that

$$K(N,k,N-x,q) = [t^k](1-t)^{N-x}(1+(q-1)t)^x.$$

Letting $P=1/q$ and thus $(1-P)/P = q-1$ we obtain

$$K(N,k,N-x,q) = \frac{1}{q^n}\kappa_n^{(P)}(x,N).$$

We shall see in the sequel that by varying the parameters n, k, x and q, we can obtain families of (ordinary) Riordan arrays defined by the corresponding Krawtchouk expressions.

Example 175. We first look at the term $K(k, n-k, r, q)$. We have

$$\begin{aligned} K(k, n-k, r, q) &= \sum_{j=0}^{n-k} (-1)^j \binom{r}{j} \binom{k-r}{n-k-j} (q-1)^{n-k-j} \\ &= \sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{k-r}{n-k-j} (q-1)^{n-k-j}. \end{aligned}$$

But this last term is the general term of the Riordan array

$$\left(\left(\frac{1-x}{1+(q-1)x} \right)^r, x(1+(q-1)x) \right). \tag{9.1}$$

The term $(-1)^{n-k} K(k, n-k, r, q)$ then represents the general term of the inverse of this Riordan array, which is given by

$$\left(\left(\frac{1+x}{1-(q-1)x} \right)^r, x(1-(q-1)x) \right).$$

The A-sequence of the array (9.1) is given by

$$A(x) = \frac{1+\sqrt{1+4(q-1)x}}{2}.$$

Thus

$$a_0 = 1, \qquad a_n = (-1)^{n-1}(q-1)^n C_{n-1}.$$

With these values, we therefore have

$$K(k+1, n-k, r, q) = K(k, n-k, r, q) + a_1 K(k+1, n-k-1, r, q) + a_2 K(k+2, n-k-2, r, q) + \ldots$$

Example 176. We next look at the family defined by $(-1)^k K(n, k, k, q)$. We have

$$\begin{aligned} (-1)^k K(n, k, k, q) &= (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-k}{k-j} (q-1)^{k-j} \\ &= \sum_{j=0}^{k} \binom{k}{j} \binom{n-k}{k-j} (1-q)^{k-j} \\ &= \sum_{j=0}^{n-k} \binom{k}{j} \binom{n-k}{j} (1-q)^j. \end{aligned}$$

Using the results of [16] (see also Chapter 10) we see that these represent the family of Riordan arrays

$$\left(\frac{1}{1-x}, \frac{x(1-qx)}{1-x} \right).$$

The A-sequence for this array is given by

$$A(x) = \frac{1+x+\sqrt{1+2x(1-2q)+x^2}}{2}.$$

For example, the matrix with general term $T(n,k) = (-1)^k K(n,k,k,-3)$ is the Riordan array $\left(\frac{1}{1-x}, \frac{x(1+3x)}{1-x}\right)$, A081578 or

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 5 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 9 & 9 & 1 & 0 & 0 & \ldots \\ 1 & 13 & 33 & 13 & 1 & 0 & \ldots \\ 1 & 17 & 73 & 73 & 7 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The A-sequence for this array has g.f. $\frac{1+x+\sqrt{1+14x+x^2}}{2}$ which expands to

$$1, 4, -12, 84, -732, 7140, -74604, \ldots$$

Thus

$$\begin{aligned} (-1)^{k+1}K(n+1,k+1,k+1,-3) &= (-1)^k K(n,k,k,-3) \\ &\quad +4(-1)^{k+1}K(n,k+1,k+1,-3) \\ &\quad -12(-1)^{k+2}K(n,k+2,k+2,-3) + \ldots \end{aligned}$$

The matrix with general term $(-1)^k K(n,k,k,2)$ is the Riordan array $\left(\frac{1}{1-x}, \frac{x(1-2x)}{1-x}\right)$ or

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 1 & -1 & -1 & 1 & 0 & 0 & \ldots \\ 1 & -2 & -2 & -2 & 1 & 0 & \ldots \\ 1 & -3 & -2 & -2 & -3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The rows of this matrix A098593 are the anti-diagonals (and a signed version of the diagonals) of the so-called Krawtchouk matrices [90, 91] which are defined as the family of $(N+1) \times (N+1)$ matrices with general term

$$K_{ij}^{(N)} = \sum_k (-1)^k \binom{j}{k} \binom{N-j}{i-k}.$$

The matrix with general term $T(n,k) = (-1)^k K(n,k,k,-1)$ is the well-known Delannoy number triangle $\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$ A008288 given by

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 5 & 5 & 1 & 0 & 0 & \ldots \\ 1 & 7 & 13 & 7 & 1 & 0 & \ldots \\ 1 & 9 & 25 & 25 & 9 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Thus in particular $(-1)^n K(2n,n,n,-1)$ is the general term of the sequence of Delannoy numbers $1, 3, 13, 63, \ldots$ A001850. We have

Proposition 177. *The array with general term $T(n,k) = [k \le n](-1)^k K(n,k,k,q)$ is the Riordan array $\left(\frac{1}{1-x}, \frac{x(1-qx)}{1-x}\right)$.*

Using results in [153], we have the following simple relationships between the terms of these matrices, where $T_{n,k}^{(q)} = T^{(q)}(n,k)$ denotes the n,k-th element of $\left(\frac{1}{1-x}, \frac{x(1-qx)}{1-x}\right)$:

$$T_{n+1,k+1}^{(q)} = T_{n,k}^{(q)} + T_{n,k+1}^{(q)} - qT_{n-1,k}^{(q)}.$$

Thus for instance the elements of the Delannoy matrix above satisfy

$$T_{n+1,k+1}^{(-1)} = T_{n,k}^{(-1)} + T_{n,k+1}^{(-1)} + T_{n-1,k}^{(-1)}.$$

Example 178. We now turn our attention to the expression $(-1)^{n-k}K(n-k, n-k, n, q)$. We have

$$\begin{aligned} (-1)^{n-k}K(n-k,n-k,n,q) &= (-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{n}{j}\binom{n-k-n}{n-k-j}(q-1)^{n-k-j} \\ &= (-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{n}{j}\binom{-k}{n-k-j}(q-1)^{n-k-j} \\ &= \sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n}{j}\binom{n-j-1}{n-k-j}(q-1)^{n-k-j} \\ &= \sum_{j=0}^{n-k}\binom{n-j-1}{n-k-j}q^{n-k-j} \\ &= [x^n]\frac{1}{1-x}\left(\frac{x}{1-qx}\right)^k. \end{aligned}$$

Thus the matrix with the general term $T(n,k;q)=(-1)^{n-k}K(n-k,n-k,n,q)$ is the Riordan array $\left(\frac{1}{1-x},\frac{x}{1-qx}\right)$. Taking the q-th inverse binomial transform of this array, we obtain

$$\left(\frac{1}{1+qx},\frac{x}{1+qx}\right)*\left(\frac{1}{1-x},\frac{x}{1-qx}\right)=\left(\frac{1}{1+(q-1)x},x\right).$$

Reversing this equality gives us

$$\left(\frac{1}{1-x},\frac{x}{1-qx}\right)=\left(\frac{1}{1-qx},\frac{x}{1-qx}\right)*\left(\frac{1}{1+(q-1)x},x\right).$$

Thus

$$(-1)^{n-k}K(n-k,n-k,n,q)=\sum_{j=k}^{n}\binom{n}{j}q^{n-j}(1-q)^{j-k}.$$

The row sums of the Riordan array $\left(\frac{1}{1-x},\frac{x}{1-qx}\right)$ have generating function

$$\frac{\frac{1}{1-x}}{1-\frac{x}{1-qx}}=\frac{1-qx}{(1-x)(1-(q+1)x)}.$$

This is thus the generating function of the sum

$$\sum_{k=0}^{n}(-1)^{n-k}K(n-k,n-k,n,q)=\sum_{k=0}^{n}\sum_{j=k}^{n}\binom{n}{j}q^{n-j}(1-q)^{j-k}=\frac{(1+q)^n-(1-q)}{q}.$$

We remark that $(-1)^kK(k,k,n,q)$ is a triangle given by the reverse of the Riordan array $\left(\frac{1}{1-x},\frac{x}{1-qx}\right)$, and will thus have the same row sums and central coefficients.

The A-sequence of this array is simply $1+qx$, which implies that

$$K(n-k,n-k,n+1,q)=-K(n-k-1,n-k-1,n,q)+qK(n-k-2,n-k-2,n,q).$$

Example 179. We now consider the expression $(-1)^{n-k}K(n-k,n-k,k,q)$. We have

$$\begin{aligned}(-1)^{n-k}K(n-k,n-k,k,q) &= (-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{k}{j}\binom{n-k-k}{n-k-j}(q-1)^{n-k-j}\\ &= \sum_{j=0}^{n-k}\binom{k}{j}\binom{n-2k}{n-k-j}(1-q)^{n-k-j}.\end{aligned}$$

This is the (n,k)-th element $T(n,k;q)$ of the Riordan array

$$\left(\frac{1}{1+(q-1)x},x(1+qx)\right).$$

Other expressions for $T(n,k;q)$ include

$$\begin{aligned} T(n,k;q) &= \sum_{j=0}^{n-k}\binom{k}{n-k-j}(1-q)^j q^{n-k-j} \\ &= \sum_{j=0}^{n-k}\sum_{i=0}^{k}\binom{k}{i}\binom{k-i}{n-k-i-j}(-1)^j(q-1)^{i+j}, \end{aligned}$$

hence these provide alternative expressions for $(-1)^{n-k}K(n-k,n-k,k,q)$.

We note that for $q=1$, we obtain the Riordan array $(1,x(1+x))$ whose inverse is the array $(1,xc(x))$. The row sums of $(1,x(1+x))$ are $F(n+1)$, thus giving us

$$\sum_{k=0}^{n}(-1)^{n-k}K(n-k,n-k,k,1)=F(n+1).$$

Similarly, we find

$$\sum_{k=0}^{n}(-1)^{n-k}K(n-k,n-k,k,0)=n+1.$$

$\sum_{k=0}^{n}(-1)^{n-k}K(n-k,n-k,k,-1)$ is the sequence $1,3,6,11,21,42,\ldots$ A024495 with generating function $\frac{1}{(1-x)^3-x^3}$.

These matrices have the interesting property that $T(2n,n;q)=1$. This is so since

$$\begin{aligned} T(2n,n;q) &= \sum_{j=0}^{2n-n}\binom{n}{2n-n-j}(1-q)^j q^{2n-n-j} \\ &= \sum_{j=0}^{n}\binom{n}{n-j}(1-q)^j q^{n-j} \\ &= \sum_{j=0}^{n}\binom{n}{j}(1-q)^j q^{n-j} \\ &= (1-q+q)^n=1. \end{aligned}$$

Thus we have

$$K(n,n,n,q)=(-1)^n.$$

The A-sequence for these arrays has generating function

$$A(x)=\frac{1+\sqrt{1+4qx}}{2}$$

and thus we have

$$a_0=1, \qquad a_n=(-1)^{n-1}q^nC_{n-1}, \qquad n>0.$$

With these values we therefore have

$$\begin{aligned} (-1)^{n-k}K(n-k,n-k,k+1,q) &= (-1)^{n-k}K(n-k,n-k,k,q) \\ &\quad +a_1(-1)^{n-k-1}K(n-k-1,n-k-1,k+1,q)+\ldots \end{aligned}$$

Example 180. We next look at the expression $(-1)^{n-k}K(n,n-k,k,q)$. We have

$$\begin{aligned}(-1)^{n-k}K(n,n-k,k,q) &= (-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{k}{j}\binom{n-k}{n-k-j}(q-1)^{n-k-j}\\ &= \sum_{j=0}^{n-k}\binom{k}{j}\binom{n-k}{n-k-j}(1-q)^{n-k-j}.\end{aligned}$$

This is the general term $T(n,k;q)$ of the Riordan array

$$\left(\frac{1}{1+(q-1)x}, \frac{x(1+qx)}{1+(q-1)x}\right).$$

Expressing $T(n,k;q)$ differently allows us to write

$$\sum_{j=0}^{n-k}\binom{k}{j}\binom{n-k}{n-k-j}(1-q)^{n-k-j} = \sum_{j=0}^{k}\binom{n}{j}\binom{n-j}{n-k-j}q^j(1-q)^{n-k-j}.$$

The central coefficients of these arrays, $T(2n,n;q)$, have generating function $e^{(2-q)x}I_0(2\sqrt{1-q}x)$ and represent the n-th terms in the expansion of $(1+(2-q)x+(1-q)x^2)^n$.

The A-sequence for this family of arrays has generating function

$$A(x) = \frac{1+(1-q)x+\sqrt{1+2x(1+q)+(q-1)^2x^2}}{2}.$$

Example 181. The expression $K(n,n-k,N,q)$ is the general term of the Riordan array

$$\left(\frac{(1-qx)^N}{1-(q-1)x}, \frac{x}{1-(q-1)x}\right).$$

This implies that

$$\sum_{j=0}^{n-k}\binom{N}{j}\binom{n-N}{n-k-j}(-1)^j(q-1)^{n-k-j} = \sum_{j=0}^{n-k}\binom{N}{j}\binom{n-j}{n-k-j}(-1)^jq^j(q-1)^{n-k-j}.$$

The A-sequence for this family of arrays is given by $1+(q-1)x$. Thus we obtain

$$K(n+1,n-k,N,q) = K(n,n-k,N,q)+(q-1)K(n,n-k-1,N,q).$$

Example 182. In this example, we indicate that summing over one of the parameters can still lead to a Riordan array. Thus the expression

$$\sum_{i=0}^{n-k}(-1)^iK(n-k,i,n,q)$$

is equivalent to the general term of the Riordan array

$$\left(\frac{1}{1-2x}, \frac{x}{1-qx}\right)$$

while the expression

$$\sum_{i=0}^{n-k} K(n-k,i,n,q)$$

is equivalent to the general term of the Riordan array

$$\left(1, \frac{x}{1+qx}\right).$$

Thus

$$\begin{aligned}\sum_{i=0}^{n-k}(-1)^i K(n-k,i,n,q) &= \sum_{i=0}^{n-k}\sum_{j=0}^{i}\binom{n}{j}\binom{k+i-j-1}{i-j}(q-1)^{i-j} \\ &= \sum_{j=0}^{n-k}\binom{j+k-1}{j}2^{n-k-j}q^j\end{aligned}$$

and

$$\sum_{i=0}^{n-k} K(n-k,i,n,q) = \binom{n-1}{n-k}(-q)^{n-k}.$$

The A-sequence for this example is given by $1+qx$, and so for example we have

$$\sum_{i=0}^{n-k}(-1)^i K(n-k,i,n+1,q) = \sum_{i=0}^{n-k}(-1)^i K(n-k,i,n,q) + q\sum_{i=0}^{n-k-1}(-1)^i K(n-k-1,i,n,q).$$

Example 183. The Riordan arrays encountered so far have all been of an elementary nature. The next example indicates that this is not always so. We make the simple change of $2n$ for n in the third parameter in the previous example. We then find that $\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,q)$ is the general term of the Riordan array

$$\left(\frac{1-2x-q(2-q)x^2}{1+qx}, \frac{x}{(1+qx)^2}\right)^{-1}.$$

For instance, $\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,1)$ represents the general term of the Riordan array

$$\left(\frac{1}{2}\left(\frac{1}{1-4x}+\frac{1}{\sqrt{1-4x}}\right), \frac{1-2x-\sqrt{1-4x}}{2x}\right)$$

while $\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,2)$ represents the general term of

$$\left(\frac{1}{\sqrt{1-8x}}, \frac{1-4x-\sqrt{1-8x}}{2x}\right).$$

The A-sequence for the first array above is $(1+x)^2$, so that we obtain

$$\begin{aligned}\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2(n+1),1) &= \sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,1)\\ &+2\sum_{i=0}^{n-k-1}(-1)^i K(n-k-1,i,2n,1)\\ &+\sum_{i=0}^{n-k-2}(-1)^i K(n-k-2,i,2n,1)\end{aligned}$$

while that of the second array is $(1+2x)^2$ and so

$$\begin{aligned}\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2(n+1),2) &= \sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,2)\\ &+4\sum_{i=0}^{n-k-1}(-1)^i K(n-k-1,i,2n,2)\\ &+4\sum_{i=0}^{n-k-2}(-1)^i K(n-k-2,i,2n,2).\end{aligned}$$

We summarize these examples in the following table.

Table 1. Summary of Riordan arrays

Krawtchouk expression	Riordan array	g.f. for A-sequence
$K(k,n-k,r,q)$	$\left(\left(\frac{1-x}{1+(q-1)x}\right)^r, x(1+(q-1)x)\right)$	$\frac{1+\sqrt{1+4(q-1)x}}{2}$
$(-1)^{n-k}K(k,n-k,r,q)$	$\left(\left(\frac{1+x}{1-(q-1)x}\right)^r, x(1-(q-1)x)\right)$	$\frac{1+\sqrt{1-4(q-1)x}}{2}$
$(-1)^k K(n,k,k,q)$	$\left(\frac{1}{1-x}, \frac{x(1-qx)}{1-x}\right)$	$\frac{1+x+\sqrt{1+2x(1-2q)+x^2}}{2}$
$(-1)^{n-k}K(n-k,n-k,k,q)$	$\left(\frac{1}{1-x}, \frac{x}{1-qx}\right)$	$1+qx$
$(-1)^{n-k}K(n-k,n-k,k,q)$	$\left(\frac{1}{1+(q-1)x}, x(1+qx)\right)$	$\frac{1+\sqrt{1+4qx}}{2}$
$(-1)^{n-k}K(n,n-k,k,q)$	$\left(\frac{1}{1+(q-1)x}, \frac{x(1+qx)}{1+(q-1)x}\right)$	$\frac{1+(1-q)x+\sqrt{1+2x(1+q)+(q-1)^2x^2}}{2}$
$K(n,n-k,N,q)$	$\left(\frac{(1-qx)^N}{1-(q-1)x}, \frac{x}{1-(q-1)x}\right)$	$1+(q-1)x$
$\sum_{i=0}^{n-k}(-1)^i K(n-k,i,n,q)$	$\left(\frac{1}{1-2x}, \frac{x}{1-qx}\right)$	$1+qx$
$\sum_{i=0}^{n-k} K(n-k,i,n,q)$	$\left(1, \frac{x}{1+qx}\right)$	$1-qx$
$\sum_{i=0}^{n-k}(-1)^i K(n-k,i,2n,q)$	$\left(\frac{1-2x-q(2-q)x^2}{1+qx}, \frac{x}{(1+qx)^2}\right)^{-1}$	$(1+qx)^2$

Chapter 10

On Integer-Sequence-Based Constructions of Generalized Pascal Triangles [1]

10.1 Introduction

In this chapter, we look at two methods of using given integer sequences to construct generalized Pascal matrices. In the first method, we look at the number triangle associated with the square matrix $\mathbf{BDB}'$, where $\mathbf{B}$ is the binomial matrix $\binom{n}{k}$ and $\mathbf{D}$ is the diagonal matrix defined by the given integer sequence. We study this construction in some depth, and characterize the sequences related to the central coefficients of the resulting triangles in a special case. We study the cases of the Fibonacci and Jacobsthal numbers in particular. The second construction is defined in terms of a generalization of $\exp(\mathbf{M})$, where $\mathbf{M}$ is a sub-diagonal matrix defined by the integer sequence in question. Our look at this construction is less detailed. It is a measure of the ubiquity of the Narayana numbers that they arise in both contexts.

The plan of the chapter is as follows. We begin with an introductory section, where we define what we will understand as a *generalized Pascal matrix*, as well as looking briefly at the binomial transform. The next section looks at the Narayana numbers, which will be used in subsequent sections. The next preparatory section looks at the reversion of the expressions $\frac{x}{1+\alpha x+\beta x^2}$ and $\frac{x(1-ax)}{1-bx}$, which are closely related to subsequent work. We then introduce the first family of generalized Pascal triangles, and follow this by looking at those elements of this family that correspond to the "power" sequences $n \to r^n$, while the section after that takes the specific cases of the Fibonacci and Jacobsthal numbers. We close the study of this family by looking at the generating functions of the columns of these triangles in the general case.

The final sections briefly study an alternative construction based on a generalized matrix exponential construction, as well as a generalized exponential Riordan array, and associated generalized Stirling and Charlier arrays.

[1] This chapter reproduces and extends the content of the published article "P. Barry, On integer-sequence-based constructions of generalized Pascal triangles, J. Integer Seq., **9** (2006), Art. 06.2.4." [16].

10.2 Preliminaries

Pascal's triangle, with general term $\binom{n}{k}$, $n, k \geq 0$, has fascinated mathematicians by its wealth of properties since its discovery [77]. Viewed as an infinite lower-triangular matrix, it is invertible, with an inverse whose general term is given by $(-1)^{n-k}\binom{n}{k}$. Invertibility follows from the fact that $\binom{n}{n} = 1$. It is *centrally symmetric*, since by definition, $\binom{n}{k} = \binom{n}{n-k}$. All the terms of this matrix are integers.

By a generalized Pascal triangle we shall understand a lower-triangular infinite integer matrix $T = T(n,k)$ with $T(n,0) = T(n,n) = 1$ and $T(n,k) = T(n,n-k)$. We shall index all matrices in this paper beginning at the $(0,0)$-th element.

We shall use transformations that operate on integer sequences during the course of this chapter. An example of such a transformation that is widely used in the study of integer sequences is the so-called *binomial transform* [230], which associates to the sequence with general term a_n the sequence with general term b_n where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k. \tag{10.1}$$

If we consider the sequence with general term a_n to be the vector $\mathbf{a} = (a_0, a_1, \ldots)$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector by the lower-triangle matrix $\mathbf{B}$ whose (n,k)-th element is equal to $\binom{n}{k}$:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & 0 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This transformation is invertible, with

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \tag{10.2}$$

We note that $\mathbf{B}$ corresponds to Pascal's triangle.

10.3 The Narayana Triangle

Example 184. An example of a well-known centrally symmetric invertible triangle that is not an element of the Riordan group is the Narayana triangle [212, 214] $\tilde{\mathbf{N}}$, defined by

$$\tilde{N}(n,k) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}$$

for $n,k \geq 0$. Other expressions for $\tilde{N}(n,k)$ are given by

$$\tilde{N}(n,k) = \binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1} = \binom{n+1}{k+1}\binom{n}{k} - \binom{n+1}{k}\binom{n}{k+1}.$$

This triangle (see A001263) begins

$$\tilde{\mathbf{N}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ 1 & 15 & 50 & 50 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We shall characterize this matrix in terms of a generalized matrix exponential construction later in this chapter. Note that in the literature, it is often the triangle $\tilde{N}(n-1,k-1) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ that is referred to as the Narayana triangle. Alternatively, the triangle $\tilde{N}(n-1,k) = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$ is referred to as the Narayana triangle. We shall denote this latter triangle by $\mathbf{N}$. We then have

$$\mathbf{N} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term

$$N(n,k) = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}.$$

Note that for $n,k \geq 1$, $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. We have, for instance,

$$\begin{aligned} \tilde{N}(n-1,k-1) &= \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \\ &= \binom{n}{k}^2 - \binom{n-1}{k}\binom{n+1}{k} \\ &= \binom{n}{k}\binom{n-1}{k-1} - \binom{n}{k-1}\binom{n-1}{k}. \end{aligned}$$

The last expression represents a 2×2 determinant of adjacent elements in Pascal's triangle. The row sums of the Narayana triangle $\mathbf{N}$ give the Catalan numbers $C_n = \binom{2n}{n}/(n+1)$, A000108.

As we see from above, the Narayana triangle has several forms. Two principal ones can be distinguished by both their Deleham representation as well as their continued fraction generating functions. Thus $\mathbf{N}$ as defined above is the Deleham array

$$[1,0,1,0,1,0,1,\ldots] \quad \Delta \quad [0,1,0,1,0,1,\ldots]$$

with generating function

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cdots}}}}}$$

(corresponding to the series reversion of $\frac{x(1-xy)}{1-(y-1)x}$). This generating function can also be expressed as

$$\frac{1-(1-y)x-\sqrt{1-2x(1+y)+(1-y)^2x^2}}{2xy}.$$

We have

$$N(n,k) = 0^{n+k} + \frac{1}{n+0^n}\binom{n}{k}\binom{n}{k+1}.$$

This triangle is A131198.

The Narayana triangle $\tilde{\tilde{\mathbf{N}}}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 1 & 0 & \ldots \\ 0 & 1 & 10 & 20 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with generating function

$$\frac{1+(1-y)x-\sqrt{1-2x(1+y)+(1-y)^2x^2}}{2x}$$

is the Deleham array

$$[0,1,0,1,0,1,\ldots] \quad \Delta \quad [1,0,1,0,1,0,1,\ldots]$$

with generating function

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cdots}}}}}}$$

or

$$\cfrac{1}{1-xy-\cfrac{x^2y}{1-x(1+y)-\cfrac{x^2y}{1-x(1+y)-\cfrac{x^2y}{1-\cdots}}}}. \tag{10.3}$$

It has general term

$$\tilde{\tilde{N}} = 0^{n+k} + \frac{1}{n+0^{nk}}\binom{n}{k}\binom{n}{k-1}$$

which corresponds to

$$[x^{n+1}]\text{Rev}\frac{x(1-x)}{1-(1-y)x}.$$

The Narayana triangle $\tilde{\mathbf{N}}$

$$\tilde{\mathbf{N}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ 1 & 15 & 50 & 50 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term

$$\tilde{N}(n,k) = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} = [x^{n+1}y^k]\text{Rev}\frac{x}{1+(1+y)x+yx^2}$$

and generating function

$$\frac{1-x(1+y)-\sqrt{1-2x(1+y)+(1-y)^2x^2}}{2x^2y}$$

is then given by

$$[0,1,0,1,0,1,\dots] \quad \Delta^{(1)} \quad [1,0,1,0,1,0,1,\dots].$$

This is A090181. We can express its generating function as a continued fraction as

$$\cfrac{1}{1-xy-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}$$

and as

$$\cfrac{1}{1-x(1+y)-\cfrac{x^2y}{1-x(1+y)-\cfrac{x^2y}{1-x(1+y)-\cfrac{x^2y}{1-\cdots}}}}. \tag{10.4}$$

An interesting identity is the following, [50, 56]:

$$\sum_{k=0}^{n-1}\frac{1}{n}\binom{n}{k}\binom{n}{k+1}x^k=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{2k}C_kx^k(1+x)^{n-2k-1} \tag{10.5}$$

where C_n is the n-th Catalan number. This identity can be interpreted in terms of Motzkin paths, where by a *Motzkin path* of length n we mean a lattice path in $\mathbf{Z}^2$ between $(0,0)$ and $(n,0)$ consisting of up-steps $(1,1)$, down-steps $(1,-1)$ and horizontal steps $(1,0)$ which never go below the x-axis. Similarly, a *Dyck* path of length $2n$ is a lattice path in $\mathbf{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-paths $(1,1)$ and down-steps $(1,-1)$ which never go below the x-axis. Finally, a (large) *Schröder* path of length n is a lattice path from $(0,0)$ to (n,n) containing no points above the line $y=x$, and composed only of steps $(0,1)$, $(1,0)$ and $(1,1)$.

For instance, the number of Schröder paths from $(0,0)$ to (n,n) is given by the large Schröder numbers $1,2,6,22,90,\ldots$ which correspond to $z=2$ for the *Narayana polynomials* [212, 214]

$$N_n(z)=\sum_{k=1}^{n}\frac{1}{n}\binom{n}{k-1}\binom{n}{k}z^k.$$

10.4 On the series reversion of $\frac{x}{1+\alpha x+\beta x^2}$ and $\frac{x(1-ax)}{1-bx}$

A number of the properties of the triangles that we will study are related to the special cases of the series reversions of $\frac{x}{1+\alpha x+\beta x^2}$ and $\frac{x(1-ax)}{1-bx}$ where $b=a-1$, $\alpha=a+1$ and $\beta=b+1$. We shall develop results relating to these reversions in full generality in this section and specialize later at the appropriate places.

Solving the equation

$$\frac{y}{1+\alpha y+\beta y^2}=x$$

yields

$$y_1 = \frac{1 - \alpha x - \sqrt{1 - 2\alpha x + (\alpha^2 - 4\beta)x^2}}{2\beta x}$$

while solving the equation

$$\frac{y(1-ay)}{1-by} = x$$

leads to

$$y_2 = \frac{1 + bx - \sqrt{(1+bx)^2 - 4ax}}{2a}.$$

We shall occasionally use the notation $y_1(\alpha, \beta)$ and $y_2(a, b)$ where relevant for these functions. Note for instance that $\frac{y_2(1,0)}{x} = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers.

Proposition 185. *Let $\alpha = a + 1$, $\beta = b + 1$, and assume that $b = a - 1$ (and hence, $\beta = \alpha - 1$). Then*

$$\frac{y_2}{x} - y_1 = 1.$$

Proof. Straight-forward calculation. □

Note that 1 is the generating function of $0^n = 1, 0, 0, 0, \ldots$.

Example 186. Consider the case $a = 2$, $b = 1$. Let $\alpha = 3$ and $\beta = 2$, so we are considering $\frac{x}{1+3x+2x^2}$ and $\frac{x(1-2x)}{1-x}$. We obtain

$$\begin{aligned} y_1(3,2) &= \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x} \\ \frac{y_2(2,1)}{x} &= \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x} \\ \frac{y_2(2,1)}{x} - y_1(3,2) &= 1. \end{aligned}$$

Thus $y_1(3,2)$ is the generating function for $0, 1, 3, 11, 45, 197, 903, 4279, \ldots$ while $\frac{y_2(2,1)}{x}$ is the generating function for $1, 1, 3, 11, 45, 197, 903, 4279, \ldots$. These are the little Schröder numbers A001003.

Example 187. We consider the case $a = 1$, $b = 1 - r$, that is, the case of $\frac{x(1-x)}{1-(1-r)x}$. We obtain

$$\begin{aligned} \frac{y_2(1, 1-r)}{x} &= \frac{1 - (r-1)x - \sqrt{(1 + (1-r)x)^2 - 4x}}{2x} \\ &= \frac{1 - (r-1)x - \sqrt{1 - 2(r+1)x + (r-1)^2x^2}}{2x} \end{aligned}$$

Example 188. We calculate the expression $\frac{y_2(1,1-r)}{rx}-\frac{1-r}{r}$. We get

$$\begin{aligned}\frac{y_2(1,1-r)}{rx}-\frac{1-r}{r} &= \frac{1-(r-1)x-\sqrt{1-2(r+1)x+(r-1)^2x^2}}{2rx}+\frac{2(r-1)x}{2rx}\\ &= \frac{1+(r-1)x-\sqrt{1-2(r+1)x+(r-1)^2x^2}}{2rx}\\ &= \frac{y_2(r,r-1)}{x}.\end{aligned}$$

In other words,

$$\frac{y_2(r,r-1)}{x}=\frac{y_2(1,1-r)}{rx}-\frac{1-r}{r}.$$

A well-known example of this is the case of the large Schröder numbers with generating function $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$ and the little Schröder numbers with generating function $\frac{1+x-\sqrt{1-6x+x^2}}{4x}$. In this case, $r=2$. Generalizations of this "pairing" for $r>2$ will be studied in a later section. For $r=1$ both sequences coincide with the Catalan numbers C_n.

Proposition 189. *The binomial transform of*

$$\frac{y_1}{x}=\frac{1-\alpha x-\sqrt{1-2\alpha x+(\alpha^2-4\beta)x^2}}{2\beta x^2}$$

is

$$\frac{1-(\alpha+1)x-\sqrt{1-2(\alpha+1)x+((\alpha+1)^2-4\beta)x^2}}{2\beta x^2}.$$

Proof. The binomial transform of $\frac{y_1}{x}$ is

$$\begin{aligned}&\frac{1}{1-x}\left\{1-\frac{\alpha x}{1-x}-\sqrt{1-\frac{2\alpha x}{1-x}+(\alpha^2-4\beta)\frac{x^2}{(1-x)^2}}\right\}/(2\beta\frac{x^2}{(1-x)^2})\\ =\;& (1-x-\alpha x-\sqrt{(1-x)^2-2\alpha x(1-x)+(\alpha^2-4\beta)x^2})/(2\beta x^2)\\ =\;& (1-(\alpha+1)x-\sqrt{1-2(\alpha+1)x+(\alpha^2+2\alpha+1-4\beta^2)x^2})/(2\beta x^2)\\ =\;& \frac{1-(\alpha+1)x-\sqrt{1-2(\alpha+1)x+((\alpha+1)^2-4\beta)x^2}}{2\beta x^2}.\end{aligned}$$

□

In other words, the binomial transform of $[x^{n+1}]\text{Rev}\frac{x}{1+ax+bx^2}$ is given by $[x^{n+1}]\text{Rev}\frac{x}{1+(a+1)x+bx^2}$.

Example 190. The binomial transform of $1, 3, 11, 45, 197, 903, \ldots$ with generating function $\frac{1-3x-\sqrt{1-6x+x^2}}{4x^2}$ is $1, 4, 18, 88, 456, 2464, 13736, \ldots$, A068764, with generating function $\frac{1-4x-\sqrt{1-8x+8x^2}}{4x^2}$. Thus the binomial transform links the series reversion of $x/(1+3x+2x^2)$ to that of $x/(1+4x+2x^2)$. We note that this can be interpreted in the context of Motzkin paths as an incrementing of the colours available for the $\mathrm{H}(1,0)$ steps.

We now look at the general terms of the sequences generated by y_1 and y_2. We use the technique of Lagrangian inversion for this. We begin with y_1. In order to avoid notational overload, we use a and b rather than α and β, hoping that confusion won't arise.

Since for y_1 we have $y = x(1 + ay + by^2)$ we can apply Lagrangian inversion to get the following expression for the general term of the sequence generated by y_1:

$$[t^n]y_1 = \frac{1}{n}[t^{n-1}](1 + at + bt^2)^n.$$

At this point we remark that there are many ways to develop the trinomial expression, and the subsequent binomial expressions. Setting these different expressions equal for different combinations of a and b and different relations between a and b can lead to many interesting combinatorial identities, many of which can be interpreted in terms of Motzkin paths. We shall confine ourselves to the derivation of two particular expressions. First of all,

$$\begin{aligned}
[t^n]y_1 &= \frac{1}{n}[t^{n-1}](1 + at + bt^2)^n \\
&= \frac{1}{n}[t^{n-1}]\sum_{k=0}^{n}\binom{n}{k}(at + bt^2)^k \\
&= \frac{1}{n}[t^{n-1}]\sum_{k=0}^{n}\binom{n}{k}t^k\sum_{j=0}^{k}\binom{k}{j}a^j b^{k-j} t^{k-j} \\
&= \frac{1}{n}[t^{n-1}]\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}a^j b^{k-j} t^{2k-j} \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{k}{n-k-1}a^{2k-n+1}b^{n-k-1} \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{k}{2k-n+1}a^{2k-n+1}b^{n-k-1}.
\end{aligned}$$

Of the many other possible expressions for $[t^n]y_1$, we cite the following examples:

$$\begin{aligned}
[t^n]y_1 &= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{k+1}{2k-n-1}a^{2k-n+1}b^{n-k-1} \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{k}{2k-n+1}b^{n-k-1}a^{2k-n+1} \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k-1}b^{k-1}a^{n-2k+1} \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k+1}\binom{n-k-1}{k+1}b^{k}a^{n-2k}.
\end{aligned}$$

We shall be interested at a later stage in generalized Catalan sequences. The following interpretation of $[t^n]y_1$ is therefore of interest.

Proposition 191.

$$[t^n]y_1 = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k a^{n-2k-1} b^k.$$

Proof.

$$\begin{aligned}
[t^n]y_1 &= \frac{1}{n}[t^{n-1}](1+at+bt^2)^n \\
&= \frac{1}{n}[t^{n-1}](at+(1+bt^2))^n \\
&= \frac{1}{n}[t^{n-1}]\sum_{j=0}^{n}\binom{n}{j}a^j t^j (1+bt^2)^{n-j} \\
&= \frac{1}{n}[t^{n-1}]\sum_{j=0}^{n}\sum_{k=0}^{n-j}\binom{n}{j}\binom{n-j}{k}a^j b^k t^{2k+j} \\
&= \frac{1}{n}\sum_{k=0}\binom{n}{n-2k-1}\binom{2k+1}{k}a^{n-2k-1}b^k \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{2k+1}\binom{2k+1}{k}a^{n-2k-1}b^k \\
&= \frac{1}{n}\sum_{k=0}\frac{n}{2k+1}\binom{n-1}{n-2k-1}\frac{2k+1}{k+1}\binom{2k}{k}a^{n-2k-1}b^k \\
&= \sum_{k=0}\binom{n-1}{2k}C_k a^{n-2k-1}b^k.
\end{aligned}$$

□

Corollary 192.

$$\begin{aligned}
C_n &= 0^n + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1}{2k}C_k 2^{n-2k-1} \\
C_{n+1} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}C_k 2^{n-2k}.
\end{aligned}$$

Proof. The sequence $C_n - 0^n$, or $0, 1, 2, 5, 14, \ldots$, has generating function

$$\frac{1-\sqrt{1-4x}}{2x} - 1 = \frac{1-2x-\sqrt{1-4x}}{2x}$$

which corresponds to $y_1(2,1)$. □

This is the formula of Touchard [220], with adjustment for the first term.

Corollary 193.

$$[t^n]y_1(r+1,r) = \sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k}\binom{n}{k+1} r^k.$$

Proof. By the proposition, we have

$$[t^n]y_1(r+1,r) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k (r+1)^{n-2k-1} r^k.$$

The result then follows from identity (10.5). □

This therefore establishes a link to the Narayana numbers.

Corollary 194.

$$[x^{n+1}]Rev\frac{x}{1+ax+bx^2} = \sum_{k=0}^{n} \binom{n}{2k} C_k a^{n-2k} b^k.$$

We note that the generating function of $[x^{n+1}]\text{Rev}\frac{x}{1+ax+bx^2}$ has the following simple continued fraction expansion:

$$\cfrac{1}{1-ax-\cfrac{bx^2}{1-ax-\cfrac{bx^2}{1-ax-\cfrac{bx^2}{1-ax-\cfrac{bx^2}{1-ax-\cdots}}}}}$$

We note that this indicates that $[x^{n+1}]\text{Rev}\frac{x}{1+ax+bx^2}$ is the a-th binomial transform of $[x^{n+1}]\text{Rev}\frac{x}{1+bx^2}$. More generally, we note that the generating function of the reversion of

$$\frac{x(1+cx)}{1+ax+bx^2}$$

is given by

$$\cfrac{x}{1-(a-c)x-\cfrac{(b-ac+c^2)x^2}{1-(a-2c)-\cfrac{(b-ac+c^2)x^2}{1-(a-2c)-\cfrac{(b-ac+c^2)x^2}{1-\cdots}}}} \tag{10.6}$$

Corollary 195. *Let $s_n(a,b)$ be the sequence with general term*

$$s_n(a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k a^{n-2k} b^k.$$

Then the binomial transform of this sequence is the sequence $s_n(a+1,b)$ with general term

$$s_n(a+1,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k (a+1)^{n-2k} b^k.$$

Proof. This is a re-interpretation of the results of Proposition 189. □

We now examine the case of $[t^n]y_2$. In this case, we have

$$y = x\frac{1-by}{1-ay}$$

so we can apply Lagrangian inversion. Again, various expressions arise depending on the order of expansion of the binomial expressions involved. For instance,

$$\begin{aligned}
[t^n]y_2 &= \frac{1}{n}[t^{n-1}]\left(\frac{1-bt}{1-at}\right)^n \\
&= \frac{1}{n}[t^{n-1}](1-bt)^n(1-at)^{-n} \\
&= \frac{1}{n}[t^{n-1}]\sum_{k=0}^{n}\sum_{j=0}\binom{n}{k}\binom{n+j-1}{j}a^j(-b)^k t^{k+j} \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k-2}{n-1}a^{n-k-1}(-b)^k.
\end{aligned}$$

A more interesting development is given by the following.

$$\begin{aligned}
[t^n]\frac{y_2}{x} &= [t^{n+1}]y_2 \\
&= \frac{1}{n+1}[t^n](1-bt)^{n+1}(1-at)^{-(n+1)} \\
&= \frac{1}{n+1}[t^n]\sum_{k=0}^{n+1}\binom{n+1}{k}(-bt)^{n+1-k}\sum_{j=0}\binom{-n-1}{j}(-at)^j \\
&= \frac{1}{n+1}[t^n]\sum_{k=0}^{n+1}\sum_{j=0}\binom{n+1}{k}\binom{n+j}{j}(-b)^{n-k+1}a^j t^{n+1-k+j} \\
&= \frac{1}{n+1}\sum_{j=0}\binom{n+1}{j+1}\binom{n+j}{j}(-b)^{n-j}a^j \\
&= \sum_{j=0}^{n}\frac{1}{j+1}\binom{n}{j}\binom{n+j}{j}a^j(-b)^{n-j}.
\end{aligned}$$

An alternative expression obtained by developing for k above is given by

$$[t^n]\frac{y_2}{x} = \sum_{k=0}^{n+1}\frac{1}{n-k+1}\binom{n}{k}\binom{n+k-1}{k-1}a^{k-1}(-b)^{n-k+1}.$$

Note that the underlying matrix with general element $\frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}$ is A088617, whose general element gives the number of Schröder paths from $(0,0)$ to $(2n,0)$, having k $U(1,1)$ steps. Recognizing that $\sum_{j=0}^{n}\frac{1}{j+1}\binom{n}{j}\binom{n+j}{j}a^j(-b)^{n-j}$ is a convolution, we can also write

$$\begin{aligned}
[t^n]\frac{y_2}{x} &= \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}a^k(-b)^{n-k}\\
&= \sum_{k=0}^{n}\frac{1}{n-k+1}\binom{n}{n-k}\binom{2n-k}{n-k}a^{n-k}(-b)^k\\
&= \sum_{k=0}^{n}\frac{1}{n-k+1}\binom{n}{k}\binom{2n-k}{n}a^{n-k}(-b)^k\\
&= \sum_{k=0}^{n}\frac{1}{n-k+1}\binom{2n-k}{k}\binom{2n-k-k}{n-k}a^{n-k}(-b)^k\\
&= \sum_{k=0}^{n}\binom{2n-k}{k}\frac{1}{n-k+1}\binom{2n-2k}{n-k}a^{n-k}(-b)^k\\
&= \sum_{k=0}^{n}\binom{2n-k}{k}C_{n-k}a^{n-k}(-b)^k\\
&= \sum_{k=0}^{n}\binom{n+k}{2k}C_k a^k(-b)^{n-k}.
\end{aligned}$$

Again we note that the matrix with general term $\binom{n}{k}\binom{2n-k}{k}\frac{1}{n-k+1}=\binom{2n-k}{k}C_{n-k}$ is A060693, whose general term counts the number of Schröder paths from $(0,0)$ to $(2n,0)$, having k peaks. This matrix can be expressed as

$$[1,1,1,1,\ldots] \quad \Delta \quad [1,0,1,0,\ldots].$$

This matrix is closely linked to the Narayana numbers. The reverse of this triangle, with general term $\binom{n+k}{2k}C_k$ is A088617. Gathering these results leads to the next proposition.

Proposition 196. $[t^n]\frac{y_2(a,b)}{x}=[t^{n+1}]Rev\frac{x(1-ax)}{1-bx}$ *is given by the equivalent expressions*

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}a^k(-b)^{n-k}\\
&= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\binom{2n-k}{n}a^{n-k}(-b)^k\\
&= \sum_{k=0}^{n}\frac{1}{n-k+1}\binom{n}{k}\binom{2n-k}{n}a^{n-k}(-b)^k\\
&= \sum_{k=0}^{n}\binom{n+k}{2k}C_k a^k(-b)^{n-k}\\
&= \sum_{k=0}^{n}\binom{2n-k}{k}C_{n-k}a^{n-k}(-b)^k.
\end{aligned}$$

A further equivalent expression is closely linked to the Narayana numbers. We have

$$[x^{n+1}]\text{Rev}\frac{x(1-ax)}{1-bx} = \sum_{k=0}^{n} N(n,k)a^k(a-b)^{n-k}.$$

Further expressions include

$$\begin{aligned}[x^{n+1}]\text{Rev}\frac{x(1-ax)}{1-bx} &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n-1}{n-k}\binom{n+k}{k}(a-b)^k b^{n-k} \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n-1}{k}\binom{2n-k}{n-k}(a-b)^{n-k}b^k.\end{aligned}$$

The generating function of this sequence can in fact be realized as the following continued fraction :

$$\cfrac{1}{1-\cfrac{(a-b)x}{1-\cfrac{ax}{1-\cfrac{(a-b)x}{1-\cfrac{ax}{1-\cdots}}}}}.$$

We summarize some of these results in **Table 1**, where $C_n = \frac{1}{n+1}\binom{2n}{n}$, and $P(x) = 1-2(r+1)x+(r-1)^2x^2$, and $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. We use the terms "Little sequence" and "large sequence" in analogy with the Schröder numbers. In [203] we note that the terms "Little Schröder", "Big Schröder" and "Bigger Schröder" are used. For instance, the numbers $1, 3, 11, 45, \ldots$ appear there as the "Bigger Schröder" numbers.

Table 1. Summary of section results

Little sequence, s_n e.g. $1, 1, 3, 11, 45, \ldots$	Large sequence, S_n e.g. $1, 2, 6, 22, 90, \ldots$	Larger sequence $s_n - 0^n$ e.g. $0, 1, 3, 11, 45, \ldots$
$\frac{x(1-rx)}{1-(r-1)x}$	$\frac{x(1-x)}{1-(1-r)x}$	$\frac{x}{1+(r+1)x+rx^2}$
$\frac{1+(r-1)x-\sqrt{P(x)}}{2rx}$	$\frac{1-(r-1)x-\sqrt{P(x)}}{2x}$	$\frac{1-(r+1)x-\sqrt{P(x)}}{2rx}$
$a_0 = 1, a_n = \sum_{k=0}^{n} N(n,k)r^k$	$a_0 = 1, a_n = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k-1}r^k$	$\sum_{k=0}^{n-1} N(n,k)r^k$
$\sum_{k=0}^{n}\binom{n+k}{2k}C_k r^k(1-r)^{n-k}$	$\sum_{k=0}^{n}\binom{n+k}{2k}C_k(r-1)^{n-k}$	$\sum_{k=0}\binom{n-1}{2k}C_k(r+1)^{n-2k-1}r^k$
$\sum_{k=0}^{n}\binom{2n-k}{k}C_{n-k}r^{n-k}(1-r)^k$	$\sum_{k=0}^{n}\binom{2n-k}{k}C_{n-k}(r-1)^k$	-

Table 2. Little and Large sequences in OEIS

r	s_n	S_n	Triangle
0	A000012	A000007	
1	A000108	A000180	A007318
2	A001003	A006318	A008288
3	A007564	A047891	A081577
4	A059231	A082298	A081578
5	A078009	A082301	A081579
6	A078018	A082302	A081580
7	A081178	A082305	A143679
8	A082147	A082366	A143681
9	A082181	A082367	A143683
10	A082148	A143749	A143684

Note that by Example 188 we can write

$$s_n = \frac{1}{r}S_n + \frac{(r-1)0^n}{r}.$$

Also we have

$$S_n = r^n\,{}_2F_1\left(-n, -n+1; 2; \frac{1}{r}\right)$$

with generating function (see Example 25)

$$g(x;r) = \cfrac{1}{1-\cfrac{rx}{1-\cfrac{x}{1-\cfrac{rx}{1-\cfrac{x}{1-\cfrac{rx}{1-\ldots}}}}}}$$

or

$$g(x,r) = \cfrac{1}{1-rx-\cfrac{rx^2}{1-x(r+1)-\cfrac{rx^2}{1-x(r+1)-\cfrac{rx^2}{1-\cdots}}}} \tag{10.7}$$

and

$$s_{n+1} = r^n\,{}_2F_1\left(-n, -n-1; 2; \frac{1}{r}\right).$$

We also note that the Hankel transform of both $s_n = s_n(r)$ and $S_n = S_n(r)$ is given by $r^{\binom{n+1}{2}}$.

We observe that the coefficient array associated with the sequence $s_n(r)$ is in fact the

Riordan array $\left(\frac{1}{1+x}, \frac{x}{(1+x)(1+rx)}\right)^{-1}$. The production array of this matrix is given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ r & r+1 & 1 & 0 & 0 & 0 & \dots \\ 0 & r & r+1 & 1 & 0 & 0 & \dots \\ 0 & 0 & r & r+1 & 1 & 0 & \dots \\ 0 & 0 & 0 & r & r+1 & r & \dots \\ 0 & 0 & 0 & 0 & r & r+1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which defines the sequence of orthogonal polynomials associated to this sequence. In terms of moment representation, we find that

$$s_n(r) = \frac{1}{2r\pi}\int_{-1-2\sqrt{r}-r}^{-1+2\sqrt{r}-r} x^n \frac{\sqrt{-x^2-2x(1+r)-(1-r)^2}}{x}dx + \frac{r-1}{r}0^n.$$

Looking at the sequences $S_n^*(r)$, where

$$S_n^*(r) = \sum_{k=0}^{n-1} \binom{n+k-1}{2k} C_k (r-1)^{n-k-1} = [x^n]\mathrm{Rev}\left(\frac{x(1-x)}{1+(r-1)x}\right),$$

we obtain the following proposition :

Proposition 197. *The Hankel transform $H_n^*(r)$ of $S_n^*(r)$ is given by*

$$H_n^*(r) = \frac{1-r^n}{r-1} r^{\binom{n}{2}}.$$

In particular, H_{n+1}^* is equal to the number of normalized polynomials of degree exactly r in $\mathbb{F}_r[x,y]$ [34].

We note finally that

$$\begin{aligned} (r-1)^n {}_2F_1(-n, n+3; 2; -\frac{1}{r-1}) &= \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} r^k \\ &= \sum_{k=0}^{n} \tilde{N}(n,k) r^k \\ &= [x^{n+1}]\mathrm{Rev}\frac{x}{1+(r+1)x+rx^2} \end{aligned}$$

while

$$(r-1)^n {}_2F_1(-n, n+2; 1; -\frac{1}{r-1}) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+1}{k+1} r^k.$$

These latter sequences have e.g.f.

$$e^{(r+1)x}(I_0(2\sqrt{r}x) + I_1(2\sqrt{r}x)/\sqrt{r}).$$

The sequence with e.g.f. $I_0(2\sqrt{r}x)+I_1(2\sqrt{r}x)/\sqrt{r}$ has general term $\binom{n}{\lfloor\frac{n}{2}\rfloor}r^{\lfloor\frac{n}{2}\rfloor}$ and hence we have

$$\begin{aligned}(r-1)^n\,{}_2F_1\left(-n,n+2;1;-\frac{1}{r-1}\right) &= \sum_{k=0}^{n}\binom{n}{k}\binom{n+1}{k+1}r^k\\ &= \sum_{k=0}^{n}(n-k+1)\tilde{N}(n,k)r^k\\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{k}{\lfloor\frac{k}{2}\rfloor}r^{\lfloor\frac{k}{2}\rfloor}(r+1)^{n-k}.\end{aligned}$$

The triangle with general term $\binom{n}{k}\binom{n+1}{k+1}$ is A103371.

10.5 Introducing the family of centrally symmetric invertible triangles

The motivation for the construction that follows comes from the following easily established proposition.

Proposition 198.

$$\binom{n}{k}=\sum_{j=0}^{\min(k,n-k)}\binom{k}{j}\binom{n-k}{j}=\sum_{j=0}^{\infty}\binom{k}{j}\binom{n-k}{j}.$$

Proof. We consider identity 5.23 of [106]:

$$\binom{r+s}{r-p+q}=\sum_j\binom{r}{p+j}\binom{s}{q+j}$$

itself a consequence of Vandermonde's convolution identity. Setting $r=k$, $s=n-k$, $p=q=0$, we obtain

$$\binom{n}{k}=\sum_j\binom{k}{j}\binom{n-k}{j}.$$

□

Now let a_n represent a sequence of integers with $a_0=1$. We define an infinite array of numbers for $n,k\ge 0$ by

$$T(n,k)=\sum_{j=0}^{\min(k,n-k)}\binom{k}{j}\binom{n-k}{j}a_j.$$

and call it *the triangle associated with the sequence a_n by this construction.* That it is a number triangle follows from the next proposition.

Proposition 199. *The matrix with general term $T(n,k)$ is an integer-valued centrally symmetric invertible lower-triangular matrix.*

Proof. All elements in the sum are integers, hence $T(n,k)$ is an integer for all $n,k \geq 0$. $T(n,k)=0$ for $k>n$ since then $n-k<0$ and hence the sum is 0. We have

$$\begin{aligned}
T(n,n-k) &= \sum_{k=0}^{\min(n-k,n-(n-k))} \binom{n-k}{j}\binom{n-(n-k)}{j} a_j \\
&= \sum_{k=0}^{\min(n-k,k)} \binom{n-k}{j}\binom{k}{j} a_j.
\end{aligned}$$

□

It is clear that Pascal's triangle corresponds to the case where a_n is the sequence $1,1,1,\ldots$.

Occasionally we shall use the above construction on sequences a_n for which $a_0=0$. In this case we still have a centrally symmetric triangle, but it is no longer invertible, since for example $T(0,0)=0$ in this case.

By an abuse of notation, we shall often use $T(n,k;a_n)$ to denote the triangle associated to the sequence a_n by the above construction, when explicit mention of a_n is required.

The associated square symmetric matrix with general term

$$T_{sq}(n,k) = \sum_{j=0}^{n} \binom{k}{j}\binom{n}{j} a_j$$

is easy to describe. We let $\mathbf{D}=\mathbf{D}(a_n)=diag(a_0,a_1,a_2,\ldots)$. Then

$$\mathbf{T}_{sq} = \mathbf{B}\mathbf{D}\mathbf{B}'$$

is the square symmetric (infinite) matrix associated to our construction. Note that when $a_n=1$ for all n, we get the square Binomial or Pascal matrix $\binom{n+k}{k}$.

Among the attributes of the triangles that we shall construct that interest us, the family of central sequences (sequences associated to $T(2n,n)$ and its close relatives) will be paramount. The central binomial coefficients $\binom{2n}{n}$, A000984, play an important role in combinatorics. We begin our examination of the generalized triangles by characterizing their 'central coefficients' $T(2n,n)$. We obtain

$$\begin{aligned}
T(2n,n) &= \sum_{j=0}^{2n-n} \binom{2n-n}{j}\binom{n}{j} a_j \\
&= \sum_{j=0}^{n} \binom{n}{j}^2 a_j.
\end{aligned}$$

For the case of Pascal's triangle with a_n given by $1, 1, 1, \ldots$ we recognize the identity $\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$. In like fashion, we can characterize $T(2n+1, n)$, for instance.

$$\begin{aligned} T(2n+1, n) &= \sum_{j=0}^{2n+1-n} \binom{2n+1-n}{j} \binom{n}{j} a_j \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{n}{j} a_j \end{aligned}$$

which generalizes the identity $\binom{2n+1}{n} = \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{n}{j}$. This is A001700. We also have

$$\begin{aligned} T(2n-1, n-1) &= \sum_{j=0}^{2n-1-n+1} \binom{2n-1-n+1}{j} \binom{n-1}{j} a_j \\ &= \sum_{j=0}^{n} \binom{n-1}{j} \binom{n}{j} a_j. \end{aligned}$$

This generalizes the equation $\binom{2n-1}{n-1} + 0^n = \sum_{j=0}^{n} \binom{n-1}{j} \binom{n}{j}$. See A088218.

In order to generalize the Catalan numbers C_n, A000108, in our context, we note that $C_n = \binom{2n}{n}/(n+1)$ has the alternative representation

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \binom{2n}{n} - \binom{2n}{n+1}.$$

This motivates us to look at $T(2n, n) - T(2n, n-1) = T(2n, n) - T(2n, n+1)$. We obtain

$$\begin{aligned} T(2n, n) - T(2n, n-1) &= \sum_{j=0}^{n} \binom{n}{j}^2 a_j - \sum_{j=0}^{2n-n+1} \binom{n-1}{j} \binom{2n-n+1}{j} a_j \\ &= \sum_{j=0}^{n} \binom{n}{j}^2 a_j - \sum_{j=0}^{n+1} \binom{n-1}{j} \binom{n+1}{j} a_j \\ &= \delta_{n,0} a_n + \sum_{j=0}^{n} (\binom{n}{j}^2 - \binom{n-1}{j} \binom{n+1}{j}) a_j \\ &= \delta_{n,0} a_0 + \sum_{j=0}^{n} \tilde{N}(n-1, j-1) a_j \end{aligned}$$

where we use the formalism $\binom{n-1}{n+1} = -1$, for $n = 0$, and $\binom{n-1}{n+1} = 0$ for $n > 0$. We assume that $\tilde{N}(n, -1) = 0$ and $\tilde{N}(-1, k) = \binom{1}{k} - \binom{0}{k}$ in the above. For instance, in the case of Pascal's triangle, where $a_n = 1$ for all n, we retrieve the Catalan numbers. We have also established a link between these generalized Catalan numbers and the Narayana numbers. We shall use the notation

$$c(n; a(n)) = T(2n, n) - T(2n, n-1) = T(2n, n) - T(2n, n+1)$$

for this sequence, which we regard as a sequence of *generalized Catalan numbers*.

Example 200. We first look at the case $a_n = 2^n$. Thus

$$T(n,k) = \sum_{j=0} \binom{k}{j}\binom{n-k}{j} 2^j$$

with matrix representation

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 5 & 5 & 1 & 0 & 0 & \dots \\ 1 & 7 & 13 & 7 & 1 & 0 & \dots \\ 1 & 9 & 25 & 25 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which is the well-known Delannoy number triangle A008288. We have

$$T(n,k) = \sum_{j=0}^{k} \binom{k}{j}\binom{n-j}{k}.$$

We shall generalize this identity later in this chapter.
As a Riordan array, this is given by

$$\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right).$$

Anticipating the general case, we examine the row sums of this triangle, given by

$$\sum_{k=0}^{n} \sum_{j=0} \binom{k}{j}\binom{n-k}{j} 2^j.$$

Using the formalism of the Riordan group, we see that this sum has generating function given by

$$\frac{\frac{1}{1-x}}{1-\frac{x(1+x)}{1-x}} = \frac{1}{1-2x-x^2}.$$

In other words, the row sums in this case are the numbers $Pell(n+1)$, A000129, [245]. We look at the inverse binomial transform of these numbers, which has generating function

$$\frac{1}{1+x}\frac{1}{1-2\frac{x}{1+x}-\frac{x^2}{(1+x)^2}} = \frac{1+x}{1-2x^2}.$$

This is the generating function of the sequence $1, 1, 2, 2, 4, 4, \dots$, A016116, which is the doubled sequence of $a_n = 2^n$.
Another way to see this result is to observe that we have the factorization

$$\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)\left(1, \frac{x(1+2x)}{1+x}\right)$$

where $(\frac{1}{1-x}, \frac{x}{1-x})$ represents the binomial transform. The row sums of the Riordan array $(1, \frac{x(1+2x)}{1+x})$ are $1, 1, 2, 2, 4, 4, \ldots$.

For this triangle, the central numbers $T(2n, n)$ are the well-known central Delannoy numbers $1, 3, 13, 63, \ldots$ or A001850, with ordinary generating function $\frac{1}{\sqrt{1-6x+x^2}}$ and exponential generating function $e^{3x}I_0(2\sqrt{2}x)$ where I_n is the n-th modified Bessel function of the first kind [243]. They represent the coefficients of x^n in the expansion of $(1+3x+2x^2)^n$. We have

$$T(2n, n; 2^n) = \sum_{k=0}^{n} \binom{n}{k}^2 2^k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}.$$

The numbers $T(2n+1, n)$ in this case are A002002, with generating function $(\frac{1-x}{\sqrt{1-6x+x^2}} - 1)/(2x)$ and exponential generating function $e^{3x}(I_0(2\sqrt{2}x) + \sqrt{2}I_1(2\sqrt{2}x))$. We note that $T(2n-1, n-1)$ represents the coefficient of x^n in $((1-x)/(1-2x))^n$. It counts the number of peaks in all Schröder paths from $(0,0)$ to $(2n, 0)$.

The numbers $T(2n, n) - T(2n, n-1)$ are $1, 2, 6, 22, 90, 394, 1806, \ldots$ or the large Schröder numbers. These are the series reversion of $\frac{x(1-x)}{1+x}$. Thus the generating function of the sequence $\frac{1}{2}(T(2n, n; 2^n) - T(2n, n-1; 2^n))$ is

$$y_2(1, -1) = \frac{1 - x - \sqrt{1-6x+x^2}}{2x}.$$

We remark that in [235], the author states that "The Schröder numbers bear the same relation to the Delannoy numbers as the Catalan numbers do to the binomial coefficients." This note amplifies on this statement, defining generalized Catalan numbers for a family of number triangles.

Example 201. We take the case $a_n = (-1)^n$. Thus

$$T(n, k) = \sum_{j=0}^{\min(k, n-k)} \binom{k}{j}\binom{n-k}{j}(-1)^j$$

with matrix representation

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 1 & -1 & -1 & 1 & 0 & 0 & \ldots \\ 1 & -2 & -2 & -2 & 1 & 0 & \ldots \\ 1 & -3 & -2 & -2 & -3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As a Riordan array, this is given by

$$\left(\frac{1}{1-x}, \frac{x(1-2x)}{1-x}\right).$$

Again, we look at the row sums of this triangle, given by

$$\sum_{k=0}^{n}\sum_{j=0}\binom{k}{j}\binom{n-k}{j}(-1)^j.$$

Looking at generating functions, we see that this sum has generating function given by

$$\frac{\frac{1}{1-x}}{1-\frac{x(1-2x)}{1-x}}=\frac{1}{1-2x+2x^2}.$$

In other words, the row sums in this case are the numbers $1, 2, 2, 0, -4, -8, -8, \ldots$ with exponential generating function $\exp(x)(\sin(x)+\cos(x))$, A009545. Taking the inverse binomial transform of these numbers, we get the generating function

$$\frac{1}{1+x}\frac{1}{1-2\frac{x}{1+x}+2\frac{x^2}{(1+x)^2}}=\frac{1+x}{1+x^2}.$$

This is the generating function of the sequence $1, 1, -1, -1, 1, 1, \ldots$ which is the doubled sequence of $a_n = (-1)^n$.

Another way to see this result is to observe that we have the factorization

$$\left(\frac{1}{1-x}, \frac{x(1-2x)}{1-x}\right)=\left(\frac{1}{1-x}, \frac{x}{1-x}\right)\left(1, \frac{x(1-x)}{1+x}\right)$$

where $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ represents the binomial transform. The row sums of the Riordan array $\left(1, \frac{x(1-x)}{1+x}\right)$ are $1, 1, -1, -1, 1, 1, \ldots$ with general term $(-1)^{\binom{n}{2}}$.

The central terms $T(2n, n)$ turn out to be an 'aerated' signed version of $\binom{2n}{n}$ given by $1, 0, -2, 0, 6, 0, -20, \ldots$ with ordinary generating function $\frac{1}{\sqrt{1+4x^2}}$ and exponential generating function $I_0(2\sqrt{-1}x)$. They represent the coefficients of x^n in $(1-x^2)^n$. We have

$$T(2n, n; (-1)^n)=\sum_{k=0}^{n}\binom{n}{k}^2(-1)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k 2^{n-k}.$$

The terms $T(2n+1, n)$ turn out to be a signed version of $\binom{n}{\lfloor n/2\rfloor}$, namely

$$1, -1, -2, 3, 6, -10, -20, 35, 70, \ldots$$

with ordinary generating function $(\frac{1+2x}{\sqrt{1+4x^2}}-1)/(2x)$ and exponential generating function $I_0(2\sqrt{-1}x)+\sqrt{-1}I_1(2\sqrt{-1}x)$.

The generalized Catalan numbers $T(2n, n) - T(2n, n-1)$ are the numbers

$$1, -1, 0, 1, 0, -2, 0, 5, 0, -14, 0, \ldots$$

with generating function $y_2(1, 2) = \frac{1+2x-\sqrt{1+4x^2}}{2x}$. This is the series reversion of $\frac{x(1-x)}{1-2x}$.

We note that the sequence $T(2(n+1), n) - T(2(n+1), n+1)$ is $(-1)^{n/2}C_{\frac{n}{2}}(1+(-1)^n)/2$ with exponential generating function $I_1(2\sqrt{-1}x)/(\sqrt{-1}x)$.

10.6 A one-parameter sub-family of triangles

The above examples motivate us to look at the one-parameter subfamily given by the set of triangles defined by the power sequences $n \to r^n$, for $r \in \mathbf{Z}$. The case $r = 1$ corresponds to Pascal's triangle, while the case $r = 0$ corresponds to the 'partial summing' triangle with 1s on and below the diagonal.

Proposition 202. *The matrix associated to the sequences $n \to r^n$, $r \in \mathbf{Z}$, is given by the Riordan array*

$$\left(\frac{1}{1-x}, \frac{x(1+(r-1)x)}{1-x}\right).$$

Proof. The general term $T(n,k)$ of the above matrix is given by

$$\begin{aligned} T(n,k) &= [x^n](1+(r-1)x)^k x^k (1-x)^{-(k+1)} \\ &= [x^{n-k}](1+(r-1)x)^k (1-x)^{-(k+1)} \\ &= [x^{n-k}] \sum_{j=0}^{k} \binom{k}{j} (r-1)^j x^j \sum_{i=0} \binom{k+i}{i} x^i \\ &= [x^{n-k}] \sum_{j=0}^{k} \sum_{i=0} \binom{k}{j} \binom{k+i}{i} (r-1)^j x^{i+j} \\ &= \sum_{j=0}^{k} \binom{k}{j} \binom{k+n-k-j}{n-k-j} (r-1)^j \\ &= \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{k} (r-1)^j \\ &= \sum_{j=0}^{k} \binom{k}{j} \binom{n-k}{j} r^j. \end{aligned}$$

where the last equality is a consequence of identity (3.17) in [209]. □

Corollary 203. *The row sums of the triangle defined by $n \to r^n$ are the binomial transforms of the doubled sequence $n \to 1, 1, r, r, r^2, r^2, \ldots$, i.e., $n \to r^{\lfloor \frac{n}{2} \rfloor}$.*

Proof. The row sums of $\left(\frac{1}{1-x}, \frac{x(1+(r-1)x)}{1-x}\right)$ are the binomial transform of the row sums of its product with the inverse of the binomial matrix. This product is

$$\left(\frac{1}{1+x}, \frac{x}{1+x}\right)\left(\frac{1}{1-x}, \frac{x(1+(r-1)x)}{1-x}\right) = \left(1, \frac{x(1+rx)}{1+x}\right).$$

The row sums of this product have generating function given by

$$\frac{1}{1-\frac{x(1+rx)}{1+x}} = \frac{1+x}{1-rx^2}.$$

This is the generating function of $1, 1, r, r, r^2, r^2 \ldots$ as required. □

We note that the generating function for the row sums of the triangle corresponding to r^n is $\frac{1}{1-2x-(r-1)x^2}$.

We now look at the term $T(2n,n)$ for this subfamily.

Proposition 204. *$T(2n,n;r^n)$ is the coefficient of x^n in $(1+(r+1)x+rx^2)^n$.*

Proof. We have $(1+(r+1)x+rx^2)=(1+x)(1+rx)$. Hence

$$\begin{aligned}
[x^n](1+(r+1)x+rx^2)^n &= [x^n](1+x)^n(1+rx)^n \\
&= [x^n]\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{k}\binom{n}{j}r^j x^{k+j} \\
&= \sum_{j=0}^{n}\binom{n}{n-j}\binom{n}{j}r^j \\
&= \sum_{j=0}^{n}\binom{n}{j}^2 r^j.
\end{aligned}$$

□

Corollary 205. *The generating function of $T(2n,n;r^n)$ is*

$$\frac{1}{\sqrt{1-2(r+1)x+(r-1)^2x^2}}.$$

Proof. Using Lagrangian inversion, we can show that

$$[x^n](1+ax+bx^2)^n = [t^n]\frac{1}{\sqrt{1-2at+(a^2-4b)t^2}}$$

(see exercises 5.3 and 5.4 in [250]). Then

$$\begin{aligned}
[x^n](1+(r+1)x+rx^2)^n &= [t^n]\frac{1}{\sqrt{1-2(r+1)t+((r+1)^2-4r)t^2}} \\
&= [t^n]\frac{1}{\sqrt{1-2(r+1)t+(r-1)^2t^2}}
\end{aligned}$$

□

Corollary 206.

$$\sum_{k=0}^{n}\binom{n}{k}^2 r^k = \sum_{k=0}^{n}\binom{n}{2k}\binom{2k}{k}(r+1)^{n-2k}r^k = \sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}(r+1)^{n-2k}r^k.$$

Proof. This follows since the coefficient of x^n in $(1+ax+bx^2)^n$ is given by [171]

$$\sum_{k=0}^{n}\binom{n}{2k}\binom{2k}{k}a^{n-2k}b^k = \sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}a^{n-2k}b^k.$$

Hence each term is equal to $T(2n,n;r^n)$. □

We now look at the sequence $T(2n-1,n-1)$.

Proposition 207. *$T(2n-1,n-1;r^n)$ is the coefficient of x^n in $\left(\frac{1-(r-1)x}{1-rx}\right)^n$*

Proof. We have $\frac{1-(r-1)x}{1-rx}=\frac{1-rx+x}{1-rx}=1+\frac{x}{1-rx}$. Hence

$$\begin{aligned}[x^n]\left(\frac{1-(r-1)x}{1-rx}\right)^n &= [x^n]\left(1+\frac{x}{1-rx}\right)^n\\ &= [x^n]\sum_{k=0}^n\sum_{k=0}^n\binom{n}{k}x^k\sum_{j=0}\binom{k+j-1}{j}r^jx^j\\ &= \sum_{j=0}\binom{n}{n-j}\binom{n-1}{j}r^j\\ &= \sum_{j=0}\binom{n}{j}\binom{n-1}{j}r^j.\end{aligned}$$

□

Corollary 208.

$$\begin{aligned}\sum_{k=0}^n\binom{n}{k}\binom{n-1}{k}r^k &= \sum_{k=0}^n\binom{n}{k}\binom{n+k-1}{k}(1-r)^{n-k}r^k\\ &= \sum_{k=0}^n\binom{n}{k}\binom{2n-k-1}{n-k}(1-r)^kr^{n-k}.\end{aligned}$$

Proof. The coefficient of x^n in $(\frac{1-ax}{1-bx})^n$ is seen to be

$$\sum_{k=0}^n\binom{n}{k}\binom{n+k-1}{k}(-a)^{n-k}r^k=\sum_{k=0}^n\binom{n}{k}\binom{2n-k-1}{n-k}(-a)^kr^{n-k}.$$

Hence all three terms in the statement are equal to $T(2n-1,n-1;r^n)$. □

We can generalize the results seen above for $T(2n,n)$, $T(2n+1,n)$, $T(2n-1,n-1)$ and $T(2n,n)-T(2n,n-1)$ as follows.

Proposition 209. *Let $T(n,k)=\sum_{k=0}^{n-k}\binom{k}{j}\binom{n-k}{j}r^j$ be the general term of the triangle associated to the power sequence $n\to r^n$.*

1. *The sequence $T(2n,n)$ has ordinary generating function $\frac{1}{\sqrt{1-2(r+1)x+(r-1)^2x^2}}$, exponential generating function $e^{(r+1)x}I_0(2\sqrt{r}x)$, and corresponds to the coefficients of x^n in $(1+(r+1)x+rx^2)^n$.*
2. *The numbers $T(2n+1,n)$ have generating function $(\frac{1-(r-1)x}{\sqrt{1-2(r+1)x+(r-1)^2x^2}}-1)/(2x)$ and exponential generating function $e^{(r+1)x}(I_0(2\sqrt{r}x)+\sqrt{r}I_1(2\sqrt{r}x))$.*

3. $T(2n-1,n-1)$ *represents the coefficient of* x^n *in* $((1-(r-1)x)/(1-rx))^n$.

4. *The generalized Catalan numbers* $c(n;r^n) = T(2n,n) - T(2n,n-1)$ *associated to the triangle have ordinary generating function* $\frac{1-(r-1)x-\sqrt{1-2(r+1)x+(r-1)^2x^2}}{2x}$.

5. *The sequence* $c(n+1;r^n)$ *has exponential generating function* $\frac{1}{\sqrt{r}x}e^{(r+1)x}I_1(2\sqrt{r}x)$.

6. *The sequence* $nc(n;r^n) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k+1}\frac{r^{n-k}}{r+1}$ *has exponential generating function*

$$\frac{1}{\sqrt{r}x}e^{(r+1)x}I_1(2\sqrt{r}).$$

7. *The sequence* $c(n;r^n) - 0^n$ *is expressible as* $\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{2k}C_k(r+1)^{n-2k-1}r^k$ *and counts the number of Motzkin paths of length* n *in which the level steps have* $r+1$ *colours and the up steps have* r *colours. It is the series reversion of* $\frac{x}{1+(r+1)x+rx^2}$.

Pascal's triangle can be generated by the well-know recurrence

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

The following proposition gives the corresponding recurrence for the case of the triangle associated to the sequence $n \to r^n$.

Proposition 210. *Let* $T(n,k) = \sum_{k=0}^{n-k}\binom{k}{j}\binom{n-k}{j}r^j$. *Then*

$$T(n,k) = T(n-1,k-1) + (r-1)T(n-2,k-1) + T(n-1,k).$$

Proof. The triangle in question has Riordan array representation

$$\left(\frac{1}{1-x}, \frac{x(1+(r-1)x)}{1-x}\right).$$

Thus the bivariate generating function of this triangle is given by

$$\begin{aligned} F(x,y) &= \frac{1}{1-x}\frac{1}{1-y\frac{x(1+(r-1)x)}{1-x}} \\ &= \frac{1}{1-x-xy-(r-1)x^2y}. \end{aligned}$$

□

In this simple case, it is possible to characterize the inverse of the triangle. We have

Proposition 211. *The inverse of the triangle associated to the sequence* $n \to r^n$ *is given by the Riordan array* $(1-u,u)$ *where*

$$u = \frac{\sqrt{1+2(2r-1)x+x^2}-x-1}{2(r-1)}.$$

Proof. Let $(g^*, \bar{f}) = (\frac{1}{1-x}, \frac{x(1+(r-1)x)}{1-x})^{-1}$. Then

$$\frac{\bar{f}(1+(r-1)\bar{f})}{1-\bar{f}} = x \Rightarrow \bar{f} = \frac{\sqrt{1+2(2r-1)x+x^2}-x-1}{2(r-1)}.$$

Since $g^* = \frac{1}{g \circ \bar{f}} = 1 - \bar{f}$ we obtain the result. □

Corollary 212. *The row sums of the inverse of the triangle associated with $n \to r^n$ are* $1, 0, 0, 0, \ldots$.

Proof. The row sums of the inverse $(1-u, u)$ have generating function given by $\frac{1-u}{1-u} = 1$. In other words, the row sums of the inverse are $0^n = 1, 0, 0, 0, \ldots$. □

Other examples of these triangles are given by A081577, A081578, A081579, and A081580.

10.7 The Jacobsthal and the Fibonacci cases

We now look at the triangles generated by sequences whose elements can be expressed in Binet form as a simple sum of powers. In the first example of this section, the powers are of integers, while in the second case (Fibonacci numbers) we indicate that the formalism can be extended to non-integers under the appropriate conditions.

Example 213. The Jacobsthal numbers $J(n+1)$, A001045, have generating function $\frac{1}{1-x-2x^2}$ and general term $J(n+1) = 2.2^n/3 + (-1)^n/3$. Using our previous examples, we see that the triangle defined by $J(n+1)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 8 & 4 & 1 & 0 & 0 & \ldots \\ 1 & 5 & 16 & 16 & 5 & 1 & 0 & \ldots \\ 1 & 6 & 27 & 42 & 27 & 6 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or A114202, is the scaled sum of the Riordan arrays discussed above, given by

$$\frac{2}{3}\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right) + \frac{1}{3}\left(\frac{1}{1-x}, \frac{x(1-2x)}{1-x}\right).$$

In particular, the k-th column of the triangle has generating function

$$\begin{aligned} g_k(x) &= \frac{x^k}{(1-x)^{k+1}}\left(\frac{2}{3}(1+x)^k + \frac{1}{3}(1-2x)^k\right) \\ &= \frac{x^k}{(1-x)^{k+1}} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{3}(2+(-2)^j)x^j. \end{aligned}$$

We recognize in the sequence $\frac{1}{3}(2+(-2)^n)$ the inverse binomial transform of $J(n+1)$. Obviously, the inverse binomial transform of the row sums of the matrix are given by

$$\frac{2}{3}2^{\lfloor \frac{n}{2} \rfloor}+\frac{1}{3}(-1)^{\lfloor \frac{n}{2} \rfloor}$$

or $1, 1, 1, 1, 3, 3, 5, 5, \ldots$, the doubled sequence of $J(n+1)$.
The terms $T(2n, n)$ for this triangle can be seen to have generating function $\frac{2}{3}\frac{1}{\sqrt{1-6x+x^2}}+\frac{1}{3}\frac{1}{\sqrt{1+4x^2}}$ and exponential generating function $\frac{2}{3}e^{3x}I_0(2\sqrt{2}x)+\frac{1}{3}I_0(2\sqrt{-1}x)$.
The generalized Catalan numbers for this triangle are

$$1, 1, 4, 15, 60, 262, 1204, 5707, 27724, \ldots$$

whose generating function is $\frac{3-\sqrt{1+4x^2}-2\sqrt{1-6x+x^2}}{6x}$.
To find the relationship between $T(n, k)$ and its 'previous' elements, we proceed as follows, where we write $T(n, k) = T(n, k; J(n+1))$ to indicate its dependence on $J(n+1)$.

$$\begin{aligned}
T(n,k;J(n+1)) &= \sum_{j=0}\binom{k}{j}\binom{n-k}{j}(\frac{2}{3}2^j+\frac{1}{3}(-1)^j)\\
&= \frac{2}{3}\sum_{j=0}\binom{k}{j}\binom{n-k}{j}2^j+\frac{1}{3}\sum_{j=0}\binom{k}{j}\binom{n-k}{j}(-1)^j\\
&= \frac{2}{3}T(n,k;2^n)+\frac{1}{3}T(n,k;(-1)^n)\\
&= \frac{2}{3}(T(n-1,k-1;2^n)+T(n-2,k-1;2^n)+T(n-1,k;2^n))\\
&\quad +\frac{1}{3}(T(n-1,k-1;(-1)^n)-2T(n-2,k-1;(-1)^n)+T(n-1,k;(-1)^n))\\
&= \frac{2}{3}T(n-1,k-1;2^n)+\frac{1}{3}T(n-1,k-1;(-1)^n)\\
&\quad +\frac{2}{3}(T(n-2,k-1;2^n)-T(n-2,k-1;(-1)^n))\\
&\quad +\frac{2}{3}T(n-1,k;2^n)+\frac{1}{3}T(n-1,k;(-1)^n)\\
&= T(n-1,k-1;J(n+1))+2T(n-2,k-1;J(n))+T(n-1,k;J(n+1)).
\end{aligned}$$

We see here the appearance of the non-invertible matrix based on $J(n)$. This begins as

$$\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots\\
0 & 2 & 2 & 0 & 0 & 0 & 0 & \ldots\\
0 & 3 & 5 & 3 & 0 & 0 & 0 & \ldots\\
0 & 4 & 9 & 9 & 4 & 0 & 0 & \ldots\\
0 & 5 & 14 & 21 & 14 & 5 & 0 & \ldots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

Example 214. We briefly look at the case of the Fibonacci sequence

$$F(n+1)=\left(\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n\right)/\sqrt{5}.$$

Again, we can display the associated triangle

$$\left(\begin{array}{cccccccc}1&0&0&0&0&0&0&\ldots\\1&1&0&0&0&0&0&\ldots\\1&2&1&0&0&0&0&\ldots\\1&3&3&1&0&0&0&\ldots\\1&4&7&4&1&0&0&\ldots\\1&5&13&13&5&1&0&\ldots\\1&6&21&31&21&6&1&\ldots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{array}\right)$$

or A114197 as a sum of scaled 'Riordan arrays' as follows:

$$\frac{1+\sqrt{5}}{2}\left(\frac{1}{1-x},\frac{x(1+(\frac{1+\sqrt{5}}{2}-1)x)}{1-x}\right)-\frac{1-\sqrt{5}}{2}\left(\frac{1}{1-x},\frac{x(1+(\frac{1-\sqrt{5}}{2}-1)x)}{1-x}\right).$$

Hence the k-th column of the associated triangle has generating function given by

$$\frac{x^k}{(1-x)^{k+1}}\left\{\frac{1+\sqrt{5}}{2}\left(1+\left(\frac{1+\sqrt{5}}{2}-1\right)x\right)^k+\frac{1-\sqrt{5}}{2}\left(1+\left(\frac{1-\sqrt{5}}{2}-1\right)x\right)^k\right\}.$$

Expanding, we find that the generating function of the k-th column of the triangle associated to $F(n+1)$ is given by

$$\frac{x^k}{(1-x)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}b_jx^j$$

where the sequence b_n is the inverse binomial transform of $F(n+1)$. That is, we have

$$b_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F(k+1)=\left(\phi(\phi-1)^n+\frac{1}{\phi}\left(-\frac{1}{\phi}-1\right)^n\right)/\sqrt{5}$$

where $\phi=\frac{1+\sqrt{5}}{2}$.
Again, the inverse binomial transform of the row sums is given by $F(\lfloor\frac{n}{2}\rfloor+1)$.
The term $T(2n,n)$ in this case is $\sum_{k=0}^{n}\binom{n}{k}^2F(k+1)$, or $1,2,7,31,142,659,\ldots$ (A114198). This has ordinary generating function given by

$$\frac{\frac{1+\sqrt{5}}{2\sqrt{5}}}{\sqrt{1-2(\frac{1+\sqrt{5}}{2}+1)x+(\frac{1+\sqrt{5}}{2}-1)^2x^2}}-\frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{\sqrt{1-2(\frac{1-\sqrt{5}}{2}+1)x+(\frac{1-\sqrt{5}}{2}-1)^2x^2}}$$

and exponential generating function

$$\frac{1+\sqrt{5}}{2\sqrt{5}}\exp\left(\frac{3+\sqrt{5}}{2}x\right)I_0\left(2\sqrt{\frac{1+\sqrt{5}}{2}}x\right)-\frac{1-\sqrt{5}}{2\sqrt{5}}\exp\left(\frac{3-\sqrt{5}}{2}x\right)I_0\left(2\sqrt{\frac{1-\sqrt{5}}{2}}x\right).$$

$T(n,k)$ satisfies the following recurrence

$$T(n,k;F(n+1))=T(n-1,k-1;F(n+1))+T(n-2,k-1;F(n))+T(n-1,k;F(n+1))$$

where the triangle associated to $F(n)$ begins

$$\left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 5 & 3 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 9 & 9 & 4 & 0 & 0 & \ldots \\ 0 & 5 & 14 & 20 & 14 & 5 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We note that all Lucas sequences [242] can be treated in similar fashion.

10.8 The general case

Proposition 215. *Given an integer sequence a_n with $a_0=1$, the centrally symmetric invertible triangle associated to it by the above construction has the following generating function for its k-th column:*

$$\frac{x^k}{1-x}\sum_{j=0}^{k}\binom{k}{j}a_j\left(\frac{x}{1-x}\right)^j=\frac{x^k}{(1-x)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}b_jx^j$$

where b_n is the inverse binomial transform of a_n.

Proof. We have

$$\begin{aligned}
[x^n]\frac{x^k}{1-x}\sum_{j=0}^{k}\binom{k}{j}a_j\left(\frac{x}{1-x}\right)^j &= [x^{n-k}]\sum_{j=0}^{k}\binom{k}{j}a_j\frac{x^j}{(1-x)^{j+1}}\\
&= \sum_j \binom{k}{j}a_j[x^{n-k-j}](1-x)^{-(j+1)}\\
&= \sum_j \binom{k}{j}a_j[x^{n-k-j}]\sum_i \binom{j+i}{i}x^i\\
&= \sum_j \binom{k}{j}a_j\binom{j+n-k-j}{n-k-j}\\
&= \sum_j \binom{k}{j}\binom{n-k}{j}a_j\\
&= T(n,k).
\end{aligned}$$

Similarly,

$$\begin{aligned}
[x^n]\frac{x^k}{(1-x)^{k+1}}\sum_{j=0}^{k}\binom{k}{j}b_jx^j &= \sum_j \binom{k}{j}b_j[x^{n-k-j}](1-x)^{-(k+1)}\\
&= \sum_j \binom{k}{j}b_j[x^{n-k-j}]\sum_i \binom{k+i}{i}x^i\\
&= \sum_j \binom{k}{j}b_j\binom{k+n-k-j}{n-k-j}\\
&= \sum_j \binom{k}{j}\binom{n-j}{k}b_j.
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{j=0} \binom{k}{j}\binom{n-k}{j} a_j &= \sum_{j=0} \binom{k}{j}\binom{n-k}{j} \sum_{i=0}^{j} \binom{j}{i} b_i \\
&= \sum_j \sum_i \binom{k}{j}\binom{n-k}{j}\binom{j}{i} b_i \\
&= \sum_j \sum_i \binom{k}{j}\binom{j}{i}\binom{n-k}{j} b_i \\
&= \sum_j \sum_i \binom{k}{i}\binom{k-i}{j-i}\binom{n-k}{j} b_i \\
&= \sum_i \binom{k}{i} b_i \sum_j \binom{k-i}{k-j}\binom{n-k}{j} \\
&= \sum_i \binom{k}{i} b_i \binom{n-i}{k} \\
&= \sum_j \binom{k}{j}\binom{n-j}{k} b_j.
\end{aligned}$$

□

Corollary 216. *The following relationship exists between a sequence a_n and its inverse binomial transform b_n:*

$$\sum_j \binom{k}{j}\binom{n-k}{j} a_j = \sum_j \binom{k}{j}\binom{n-j}{k} b_j.$$

It is possible of course to reverse the above proposition to give us the following:

Proposition 217. *Given a sequence b_n, the product of the triangle whose k-th column has ordinary generating function*

$$\frac{x^k}{(1-x)^{k+1}} \sum_{j=0}^{k} \binom{k}{j} b_j x^j$$

by the binomial matrix is the centrally symmetric invertible triangle associated to the binomial transform of b_n.

10.9 Exponential-factorial triangles

In this section, we briefly describe an alternative method that produces generalized Pascal matrices, based on suitably chosen sequences. For this, we recall that the binomial matrix

$\mathbf{B}$ may be represented as

$$\mathbf{B} = \exp \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 5 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while if we write $a(n) = n$ then the general term $\binom{n}{k}$ of this matrix can be written as

$$\binom{n}{k} = \frac{\prod_{j=1}^{k} a(n-j+1)}{\prod_{j=1}^{k} a(j)} = \frac{\prod_{j=1}^{n} a(j)}{\prod_{j=1}^{k} a(j) \prod_{j=1}^{n-k} a(j)}.$$

Furthermore,

$$\mathbf{B} = \sum_{k=0} \frac{\mathbf{M}^k}{\prod_{j=1}^{k} a(j)}$$

where $\mathbf{M}$ is the sub-diagonal matrix formed from the elements of $a(n)$.

We shall see that by generalizing this construction to suitably chosen sequences $a(n)$ where $a(0) = 0$ and $a(1) = 1$, we can obtain generalized Pascal triangles, some of which are well documented in the literature. Thus we let $T(n,k)$ denote the matrix with general term

$$T(n,k) = \frac{\prod_{j=1}^{k} a(n-j+1)}{\prod_{j=1}^{k} a(j)} = \frac{\prod_{j=1}^{n} a(j)}{\prod_{j=1}^{k} a(j) \prod_{j=1}^{n-k} a(j)} = \binom{n}{k}_a.$$

Proposition 218. $T(n, n-k) = T(n,k)$, $T(n,1) = a(n)$, $T(n+1,1) = T(n+1,n) = a(n+1)$

Proof. To prove the first assertion, we assume first that $k \le n-k$. Then

$$\begin{aligned} T(n,k) &= \frac{a(n)\dots a(n-k+1)}{a(1)\dots a(k)} \\ &= \frac{a(n)\dots a(n-k+1)}{a(1)\dots a(k)} \frac{a(n-k)\dots a(k+1)}{a(k+1)\dots a(n-k)} \\ &= T(n, n-k). \end{aligned}$$

Secondly, if $k > n-k$, we have

$$\begin{aligned} T(n, n-k) &= \frac{a(n)\dots a(k+1)}{a(1)\dots a(n-k)} \\ &= \frac{a(n)\dots a(k+1)}{a(1)\dots a(n-k)} \frac{a(k)\dots a(n-k+1)}{a(n-k+1)\dots a(k)} \\ &= T(n,k). \end{aligned}$$

Next, we have

$$\begin{aligned} T(n,1) &= \frac{\prod_{j=1}^{1} a(n-j+1)}{\prod_{j=1}^{1} a(j)} \\ &= \frac{a(n-1+1)}{a(1)} = a(n). \end{aligned}$$

since $a(1) = 1$. Similarly,

$$\begin{aligned} T(n+1,1) &= \frac{\prod_{j=1}^{1} a(n+1-j+1)}{\prod_{j=1}^{1} a(j)} \\ &= \frac{a(n+1-1+1)}{a(1)} = a(n+1). \end{aligned}$$

□

Introducing the notation

$$a!(n) = \prod_{k=1}^{n} a(j)$$

we can write

$$\binom{n}{k}_a = \frac{a!(n)}{a!(k)a!(n-k)}.$$

We have

$$\begin{aligned} \binom{n}{k}_a &= \frac{a!(n)}{a!(k)a!(n-k)} \\ &= \frac{\prod_{j=1}^{k} a(n-j+1)}{\prod_{j=1}^{k} a(j)} \\ &= \frac{\prod_{j=1}^{k} a(n-k+j)}{\prod_{j=1}^{k} a(j)}. \end{aligned}$$

Along with the notation $a!(n)$, we find it convenient to define the $a-$*exponential* as the power series

$$E_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{a!(k)}.$$

Thus for those choices of the sequence $a(n)$ for which the values of $T(n,k)$ are integers, $T(n,k)$ represents a generalized Pascal triangle with $T(n,1) = a(n+1)$. We shall use the notation $\mathbf{P}_{a(n)}$ to denote the triangle constructed as above.

We define the *generalized Catalan sequence associated to* $a(n)$ *by this construction* to be the sequence with general term

$$\frac{T(2n,n)}{a(n+1)}.$$

Example 219. The Fibonacci numbers. The matrix $\mathbf{P}_{F(n)}$ with general term

$$\frac{\prod_{j=1}^{k} F(n-j+1)}{\prod_{j=1}^{k} F(j)}$$

which can be expressed as

$$\sum_{k=0} \frac{\mathbf{M}_F^k}{\prod_{j=1}^{k} F(j)} = E_F(\mathbf{M}_F)$$

where $\mathbf{M}_F$ is the sub-diagonal matrix generated by $F(n)$:

$$\mathbf{M}_F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 5 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the much studied Fibonomial matrix, A010048, [127, 135, 190, 236]. For instance, the generalized Catalan numbers associated to this triangle are the Fibonomial Catalan numbers, A003150.

Example 220. Let $a(n) = \frac{2^n}{2} - \frac{0^n}{2}$. The matrix $\mathbf{P}_{a(n)}$ with general term

$$\frac{\prod_{j=1}^{k} a(n-j+1)}{\prod_{j=1}^{k} a(j)}$$

which can be expressed as

$$\sum_{k=0} \frac{\mathbf{M}^k}{\prod_{j=1}^{k} a(j)} = E_a(\mathbf{M})$$

where $\mathbf{M}$ is the sub-diagonal matrix generated by $a(n)$

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 8 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 16 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 4 & 4 & 1 & 0 & 0 & \dots \\ 1 & 8 & 16 & 8 & 1 & 0 & \dots \\ 1 & 16 & 64 & 64 & 16 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A117401. For this matrix, we have $T(2n,n) = 2^{n^2}$ and $c(n; a(n)) = 2^{n(n-1)}$. This is easily generalized to the sequence $n \longrightarrow \frac{k^n}{k} - \frac{0^n}{k}$. For this sequence, we obtain $T(2n,n) = k^{n^2}$ and $c(n) = k^{n(n-1)}$.

Example 221. We take the case $a(n) = \lfloor \frac{n+1}{2} \rfloor$. In this case, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 1 & 0 & 0 & \dots \\ 1 & 2 & 4 & 2 & 1 & 0 & \dots \\ 1 & 3 & 6 & 6 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which has general term

$$\binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor}\binom{\lceil \frac{n}{2} \rceil}{\lceil \frac{k}{2} \rceil}.$$

This is the triangle

$$[0, 1, 0, -1, 0, 1, 0, -1, \dots] \quad \Delta^{(1)} \quad [1, 0, -1, 0, 1, 0, -1, \dots].$$

This triangle counts the number of symmetric Dyck paths of semi-length n with k peaks (A088855). The row sums of this array are given by $\binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}$ (which has Hankel transform $(-1)^{\binom{n+1}{2}}$). We note that for this triangle, $T(2n,n)$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}^2$ while $T(2n,n) - T(2n,n-1)$ is the sequence

$$1, 0, 2, 0, 12, 0, 100, 0, 980, 0, 10584 \dots$$

(A000888 aerated). We note that the triangle

$$[0, 1, 0, -1, 0, 1, 0, -1, \dots] \quad \Delta \quad [1, 0, -1, 0, 1, 0, -1, \dots]$$

begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 2 & 1 & 0 & \dots \\ 0 & 1 & 2 & 4 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and has row sums equal to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Example 222. The Jacobsthal numbers. Let $a(n) = J(n) = \frac{2^n}{3} - \frac{(-1)^n}{3}$. We form the matrix with general term

$$\frac{\prod_{j=1}^{k} J(n-j+1)}{\prod_{j=1}^{k} J(j)}$$

which can be expressed as

$$\sum_{k=0} \frac{\mathbf{M}_J^k}{\prod_{j=1}^{k} J(j)} = E_J(\mathbf{M}_J)$$

where $\mathbf{M}_J$ is the sub-diagonal matrix generated by $J(n)$:

$$\mathbf{M}_J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 5 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 11 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We obtain the matrix

$$\mathbf{P}_{J(n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 5 & 15 & 5 & 1 & 0 & \ldots \\ 1 & 11 & 55 & 55 & 11 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We recognize in this triangle the unsigned version of the q-binomial triangle for $q = -2$, A015109, whose k-th column has generating function

$$x^k \frac{1}{\prod_{j=0}^{k}(1-(-2)^j x)}.$$

Using the above notation, this latter signed triangle is therefore $\mathbf{P}_{(-1)^n J(n)}$. Note that

$$\frac{x}{(1-x)(1+2x)} = \frac{x}{1+x-2x^2}$$

is the generating function for $(-1)^n J(n)$.
The generating function of the k-th column of $\mathbf{P}_{J(n)}$ is given by

$$x^k \prod_{j=0}^{k} \frac{1}{(1-(-1)^{(j+k \bmod 2)} 2^j x)}.$$

The generalized Catalan numbers for $\mathbf{P}_{J(n)}$ are given by $\frac{\mathbf{P}_{J(n)}(2n,n)}{J(n+1)}$. These are A015056

$$1, 1, 5, 77, 5117, 1291677, \dots$$

We can generalize these results to the following:

Proposition 223. *Let $a(n)$ be the solution to the recurrence*

$$a(n) = (r-1)a(n-1) + r^2 a(n-2), \qquad a(0) = 0, \qquad a(1) = 1.$$

Then $\mathbf{P}_{a(n)}$ is a generalized Pascal triangle whose k-th column has generating function given by

$$x^k \prod_{j=0}^{k} \frac{1}{(1-(-1)^{(j+k \bmod 2)} r^j x)}.$$

Example 224. The Narayana and related triangles. The Narayana triangle $\tilde{N}$ is a generalized Pascal triangle in the sense of this section. It is known that the generating function of its k-th column is given by

$$x^k \frac{\sum_{j=0}^{k} N(k,j) x^j}{(1-x)^{2k+1}}.$$

Now $a(n) = \tilde{N}(n,1) = \binom{n+1}{2}$ satisfies $a(0) = 0$, $a(1) = 1$. It is not difficult to see that, in fact, $\tilde{\mathbf{N}} = \mathbf{P}_{\binom{n+1}{2}}$. See [115]. $T(2n,n)$ for this triangle is A000891, with exponential generating function $I_0(2x)I_1(2x)/x$. We note that in this case, the numbers generated by $\tilde{N}(2n,n)/a(n+1)$ do not produce integers. However the sequence $\tilde{N}(2n,n) - \tilde{N}(2n,n+1)$ turns out to be the product of successive Catalan numbers $C_n C_{n+1}$. This is A005568. Note also that by the definition of $\binom{n}{k}_a$, the sequence $\binom{n+1}{2} = \frac{n(n+1)}{2}$ can be replaced by any multiple of $n(n+1)$.
The triangle $\mathbf{P}_{\binom{n+2}{3}}$ is A056939 with matrix

$$\mathbf{P}_{\binom{n+2}{3}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 4 & 1 & 0 & 0 & 0 & \dots \\ 1 & 10 & 10 & 1 & 0 & 0 & \dots \\ 1 & 20 & 50 & 20 & 1 & 0 & \dots \\ 1 & 35 & 175 & 175 & 35 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The k-th column of this matrix has generating function

$$x^k \frac{\sum_{j=0}^{k} N_3(k,j) x^j}{(1-x)^{3k+1}}$$

where $N_3(n,k)$ is the triangle of 3-Narayana numbers, [214], A087647. $\mathbf{P}_{\binom{n+3}{4}}$ is the number triangle A056940.
The product $\mathbf{P}_{\binom{n+1}{2}} \mathbf{P}_{\binom{n}{1}} = \tilde{\mathbf{N}}\mathbf{B}$ is A126216 which counts certain Schröder paths.

10.10 A generalized Riordan array

In Chapter 11, we will study generalized Pascal triangles defined by exponential Riordan arrays. The basic example it that of Pascal's triangle itself, which is defined by

$$\mathbf{B} = [e^x, x].$$

In this section, we generalize this notation to the case of $E_a(x)$ defined above as

$$E_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{a!(k)}.$$

For this, we define the notation

$$[g(x), f(x)]_a$$

to represent the array whose $(n,k)-$th element is given by

$$\frac{a!(n)}{a!(k)}[x^n]g(x)f(x)^k.$$

Proposition 225. *$\binom{n}{k}_a$ is the $(n,k)-$th element of $[E_a(x), x]_a$.*

Proof. We have

$$\begin{aligned}
\frac{a!(n)}{a!(k)}[x^n]E_a(x)x^k &= \frac{a!(n)}{a!(k)}[x^{n-k}]\sum_{j=0}^{\infty}\frac{x^j}{a!(j)} \\
&= \frac{a!(n)}{a!(k)}\frac{1}{a!(n-k)} \\
&= \binom{n}{k}_a.
\end{aligned}$$

□

Using this notation, we can for example write

$$[E_F(x), x]_F = E_F(\mathbf{M}_F).$$

Example 226. The Narayana triangle $\tilde{\mathbf{N}}$ can be defined as

$$\tilde{\mathbf{N}} = [E_{\binom{n+1}{2}}, x]_{\binom{n+1}{2}} = \binom{n}{k}_{\binom{n+1}{2}}.$$

The foregoing suggests the following extension to our methods for constructing Pascal-like matrices.

Proposition 227. *The array with general $(n,k)-$th element*

$$\frac{a!(n)}{a!(k)}[x^n]E_a(x)(1+\alpha a(k)x),$$

for general integer α, is Pascal-like.

Proof. We have

$$\begin{aligned}\frac{a!(n)}{a!(k)}[x^n]E_a(x)(1+\alpha a(k)x) &= \frac{a!(n)}{a!(k)a!(n-k)} + \frac{\alpha a!(n)}{a!(k)}[x^{n-k-1}]\sum_{j=0} \frac{x^j}{a!(j)} \\ &= \frac{a!(n)}{a!(k)a!(n-k)} + \alpha\frac{a!(n)}{a!(k)}\frac{1}{a!(n-k-1)} \\ &= \frac{a!(n)}{a!(k)a!(n-k)}(1+\alpha a(k)a(n-k)).\end{aligned}$$

$\square$

Thus each Pascal-like triangle P_a, for suitable $a(n)$, defines a family of Pascal-like triangles $P_{a,\alpha}$ with general term

$$\frac{a!(n)}{a!(k)a!(n-k)}(1+\alpha a(k)a(n-k)).$$

Example 228. We take the simple sequence $a(n)$ where $a(0) = 0$, $a(1) = 1$ and $a(n) = 2$ for $n > 1$. Letting $\alpha = 0 \ldots 5$, we get the following family of Pascal-like triangles:

$$\begin{array}{cccccc|cccccc|cccccc} 1 &&&&&& 1 &&&&&& 1 \\ 1 & 1 &&&&& 1 & 1 &&&&& 1 & 1 \\ 1 & 2 & 1 &&&& 1 & 4 & 1 &&&& 1 & 6 & 1 \\ 1 & 2 & 2 & 1 &&& 1 & 6 & 6 & 1 &&& 1 & 10 & 10 & 1 \\ 1 & 2 & 2 & 2 & 1 && 1 & 6 & 10 & 6 & 1 && 1 & 10 & 18 & 10 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 & 6 & 10 & 10 & 6 & 1 & 1 & 10 & 18 & 18 & 10 & 1 \end{array}$$

$$\begin{array}{cccccc|cccccc|cccccc} 1 &&&&&& 1 &&&&&& 1 \\ 1 & 1 &&&&& 1 & 1 &&&&& 1 & 1 \\ 1 & 8 & 1 &&&& 1 & 10 & 1 &&&& 1 & 12 & 1 \\ 1 & 14 & 14 & 1 &&& 1 & 18 & 18 & 1 &&& 1 & 22 & 22 & 1 \\ 1 & 14 & 26 & 14 & 1 && 1 & 18 & 34 & 18 & 1 && 1 & 22 & 42 & 22 & 1 \\ 1 & 14 & 26 & 26 & 14 & 1 & 1 & 18 & 34 & 34 & 18 & 1 & 1 & 22 & 42 & 42 & 22 & 1 \end{array}$$

Example 229. For the Fibonomial matrix $\mathbf{P}_{F(n)} = \mathbf{P}_{F,0}$ (A010048) given by

$$\mathbf{P}_{F(n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 2 & 1 & 0 & 0 & \ldots \\ 1 & 3 & 6 & 3 & 1 & 0 & \ldots \\ 1 & 5 & 15 & 15 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we have as the next member of the family the matrix

$$\mathbf{P}_{F,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 4 & 4 & 1 & 0 & 0 & \dots \\ 1 & 9 & 12 & 9 & 1 & 0 & \dots \\ 1 & 20 & 45 & 45 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A154218.

Proposition 230. *We have*

$$\mathbf{P}_{a,\alpha} = \mathbf{P}_a \mathbf{L}_{a,\alpha}$$

where $\mathbf{L}_{a,\alpha}$ *is the matrix*

$$\mathbf{L}_{a,\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \alpha a(1)a(2) & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & \alpha a(2)a(3) & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \alpha a(3)a(4) & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \alpha a(4)a(5) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 231. We can define Pascal's triangle $\mathbf{B}$ as $\mathbf{P}_a$ where $a(n) = n$. In this case $\mathbf{L_{a,1}}$ is given by

$$\mathbf{L}_{n,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 6 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 10 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which gives us

$$\mathbf{P}_{n,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 4 & 1 & 0 & 0 & 0 & \dots \\ 1 & 9 & 9 & 1 & 0 & 0 & \dots \\ 1 & 16 & 30 & 16 & 1 & 0 & \dots \\ 1 & 25 & 70 & 70 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now note that $a(n) = n/2$ will also produce $\mathbf{B}$. However, in this case

$$\mathbf{L}_{n/2,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 5 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which yields

$$\mathbf{P}_{n/2,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 18 & 10 & 1 & 0 & \dots \\ 1 & 15 & 40 & 40 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

10.11 A note on generalized Stirling matrices

We have seen that the (signed) Stirling numbers of the first kind are elements of the exponential Riordan array $[1, \ln(1 + x)]$ which begins

$$\tilde{\mathbf{s}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & -3 & 1 & 0 & 0 & \dots \\ 0 & -6 & 11 & -6 & 1 & 0 & \dots \\ 0 & 24 & -50 & 35 & -10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with inverse given by $[1, e^x - 1]$ which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 3 & 1 & 0 & 0 & \dots \\ 0 & 1 & 7 & 6 & 1 & 0 & \dots \\ 0 & 1 & 15 & 25 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\tilde{\mathbf{s}}$ is the coefficient array of the polynomials $P_n(x)$ defined by the falling factorials

$$P_n(x) = (x)_n = \prod_{k=0}^{n-1}(x - k).$$

We now wish to extend these results to the polynomial family $P_n(x;a)$ that depends on a sequence $a(n)$ (always with $a(0)=0$, $a(1)=1$), defined by

$$P_n(x;a)=\prod_{k=0}^{n-1}(x-a(k)).$$

Example 232. We let $a(n)=2^{n-1}-0^n/2$. We find that the coefficient array of $P_n(x;a)$, which begins

$$\tilde{\mathbf{s}}_a=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & -3 & 1 & 0 & 0 & \dots \\ 0 & -8 & 14 & -7 & 1 & 0 & \dots \\ 0 & 64 & -120 & 70 & -15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has inverse which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 3 & 1 & 0 & 0 & \dots \\ 0 & 1 & 7 & 7 & 1 & 0 & \dots \\ 0 & 1 & 15 & 35 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is an augmented version of the matrix of Gaussian binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for $q=2$, A022166. If $T_{n,k}$ denotes the general term of this array then we have

$$T_{n,k}=T_{n-1,k-1}+\frac{2^k-0^k}{2}T_{n-1,k}$$

and

$$T_{n,k}=[x^n]\frac{x^k}{\prod_{j=0}^{k}(1-(2^{j-1}-0^j/2)x)}.$$

This matrix has production matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 8 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 16 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The production matrix for the internal triangle A022166 is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 4 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 8 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 16 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 32 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This triangle is the inverse of the coefficient array of

$$\prod_{k=1}^{n}(x-a(k)).$$

We have in general the result

Proposition 233. *Let* $a(n;q)=\frac{q^n-0^n}{q}$*. Then the inverse coefficient array of the polynomial family* $P_n(x;a)=\prod_{k=0}^{n-1}(x-a(k;q))$ *is the array with general term* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ *augmented as above. Furthermore, this matrix has production matrix*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & q & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & q^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & q^3 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & q^4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus if $T_{n,k}$ denotes the general element of the inverse coefficient array of the family of polynomials $P_n(x;a)$ we have

$$T_{n+1,k+1}=\begin{bmatrix} n \\ k \end{bmatrix}_q.$$

We also have

$$T_{n,k}=T_{n-1,k-1}+a(k;q)T_{n-1,k}$$

and

$$T_{n,k}=[x^n]\frac{x^k}{\prod_{j=0}^{k}(1-a(j;q)x)}.$$

We note that in general this matrix is *not* equal to the matrix $[1,E_a(x)-1]_a$, using the notation of the previous section. For example, when $q = 2$, the matrix $[1, E_a(x) - 1]_a$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 4 & 1 & 0 & 0 & \dots \\ 0 & 1 & 16 & 12 & 1 & 0 & \dots \\ 0 & 1 & 80 & 144 & 32 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 234. We take the example of the Fibonacci numbers, i.e., $a(n) = F(n)$. We find that the polynomial coefficient array for $P_n(x, F)$, which begins

$$\tilde{\mathbf{s}}_F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -2 & 1 & 0 & 0 & \dots \\ 0 & -2 & 5 & -4 & 1 & 0 & \dots \\ 0 & 6 & -17 & 17 & -7 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has inverse

$$\tilde{\mathbf{S}}_F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 3 & 4 & 1 & 0 & \dots \\ 0 & 1 & 4 & 11 & 7 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is an augmented version of A111669. This matrix therefore satisfies

$$T_{n,k} = T_{n-1,k-1} + F(k) T_{n-1,k}.$$

We have

$$T_{n,k} = [x^n] \frac{x^k}{\prod_{j=0}^k (1 - F(j)x)}.$$

The production matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 3 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while that of the internal triangle, A111669, is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 5 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

10.12 Generalized Charlier polynomials

In section 8.9 we defined the (unsigned) Charlier array to be

$$\mathbf{Ch} = \mathbf{B} \cdot \mathbf{s}.$$

We now assume that a_n is an integer sequence, with $a_0 = 0$ and $a_1 = 1$. We define the *generalized Charlier array associated to a_n* to be the array

$$\mathbf{Ch}_a = \mathbf{B}_a \cdot \mathbf{s}$$

where $\mathbf{B}_a$ is the Pascal-like array with general term $\binom{n}{k}_a$. We then define the *generalized Charlier polynomials associated to a_n* to be the polynomials with coefficient array $\mathbf{Ch}_a$. By the properties of $\mathbf{s}$ and $\mathbf{B}_a$, we easily obtain the following :

Proposition 235. *Let $P_n^{(a)}(x)$ be the generalized Charlier polynomials associated to the sequence a_n. Then*

$$P_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k}_a (x)_k.$$

The n-th term of the row sums of $\mathbf{Ch}_a$ is given by

$$P_n^{(a)}(1) = \sum_{k=0}^{n} \binom{n}{k}_a k!$$

Example 236. We take the case of a_n equal to the sequence $0, 1, 2, 2, 2, 2, \dots$. We find that $\mathbf{Ch}_a$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 5 & 1 & 0 & 0 & \dots \\ 1 & 14 & 19 & 8 & 1 & 0 & \dots \\ 1 & 44 & 80 & 49 & 12 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This has row sums $\sum_{k=0}^{n} \binom{n}{k}_a k!$ where $\mathbf{B}_a = (\binom{n}{k}_a)$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 1 & 0 & 0 & \dots \\ 1 & 2 & 2 & 2 & 1 & 0 & \dots \\ 1 & 2 & 2 & 2 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is the sequence $1, 2, 5, 13, 43, 187, 1027, \ldots$. The first differences of this sequence yield the sequence $1, 1, 3, 8, 30, 144, \ldots$ or A059171, the size of the largest conjugacy class in S_n, the symmetric group on n symbols. Now we note that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 1 & 0 & 0 & \dots \\ 1 & 2 & 2 & 2 & 1 & 0 & \dots \\ 1 & 2 & 2 & 2 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus if we denote the sequence $1, 1, 3, 8, 30, 144, \ldots$ by d_n, we have

$$d_0 = 1, \qquad d_n = (n-1)! + n! - \binom{1}{n}, \qquad n \geq 1.$$

This implies that the row sums of $\mathbf{Ch}_a$ are given by

$$P_n^{(a)}(1) = 0^n + \sum_{k=1}^{n} (k! + (k-1)!).$$

We can generalize this result to cover the sequence a_n given by $0, 1, r, r, r, \ldots$. We find that the row sums of $\mathbf{Ch}_a$ in this case are given by

$$P_n^{(a)}(1) = (r-1)0^n + \sum_{k=1}^{n} (k! + (r-1)(k-1)!) - (r-2).$$

The first differences d_n of this sequence are then given by

$$d_0 = 1, \qquad d_n = (r-1)(n-1)! + n! - (r-1)\binom{1}{n}, \qquad n \geq 1.$$

Chapter 11

Generalized Pascal Triangles Defined by Exponential Riordan Arrays [1]

11.1 Introduction

In the Chapter 10 (and see [17]), we studied a family of generalized Pascal triangles whose elements were defined by Riordan arrays, in the sense of [202, 208]. In this chapter, we use so-called "exponential Riordan arrays" to define another family of generalized Pascal triangles. These number triangles are easy to describe, and important number sequences derived from them are linked to both the Hermite and Laguerre polynomials, as well as being related to the Narayana and Lah numbers.

We begin by looking at Pascal's triangle, the binomial transform, the Narayana numbers, and briefly summarize those features of the Hermite and Laguerre polynomials that we will require. We then introduce the family of generalized Pascal triangles based on exponential Riordan arrays, and look at a simple case in depth. We finish by enunciating a set of general results concerning row sums, central coefficients and generalized Catalan numbers for these triangles.

11.2 Preliminaries

Pascal's triangle, with general term $C(n,k)=\binom{n}{k}$, $n,k\geq 0$, has fascinated mathematicians by its wealth of properties since its discovery [77]. Viewed as an infinite lower-triangular matrix, it is invertible, with an inverse whose general term is given by $(-1)^{n-k}\binom{n}{k}$. Invertibility follows from the fact that $\binom{n}{n}=1$. It is *centrally symmetric*, since by definition, $\binom{n}{k}=\binom{n}{n-k}$. All the terms of this matrix are integers.

By a *generalized Pascal triangle* (or *Pascal-like* triangle) we shall understand a lower-triangular infinite integer matrix $T=T(n,k)$ with $T(n,0)=T(n,n)=1$ and $T(n,k)=T(n,n-k)$. We index all matrices in this paper beginning at the $(0,0)$-th element.

[1] This chapter reproduces and extends the content of the published article "P. Barry, On a family of generalized Pascal triangles defined by exponential Riordan arrays, J. Integer Seq., **10** (2007), Art. 7.3.5." [17].

We shall encounter transformations that operate on integer sequences during the course of this chapter. An example of such a transformation that is widely used in the study of integer sequences is the so-called Binomial transform [230], which associates to the sequence with general term a_n the sequence with general term b_n where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k. \tag{11.1}$$

If we consider the sequence with general term a_n to be the vector $\mathbf{a} = (a_0, a_1, \ldots)$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector by the lower-triangle matrix $\mathbf{B}$ whose (n,k)-th element is equal to $\binom{n}{k}$:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & 0 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This transformation is invertible, with

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \tag{11.2}$$

We note that $\mathbf{B}$ corresponds to Pascal's triangle. Its row sums are 2^n, while its diagonal sums are the Fibonacci numbers $F(n+1)$. If $\mathbf{B}^m$ denotes the $m-$th power of $\mathbf{B}$, then the $n-$th term of $\mathbf{B}^m\mathbf{a}$ where $\mathbf{a} = (a_n)_{n\geq 0}$ is given by $\sum_{k=0}^{n} m^{n-k}\binom{n}{k}a_k$.

As an exponential Riordan array, $\mathbf{B}$ represents the element $[e^x, x]$.

We note at this juncture that the exponential Riordan group, as well as the group of 'standard' Riordan arrays [202] can be cast in the more general context of matrices of type $R^q(\alpha_n, \beta_k; \phi, f, \psi)$ as found in [81, 83, 82]. Specifically, a matrix $C = (c_{nk})_{n,k=0,1,2,\ldots}$ is of type $R^q(\alpha_n, \beta_k; \phi, f, \psi)$ if its general term is defined by the formula

$$c_{nk} = \frac{\beta_k}{\alpha_n}\mathbf{res}_x(\phi(x) f^k(x) \psi^n(x) x^{-n+qk-1})$$

where $\mathbf{res}_x A(x) = a_{-1}$ for a given formal power series $A(x) = \sum_j a_j x^j$ is the formal residue of the series.

For the exponential Riordan arrays in this chapter, we have $\alpha_n = \frac{1}{n!}$, $\beta_k = \frac{1}{k!}$, and $q = 1$.

Example 237. The Binomial matrix $\mathbf{B}$ is the element $[e^x, x]$ of the exponential Riordan group. More generally, $\mathbf{B}^m$ is the element $[e^{mx}, x]$ of the Riordan group. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $[e^{-mx}, x]$.

Example 238. The exponential generating function of the row sums of the matrix $[g, f]$ is obtained by applying $[g, f]$ to e^x, the e.g.f. of the sequence $1, 1, 1, \ldots$. Hence the row sums of $[g, f]$ have e.g.f. $g(x)e^{f(x)}$.

Example 239. An example of a well-known centrally symmetric invertible triangle is the Narayana triangle $\tilde{\mathbf{N}}$, [212, 213], defined by

$$\tilde{N}(n,k) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}$$

for $n, k \geq 0$. Other expressions for $\tilde{N}(n,k)$ are given by

$$\tilde{N}(n,k) = \binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1} = \binom{n+1}{k+1}\binom{n}{k} - \binom{n+1}{k}\binom{n}{k+1}.$$

This triangle begins

$$\tilde{\mathbf{N}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & \ldots \\ 1 & 15 & 50 & 50 & 15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that in the literature, it is often the triangle $\tilde{N}(n-1,k-1) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ that is referred to as the Narayana triangle. Alternatively, the triangle $\tilde{N}(n-1,k) = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$ is referred to as the Narayana triangle. We shall denote this latter triangle by $N(n,k)$. We then have

$$\mathbf{N} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with row sums equal to the Catalan numbers C_n.

Note that for $n, k \geq 1$, $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. We have, for instance,

$$\begin{aligned} \tilde{N}(n-1,k-1) &= \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \\ &= \binom{n}{k}^2 - \binom{n-1}{k}\binom{n+1}{k} \\ &= \binom{n}{k}\binom{n-1}{k-1} - \binom{n}{k-1}\binom{n-1}{k}. \end{aligned}$$

The last expression represents a 2×2 determinant of adjacent elements in Pascal's triangle. Further details on the Narayana triangle are in Chapter 10.

The Hermite polynomials $H_n(x)$ [238] are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

They obey $H_n(-x) = (-1)^n H_n(x)$ and can be defined by the recurrence

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{11.3}$$

They have a generating function given by

$$e^{2tx-x^2} = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} x^n.$$

We have

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-2)^k \frac{(2k)!}{2^k k!} (2x)^{n-2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \frac{(2k)!}{k!} (2x)^{n-2k}.$$

A property that is related to the binomial transform is the following:

$$\sum_{k=0}^{n} \binom{n}{k} H_k(x)(2z)^{n-k} = H_n(x+z).$$

From this, we can deduce the following proposition.

Proposition 240. *For fixed x and $y \neq 0$, the binomial transform of the sequence $n \to H_n(x)y^n$ is the sequence $n \to y^n H_n(x + \frac{1}{2y})$.*

Proof. Let $z = \frac{1}{2y}$. Then $2z = \frac{1}{y}$ and hence

$$\sum_{k=0}^{n} \binom{n}{k} H_k(x)(y)^{k-n} = H_n\left(x + \frac{1}{2y}\right).$$

That is,

$$\sum_{k=0}^{n} \binom{n}{k} H_k(x)y^k = y^n H_n\left(x + \frac{1}{2y}\right)$$

as required. $\square$

The Laguerre polynomials $L_n(x)$ [241] are defined by

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} x^n e^{-x}.$$

They have generating function

$$\frac{\exp(-\frac{tx}{1-x})}{1-x} = \sum_{n=0}^{\infty} \frac{L_n(t)}{n!} x^n.$$

They are governed by the following recurrence relationship:

$$(n+1)L_{n+1}(t) = (2n+1-t)L_n(t) - nL_{n-1}(t). \tag{11.4}$$

11.3 Introducing the family of centrally symmetric invertible triangles

We recall that the Binomial matrix $\mathbf{B}$, or Pascal's triangle, is the element $[e^x, x]$ of the Riordan group. For a given integer r, we shall denote by $\mathbf{B}_r$ the element $[e^x, x(1+rx)]$ of the Riordan group. We note that $\mathbf{B} = \mathbf{B}_0$. We can characterize the general element of $\mathbf{B}_r$ as follows.

Proposition 241. *The general term $B_r(n,k)$ of the matrix $\mathbf{B}_r$ is given by*

$$B_r(n,k) = \frac{n!}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{r^j}{(n-k-j)!}.$$

Proof. We have

$$\begin{aligned} B_r(n,k) &= \frac{n!}{k!}[x^n](e^x(x(1+rx)^k) \\ &= \frac{n!}{k!}[x^n]\sum_{i=0}^{\infty}\frac{x^i}{i!}x^k\sum_{j=0}^{k}\binom{k}{j}r^jx^j \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{i=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{r^j}{i!}x^{i+j} \\ &= \frac{n!}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{r^j}{(n-k-j)!}. \end{aligned}$$

□

From the above expression we can easily establish that $B_r(n,k) = B_r(n, n-k)$ and $B_r(n,0) = B_r(n,n) = 1$. We also have

Proposition 242.

$$B_r(n,k) = \sum_{j=0}^{n}\frac{j!}{k!}\binom{n}{j}\binom{k}{j-k}r^{j-k}.$$

Proof. By definition, $\mathbf{B}_r$ is the Riordan array $[e^x, x(1+rx)] = [e^x, x][1, x(1+rx)]$. But the general term of $[1, x(1+rx)]$ is easily seen to be $\frac{n!}{k!}\binom{k}{n-k}r^{n-k}$. The result follows since the general term of $[e^x, x]$ is $\binom{n}{k}$. □

An alternative derivation of these results can be obtained be observing that the matrix $\mathbf{B}_r$

may be defined as the array $R^1(\frac{1}{n!}, \frac{1}{k!}; e^x, (1+rx), 1)$. Then we have

$$\begin{aligned} B_r(n,k) &= \frac{1/k!}{1/n!}\mathbf{res}_x(e^x(1+rx)^k x^{-n+k-1}) \\ &= \frac{n!}{k!}\mathbf{res}_x(\sum_{i=0}^{\infty}\frac{x^i}{i!}\sum_{j=0}^{k}\binom{k}{j}r^j x^j x^{-n+k-1}) \\ &= \frac{n!}{k!}\mathbf{res}_x(\sum_{i=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{r^j}{i!}x^{i+j-n+k-1}) \\ &= \frac{n!}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{r^j}{(n-k-j)!}. \end{aligned}$$

Thus $\mathbf{B_r}$ is a centrally symmetric lower-triangular matrix with $B_r(n,0) = B_r(n,n) = 1$. In this sense $\mathbf{B}_r$ can be regarded as a generalized Pascal matrix. Note that by the last property, this matrix is invertible.

Proposition 243. *The inverse of* $\mathbf{B_r}$ *is the element* $[e^{-u}, u]$ *of the Riordan group, where*

$$u = \frac{\sqrt{1+4rx}-1}{2r}.$$

Proof. Let $[g^*, \bar{f}]$ be the inverse of $[e^x, x(1+rx)]$. Then

$$[g^*, \bar{f}][e^x, x(1+rx)] = [1, x] \Rightarrow \bar{f}(1+r\bar{f}) = x.$$

Solving for $\bar{f}$ we get

$$\bar{f} = \frac{\sqrt{1+4rx}-1}{2r}.$$

But $g^* = \frac{1}{g\circ\bar{f}} = e^{-\bar{f}}$. □

This result allows us to easily characterize the row sums of the inverse $\mathbf{B}_r^{-1}$.

Corollary 244. *The row sums of the inverse triangle* $\mathbf{B}_r^{-1}$ *are given by* $0^n = 1, 0, 0, 0, \ldots$.

Proof. We have $\mathbf{B}_r^{-1} = [e^{-u}, u]$ as above. Hence the e.g.f. of the row sums of $\mathbf{B}_r^{-1}$ is $e^{-u}e^u = 1$. The result follows from this. □

Example 245. $\mathbf{B}_1 = [e^x, x(1+x)]$ is given by

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 9 & 9 & 1 & 0 & 0 & \ldots \\ 1 & 16 & 42 & 16 & 1 & 0 & \ldots \\ 1 & 25 & 130 & 130 & 25 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The row sums of $\mathbf{B}_1$ are

$$1, 2, 6, 20, 76, 312, 1384, 6512, 32400, \ldots$$

or A000898.
From the above, the terms of this sequence are given by

$$s_1(n) = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(n-k-j)!}$$

with e.g.f. $g(x)e^{f(x)} = e^x e^{x(1+x)} = e^{2x+x^2}$. What is less evident is that

$$s_1(n) = H_n(-i)i^n$$

where $i = \sqrt{-1}$. This follows since

$$\begin{aligned} e^{2x+x^2} &= e^{2(-i)(ix)-(ix)^2} \\ &= \sum_{n=0} \frac{H_n(-i)}{n!}(ix)^n \\ &= \sum_{n=0} \frac{H_n(-i)i^n}{n!} x^n \end{aligned}$$

and hence e^{2x+x^2} is the e.g.f. of $H_n(-i)i^n$. We therefore obtain the identity

$$H_n(-i)i^n = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(n-k-j)!}.$$

We can characterize the row sums of $\mathbf{B}_1$ in terms of the diagonal sums of another related special matrix. For this, we recall [231] that

$$\text{Bessel}(n,k) = \frac{(n+k)!}{2^k (n-k)!k!} = \binom{n+k}{2k} \frac{(2k)!}{2^k k!} = \binom{n+k}{2k} (2k-1)!!$$

defines the triangle A001498 of coefficients of Bessel polynomials that begins

$$\mathbf{Bessel} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 15 & 15 & 0 & 0 & \ldots \\ 1 & 10 & 45 & 105 & 105 & 0 & \ldots \\ 1 & 15 & 105 & 420 & 945 & 945 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This triangle has bi-variate o.g.f. given by the continued fraction

$$\cfrac{1}{1-x-\cfrac{xy}{1-x-\cfrac{2xy}{1-x-\cfrac{3xy}{1-x-\cfrac{4xy}{1-x-\cdots}}}}}$$

We then have

Proposition 246. *The row sums of the matrix* $\mathbf{B}_1$ *are equal to the diagonal sums of the matrix with general term* $\text{Bessel}(n,k)2^n$*. That is*

$$H_n(-i)i^n = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(n-k-j)!} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k,k)2^{n-k}.$$

Proof. We shall prove this in two steps. First, we shall show that

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k,k)2^{n-k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2k)!}{k!} \binom{n}{2k} 2^{n-2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (2k-1)!! \binom{n}{2k} 2^{n-k}.$$

We shall then show that this is equal to $H_n(-i)i^n$. Now

$$\begin{aligned}
\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k,k)2^{n-k} &= \sum_{k=0}^{n} \text{Bessel}\left(n-\frac{k}{2},\frac{k}{2}\right) 2^{n-\frac{k}{2}}(1+(-1)^k)/2 \\
&= \sum_{k=0}^{n} \frac{(n-\frac{k}{2}+\frac{k}{2})!2^{n-\frac{k}{2}}}{2^{\frac{k}{2}}(n-\frac{k}{2}-\frac{k}{2})!(\frac{k}{2})!}(1+(-1)^k)/2 \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{2^{n-k}}{(\frac{k}{2})!}(1+(-1)^k)/2 \\
&= \sum_{k=0}^{n} \frac{k!}{(\frac{k}{2})!} \binom{n}{k} 2^{n-k}(1+(-1)^k)/2 \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2k)!}{k!} \binom{n}{2k} 2^{n-2k} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2k)!}{2^k k!} \binom{n}{2k} 2^{n-k} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (2k-1)!! \binom{n}{2k} 2^{n-k}.
\end{aligned}$$

establishes the first part of the proof. The second part of the proof is a consequence of the following more general result, when we set $a=2$ and $b=1$. □

Proposition 247. *The sequence with e.g.f.* e^{ax+bx^2} *has general term* u_n *given by*

$$u_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k (k+1)! a^{n-2k} b^k.$$

Proof. We have

$$\begin{aligned} n![x^n]e^{ax+bx^2} &= n![x^n]e^{ax}e^{bx^2} \\ &= n![x^n] \sum_{i=0}^{\infty} \frac{a^i x^i}{i!} \sum_{k=0}^{\infty} \frac{b^k x^{2k}}{k!} \\ &= n![x^n] \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{a^i b^k}{i!k!} x^{i+2k} \\ &= n! \sum_{k=0}^{\infty} \frac{a^{n-2k} b^k}{(n-2k)!k!} \\ &= \sum_{k=0}^{\infty} \frac{n!}{(n-2k)!(2k)!} \frac{(2k)!}{k!} a^{n-2k} b^k \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k. \end{aligned}$$

□

Corollary 248.

$$H_n(-\frac{a}{2\sqrt{b}} i)(\sqrt{b} i)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k.$$

Corollary 249. *Let* u_n *be the sequence with e.g.f.* e^{ax+bx^2}*. Then* u_n *satisfies the recurrence*

$$u_n = a u_{n-1} + 2(n-1) b u_{n-2}$$

with $u_0 = 1$*,* $u_1 = a$*.*

Proof. Equation 11.3 implies that

$$H_n(x) = 2x H_{n-1}(x) - 2(n-1) H_{n-2}(x).$$

Thus

$$H_n\left(-\frac{a}{2\sqrt{b}} i\right) = -2\frac{a}{2\sqrt{b}} i H_{n-1}\left(-\frac{a}{2\sqrt{b}} i\right) - 2(n-1) H_{n-2}\left(-\frac{a}{2\sqrt{b}} i\right).$$

Now multiply both sides by $(\sqrt{b} i)^n$ to obtain

$$u_n = a u_{n-1} + 2(n-1) b u_{n-2}.$$

Since

$$u_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k$$

we obtain the initial values $u_0 = 1$, $u_1 = a$. □

Corollary 250. *The binomial transform of* $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k$ *is given by*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} (a+1)^{n-2k} b^k.$$

Proof. The e.g.f. of the binomial transform of the sequence with e.g.f. e^{ax+cx^2} is $e^x e^{ax+bx^2} = e^{(a+1)x+bx^2}$. $\square$

Equivalently, the binomial transform of $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k (k+1)! a^{n-2k} b^k$ is given by

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k (k+1)! (a+1)^{n-2k} b^k.$$

We note that in the last chapter, we showed that the binomial transform of $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k a^{n-2k} b^k$ is given by

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k (a+1)^{n-2k} b^k.$$

Corollary 251. *The row sums of* $\mathbf{B}_1$ *satisfy the recurrence equation*

$$u_n = 2u_{n-1} + 2(n-1)u_{n-2}$$

with $u_0 = 1$, $u_1 = 2$.

Since the triangle with general term Bessel$(n,k)2^n$ has g.f.

$$\cfrac{1}{1-2x-\cfrac{2xy}{1-2x-\cfrac{4xy}{1-2x-\cfrac{6xy}{1-2x-\cfrac{8xy}{1-2x-\cdots}}}}}$$

we see that the row sums of $\mathbf{B_1}$ have g.f. given by the continued fraction

$$\cfrac{1}{1-2x-\cfrac{2x^2}{1-2x-\cfrac{4x^2}{1-2x-\cfrac{6x^2}{1-2x-\cfrac{8x^2}{1-2x-\cdots}}}}}$$

We note that this is the second binomial transform of the aerated quadruple factorial numbers $\frac{(2n)!}{n!} = (n+1)!C_n = n!\binom{2n}{n}$, whose g.f. is given by

$$\cfrac{1}{1-\cfrac{2x}{1-\cfrac{4x}{1-\cfrac{6x}{1-\cfrac{8x}{1-\cdots}}}}}$$

The row sums of $\mathbf{B}_1$ can thus be expressed as

$$\begin{aligned} s_n &= \sum_{k=0}^{n}\binom{n}{k}2^{n-k}\frac{k!}{(k/2)!}\frac{1+(-1)^k}{2} \\ &= \sum_{k=0}^{n}\binom{n}{k}2^{n-k}(k/2)!\binom{k}{k/2}\frac{1+(-1)^k}{2} \\ &= \sum_{k=0}^{n}\binom{n}{k}2^{n-k}(k/2+1)!C_{\frac{k}{2}}\frac{1+(-1)^k}{2}. \end{aligned}$$

We can use **Proposition** 240 to study the inverse binomial transform of $s_1(n)$. By that proposition, the inverse binomial transform of $H_n(-i)i^n$ is given by $i^nH_n(-i+\frac{1}{2i}) = H_n(-\frac{i}{2})i^n$. This is the sequence

$$1, 1, 3, 7, 25, 81, 331, 1303, 5937, \ldots$$

with e.g.f. e^{x+x^2}. This is A047974 which satisfies the recurrence $a_n = a_{n-1} + 2(n-1)a_{n-2}$. It is in fact equal to $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k, k)2^k$. The second inverse binomial transform of $s_1(n)$ is the sequence

$$1, 0, 2, 0, 12, 0, 120, 0, 1680, 0, 30240, \ldots$$

with e.g.f. e^{x^2}. This is an "aerated" version of the quadruple factorial numbers $C_n(n+1)! = \frac{(2n)!}{n!}$, or A001813.

We now look at the central coefficients $B_1(2n, n)$ of $\mathbf{B}_1$. We have

$$\begin{aligned} B_1(2n,n) &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j)!} \\ &= C_n(n+1)!\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j)!} \\ &= C_n(n+1)\sum_{j=0}^{n}\binom{n}{j}^2 j! \\ &= C_n(n+1)!L_n(-1). \end{aligned}$$

Hence

$$\frac{B_1(2n,n)}{C_n(n+1)!} = L_n(-1).$$

We note that this is the rational sequence $1, 2, \frac{7}{2}, \frac{17}{3}, \frac{209}{24}, \ldots$. Two other ratios are of interest.

1. $\frac{B_1(2n,n)}{C(2n,n)} = n! L_n(-1)$ is A002720. It has e.g.f. $\frac{1}{1-x} \exp\left(\frac{x}{1-x}\right)$. It is equal to the number of partial permutations of an n-set, as well as the number of matchings in the bipartite graph $K(n,n)$. Using Equation (11.4) we can show that these numbers obey the following recurrence:
$$u_n = 2nu_{n-1} - (n-1)^2 u_{n-2}$$
with $u_0 = 1$, $u_1 = 2$.

2. $\frac{B_1(2n,n)}{C_n} = (n+1)! L_n(-1)$ is A052852$(n+1)$. It has e.g.f. given by
$$\frac{d}{dx} \frac{x}{1-x} \exp\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^3} \exp\left(\frac{x}{1-x}\right).$$
Again using Equation (11.4) we can show that these numbers obey the following recurrence:
$$v_n = 2(n+1)v_{n-1} - (n^2-1)v_{n-2}$$
with $v_0 = 1$, $v_1 = 4$.

This sequence counts the number of $(121, 212)$-avoiding n-ary words of length n. Specifically,
$$\frac{B_1(2n,n)}{C_n} = f_{121,212}(n+1, n+1)$$
where
$$f_{121,212}(n,k) = \sum_{j=0}^{k} \binom{k}{j} \binom{n-1}{j-1} j!$$
is defined in [45].

From this last point, we find the following expression
$$B_1(2n,n) = C_n \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{n}{j-1} j!. \tag{11.5}$$

Based on the fact that
$$C_n = \binom{2n}{n} - \binom{2n}{n-1}$$
we define
$$C_1(n) = B_1(2n,n) - B_1(2n,n-1) = B_1(2n,n) - B_1(2n,n+1).$$

We call $C_1(n)$ the *the generalized Catalan numbers associated with the triangle* $\mathbf{B}_1$. We calculate $B_1(2n, n-1)$ as follows:

$$\begin{aligned} B_1(2n, n-1) &= \frac{(2n)!}{(n-1)!}\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{1}{(n-j+1)!} \\ &= \frac{(2n)!}{(n-1)!}\sum_{j=0}^{n-1}\binom{n}{j}\frac{n-j}{n}\frac{1}{(n-j+1)!} \\ &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j)!}\frac{n-j}{n-j+1}. \end{aligned}$$

Hence

$$\begin{aligned} B_1(2n, n) - B_1(2n, n-1) &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j)!}\left(1 - \frac{n-j}{n-j+1}\right) \\ &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j)!}\frac{1}{n-j+1} \\ &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j+1)!}. \end{aligned}$$

Starting from the above, we can find many expressions for $C_1(n)$. For example,

$$\begin{aligned} C_1(n) &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j+1)!} \\ &= C_n\sum_{j=0}^{n}\binom{n}{j}\frac{(n+1)!}{(n+1-j)!} \\ &= C_n\sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j}j! \\ &= C_n\sum_{j=0}^{n}\binom{n}{j}^2\frac{n+1}{n-j+1}j! \\ &= C_n\sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j+1}\frac{(j+1)!}{n-j+1} \end{aligned}$$

where we have used the fact that $\frac{(2n)!}{n!} = C_n(n+1)!$. This is the sequence A001813 of quadruple factorial numbers with e.g.f. $\frac{1}{\sqrt{1-4x}}$.

Recognizing that the terms after C_n represent convolutions, we can also write

$$\begin{aligned} C_1(n) &= C_n \sum_{j=0}^{n} \binom{n}{j} \frac{(n+1)!}{(j+1)!} \\ &= C_n \sum_{j=0}^{n} \binom{n}{j} \binom{n+1}{j+1} (n-j)! \\ &= C_n \sum_{j=0}^{n} \binom{n}{j}^2 \frac{n+1}{j+1} (n-j)!. \end{aligned}$$

We note that the first expression immediately above links $C_1(n)$ to the Lah numbers A008297 (see Chapter 8).

The ratio $\frac{C_1(n)}{C_n}$, or $\sum_{j=0}^{n} \binom{n}{j} \frac{(n+1)!}{(k+1)!}$, is the sequence

$$1, 3, 13, 73, 501, 4051, 37633, 394353, 4596553, \ldots$$

or A000262$(n+1)$. This is related to the number of partitions of $[n] = \{1, 2, 3, \ldots, n\}$ into any number of lists, where a list means an ordered subset. It also has applications in quantum physics [31]. The sequence has e.g.f.

$$\frac{d}{dx} e^{\frac{x}{1-x}} = \frac{e^{\frac{x}{1-x}}}{(1-x)^2},$$

which represents the row sums of the Riordan array $\left[\frac{1}{(1-x)^2}, \frac{x}{1-x}\right] = \mathbf{Lag}^{(1)}$. We can in fact describe this ratio in terms of the Narayana numbers $\tilde{N}(n, k)$ as follows:

$$\begin{aligned} \frac{C_1(n)}{C_n} &= \sum_{j=0}^{n} \binom{n}{j} \binom{n+1}{j+1} (n-j)! \\ &= \sum_{j=0}^{n} \frac{n-j+1}{n+1} \binom{n+1}{j} \binom{n+1}{j+1} (n-j)! \\ &= \sum_{j=0}^{n} \frac{1}{n+1} \binom{n+1}{j} \binom{n+1}{j+1} (n-j+1)! \\ &= \sum_{j=0}^{n} \tilde{N}(n,j)(n-j+1)! \\ &= \sum_{j=0}^{n} \tilde{N}(n,n-j)(j+1)! \\ &= \sum_{j=0}^{n} \tilde{N}(n,j)(j+1)! \end{aligned}$$

Hence we have

$$\frac{C_1(n)}{C_n} = \sum_{j=0}^{n} \tilde{N}(n,j)(n-j+1)! = \sum_{j=0}^{n} \tilde{N}(n,j)(j+1)! = \sum_{j=0}^{n} \binom{n}{j} \frac{(n+1)!}{(j+1)!}.$$

We can generalize Proposition 247 as follows:

Proposition 252. *The sequence with e.g.f. $e^{ax+bx^2/2}$ has general term u_n given by*

$$u_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathit{Bessel}(n-k,k) a^{n-2k} b^k.$$

Proof. We have

$$\begin{aligned} n![x^n]e^{ax+bx^2/2} &= n![x^n] \sum_{i=0}^{\infty} \frac{a^i x^i}{i!} \sum_{k=0}^{\infty} \frac{b^k x^k}{2^k k!} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k k!} a^{n-2k} b^k \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k,k) a^{n-2k} b^k \\ &= \sum_{k=0}^{n} \text{Bessel}(k,n-k) a^{2k-n} b^{n-k}. \end{aligned}$$

□

Corollary 253. *Let u_n be the sequence with e.g.f. $e^{ax+bx^2/2}$. Then u_n satisfies the recurrence*

$$u_n = a u_{n-1} + (n-1) b u_{n-2}$$

with $u_0 = 1$, $u_1 = a$.

Corollary 254. *Let u_n be the sequence with e.g.f. $e^{ax+bx^2/2}$. Then*

$$u_n = \sum_{k=0}^{n} \mathit{Bessel}(k,n-k) a^{2k-n} b^{n-k}.$$

Corollary 255. *Let u_n be the sequence with e.g.f. $e^{ax+bx^2/2}$. Then u_n has generating function expressible as the continued fraction*

$$\cfrac{1}{1-ax-\cfrac{bx^2}{1-ax-\cfrac{2bx^2}{1-ax-\cfrac{3bx^2}{1-ax-\cdots}}}}.$$

We close this section by remarking that the triangle A001497 with general term

$$[k \le n]\text{Bessel}(n,n-k)$$

has bi-variate g.f. given by

$$\cfrac{1}{1-xy-\cfrac{x}{1-xy-\cfrac{2x}{1-xy-\cfrac{3x}{1-\cdots}}}}$$

This triangle begins

$$(\text{Bessel}(n,n-k)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 3 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 15 & 15 & 6 & 1 & 0 & 0 & \ldots \\ 105 & 105 & 45 & 10 & 1 & 0 & \ldots \\ 945 & 945 & 420 & 105 & 15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

11.4 The General Case

We shall now look at the row sums, central coefficients and generalized Catalan numbers associated with the general matrix $\mathbf{B}_r$. In what follows, proofs follow the methods developed in the last section.

Proposition 256. *The row sums $s_r(n)$ of $\mathbf{B}_r$ are given by $H_n(-\frac{i}{\sqrt{r}})(\sqrt{r}i)^n$.*

Proof. The row sums of $\mathbf{B}_r$ are given by the sequence

$$\sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{r^j}{(n-k-j)!}$$

with e.g.f. $g(x)e^{f(x)} = e^x e^{x(1+rx)} = e^{2x+rx^2}$. Now

$$\begin{aligned} e^{2x+rx^2} &= e^{2(\frac{-i}{\sqrt{r}})(i\sqrt{r}x)-(i\sqrt{r}x)^2} \\ &= \sum_{n=0}^{\infty} \frac{H_n(-\frac{i}{\sqrt{r}})}{n!}(i\sqrt{r}x)^n \\ &= \sum_{n=0}^{\infty} \frac{H_n(-\frac{i}{\sqrt{r}})(i\sqrt{r})^n}{n!} x^n \end{aligned}$$

$\square$

Corollary 257. *We have the identity*

$$\sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{r^j}{(n-k-j)!} = H_n\left(-\frac{i}{\sqrt{r}}\right)(\sqrt{r}i)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} 2^{n-2k} r^k.$$

As before, we can rewrite this using the fact that $\frac{(2k)!}{k!} = C(k)(k+1)! = 2^k(2k-1)!!$.

We note that the second inverse binomial transform of $s_r(n)$ has e.g.f. e^{rx^2}.

Proposition 258. *The row sums of* $\mathbf{B}_r$ *are equal to the diagonal sums of the matrix with general term* $\text{Bessel}(n,k)2^n r^k$*. That is,*

$$\sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{r^j}{(n-k-j)!} = H_n\left(-\frac{i}{\sqrt{r}}\right)(\sqrt{r}i)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k,k)2^{n-k}r^k.$$

Proposition 259. *The row sums of* $\mathbf{B_r}$ *obey the recurrence*

$$u_n = 2u_{n-1} + 2r(n-1)u_{n-2}$$

with $u_0 = 1$*,* $u_1 = 2$*.*

We now turn our attention to the central coefficients of $\mathbf{B}_r$.

Proposition 260. $B_r(2n,n) = C_n(n+1)! \sum_{j=0}^{n} \binom{k}{j}^2 j! r^j$

Proof. The proof is the same as the calculation for $B_1(2n,n)$ in Example 245, with the extra factor of r^j to be taken into account. □

Corollary 261.

$$\frac{B_r(2n,n)}{C_n(n+1)!} = r^n L_n\left(-\frac{1}{r}\right) = \sum_{j=0}^{n} \binom{n}{j} \frac{r^j}{(n-j)!}$$

for $r \neq 0$*.*

We note that the above expressions are not integers in general.

For instance, $\frac{B_2(2n,n)}{C(2n,n)} = n!2^n L_n\left(-\frac{1}{2}\right)$ is A025167, and $\frac{B_3(2n,n)}{C(2n,n)} = n!3^n L_n\left(-\frac{1}{3}\right)$ is A102757. In general, we have

Proposition 262. $\frac{B_r(2n,n)}{C(2n,n)} = n!r^n L_n(-1/r)$ *has e.g.f.* $\frac{1}{1-rx}\exp(\frac{x}{1-rx})$*, and satisfies the recurrence relation*

$$u_n = ((2n-1)r+1)u_{n-1} - r^2(n-1)^2 u_{n-2}$$

with $u_0 = 1$*,* $u_1 = r+1$*.*

Proof. We have

$$\begin{aligned}
n![x^n]\frac{e^{\frac{x}{1-rx}}}{(1-rx)} &= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}\frac{x^i}{(1-rx)^i}(1-rx)^{-1} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i(1-rx)^{-i}(1-rx)^{-1} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i(1-rx)^{-(i+1)} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i\sum_{j=0}\binom{-(i+1)}{j}(-1)^j r^j x^j \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}\sum_{j=0}\binom{i+j}{j}r^j x^{i+j} \\
&= n!\sum_{j=0}^{n}\binom{n}{j}\frac{r^j}{(n-j)!}.
\end{aligned}$$

To prove the second assertion, we use Equation (11.4) with $t=-\frac{1}{r}$. Multiplying by $n!r^{n+1}$, we obtain

$$(n+1)!r^{n+1}L_{n+1}\left(-\frac{1}{r}\right)=(2n+1+\frac{1}{r})r^{n+1}n!L_n\left(-\frac{1}{r}\right)-r^{n+1}n^2(n-1)!L_{n-1}\left(-\frac{1}{r}\right).$$

Simplifying, and letting $n\to n-1$, gives the result. $\square$

Corollary 263. $\frac{B_r(2n,n)}{C_n}=(n+1)!r^nL_n(-1/r)$ *has e.g.f.* $\frac{d}{dx}\frac{x}{1-rx}\exp(\frac{x}{1-rx})$, *and satisfies the recurrence*

$$w_n=((2n-1)+r)\frac{n+1}{n}w_{n-1}-r^2(n^2-1)w_{n-2}$$

for $n>1$, *with* $w_0=1$ *and* $w_1=2r+2$.

We can generalize Equation (11.5) to get

$$B_r(2n,n)=C_n\sum_{j=0}^{n+1}\binom{n+1}{j}\binom{n}{j-1}j!r^{j-1}.$$

We define *the generalized Catalan numbers associated with the triangles* $\mathbf{B}_r$ to be the numbers

$$C_r(n)=B_r(2n,n)-B_r(2n,n-1).$$

Using the methods of **Example** 245, we have

Proposition 264. *We have the following equivalent expressions for $C_r(n)$:*

$$\begin{aligned}
C_r(n) &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{r^j}{(n-j+1)!}\\
&= C_n\sum_{j=0}^{n}\binom{n}{j}\frac{(n+1)!}{(j+1)!}r^{n-j}\\
&= C_n\sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j+1}(n-j)!r^{n-j}\\
&= C_n\sum_{j=0}^{n}\frac{n+1}{j+1}\binom{n}{j}^2(n-j)!r^{n-j}\\
&= C_n\sum_{j=0}^{n}\tilde{N}(n,j)(j+1)!r^j.
\end{aligned}$$

For instance, $C_2(n)/C_n$ is A025168.

Proposition 265. $\frac{C_r(n)}{C_n}$ *has e.g.f.*

$$\frac{d}{dx}e^{\frac{x}{1-rx}} = \frac{e^{\frac{x}{1-rx}}}{(1-rx)^2}.$$

Proof. We have

$$\begin{aligned}
n![x^n]\frac{e^{\frac{x}{1-rx}}}{(1-rx)^2} &= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}\frac{x^i}{(1-rx)^i}(1-rx)^{-2}\\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i(1-rx)^{-i}(1-rx)^{-2}\\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i(1-rx)^{-(i+2)}\\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i\sum_{j=0}^{\infty}\binom{-(i+2)}{j}(-1)^jr^jx^j\\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}\sum_{j=0}^{\infty}\binom{i+j+1}{j}r^jx^{i+j}\\
&= n!\sum_{j=0}^{\infty}\binom{n+1}{j}\frac{r^j}{(n-j)!}\\
&= (n+1)!\sum_{j=0}^{n}\binom{n}{j}\frac{r^j}{(n-j+1)!}.
\end{aligned}$$

□

11.5 The case $r = \frac{1}{2}$

The assumption so far has been that r is an integer. In this section, we indicate that $r = \frac{1}{2}$ also produces a generalized Pascal triangle. We have $\mathbf{B}_{\frac{1}{2}} = (e^x, x(1+x/2))$. This begins

$$\mathbf{B}_{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 21 & 10 & 1 & 0 & \dots \\ 1 & 15 & 55 & 55 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is triangle A100862. Quoting from A100862, $B_{\frac{1}{2}}(n,k)$ "is the number of k-matchings of the corona $K'(n)$ of the complete graph $K(n)$ and the complete graph $K(1)$; in other words, $K'(n)$ is the graph constructed from $K(n)$ by adding for each vertex v a new vertex v' and the edge vv'". The row sums of this triangle, A005425, are given by

$$1, 2, 5, 14, 43, 142, 499, 1850, 7193, \dots$$

These have e.g.f. $e^{2x+x^2/2}$ and general term

$$H_n(-\sqrt{2}i)(i/\sqrt{2})^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} 2^{n-3k}.$$

They obey the recurrence

$$u_n = 2u_{n-1} + (n-1)u_{n-2}$$

with $u_0 = 1$, $u_1 = 2$. This sequence is thus the second binomial transform of the aerated double factorial numbers (see A001147) and the binomial transform of the involution numbers A000085. They have g.f. given by the continued fraction

$$\cfrac{1}{1-2x-\cfrac{x^2}{1-2x-\cfrac{2x^2}{1-2x-\cfrac{3x^2}{1-2x-\cfrac{4x^2}{1-2x-\cdots}}}}}$$

[32] provides an example of their use in quantum physics. Using Proposition 240 or otherwise, we see that the inverse binomial transform of this sequence, with e.g.f. $e^{x+x^2/2}$, is given by

$$H_n\left(-\sqrt{2}i + \frac{i}{\sqrt{2}}\right)(i/\sqrt{2})^n = H_n\left(-\frac{i}{\sqrt{2}}\right)(i/\sqrt{2})^n.$$

This is the sequence
$$1, 1, 2, 4, 10, 26, 76, 232, 765, \ldots$$
or A000085. It has many combinatorial interpretations, including for instance the number of matchings in the complete graph $K(n)$. These numbers are the diagonal sums of the Bessel triangle **Bessel**:
$$H_n\left(-\frac{i}{\sqrt{2}}\right)(i/\sqrt{2})^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k, k).$$
Alternatively they are the row sums of the aerated Bessel triangle beginning
$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 1 & 0 & 0 & \ldots \\ 3 & 0 & 6 & 0 & 1 & 0 & \ldots \\ 0 & 15 & 0 & 10 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$
with general term
$$Bessel^*\left(\frac{n+k}{2}, k\right)\frac{1+(-1)^{n+k}}{2}$$
where
$$Bessel^*(n, k) = \frac{(2n-k)!}{k!(n-k)!2^{n-k}}.$$
This sequence has g.f. given by the continued fraction
$$\cfrac{1}{1-x-\cfrac{x^2}{1-x-\cfrac{2x^2}{1-x-\cfrac{3x^2}{1-x-\cfrac{4x^2}{1-x-\cdots}}}}}$$
The aerated triangle above has g.f. given by the bi-variate continued fraction
$$\cfrac{1}{1-xy-\cfrac{x^2}{1-xy-\cfrac{2x^2}{1-xy-\cfrac{3x^2}{1-xy-\cfrac{4x^2}{1-xy-\cdots}}}}}$$
As we have seen, the row sums of $\mathbf{B}_{\frac{1}{2}}$ are the second binomial transform of the sequence
$$1, 0, 1, 0, 3, 0, 15, 0, 105, 0, \ldots$$

with e.g.f. $e^{x^2/2}$ and o.g.f. equal to

$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{2x^2}{1-\cfrac{3x^2}{1-\cfrac{4x^2}{1-\cdots}}}}}$$

This is an "aerated" version of the double factorial numbers $(2n-1)!!$, or A001147. These count the number of perfect matchings in the complete graph $K(2n)$. The row sums count the number of $12-3$ and $214-3$-avoiding permutations, as well as the number of matchings of the corona $K'(n)$ of the complete graph $K(n)$ and the complete graph $K(1)$. We note that the exponential Riordan array $[1, x(1+\frac{x}{2})]$ has general term

$$Bessel(k, n-k).$$

This array starts

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 6 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 15 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is 122848. The product of $\mathbf{B}$ and this matrix is then equal to $\mathbf{B}_{\frac{1}{2}}$, that is,

$$\mathbf{B}_{\frac{1}{2}} = \mathbf{B} \cdot (Bessel(k, n-k)).$$

The example of $\mathbf{B}_{\frac{1}{2}}$ prompts us to define a new family $\tilde{\mathbf{B}}_r$ where $\tilde{\mathbf{B}}_r$ is the element $[e^x, x(1+\frac{rx}{2})]$ of the exponential Riordan group. Then we have $\tilde{\mathbf{B}}_0 = \mathbf{B}$, $\tilde{\mathbf{B}}_1 = \mathbf{B}_{\frac{1}{2}}$, $\tilde{\mathbf{B}}_2 = \mathbf{B}_1$ etc. We can then show that $\tilde{\mathbf{B}}_r$ is the product of the binomial matrix $\mathbf{B}$ and the matrix with general term $\text{Bessel}(k, n-k)r^{n-k}$:

$$\tilde{\mathbf{B}}_r = \mathbf{B} \cdot \left(Bessel(k, n-k)r^{n-k}\right).$$

We have

$$\tilde{\mathbf{B}}_r(n,k) = \frac{n!}{k!}\sum_{j=0}^{k} \frac{1}{2^j}\binom{n}{k}\frac{r^j}{(n-k-j)!} = \sum_{j=0}^{n}\binom{n}{j}\frac{j!r^{j-k}}{(2k-j)!2^{j-k}(j-k)!}.$$

Thus

$$\tilde{\mathbf{B}}_r(n,k) = \binom{n}{k}\sum_{j=0}^{n}\binom{n-k}{n-j}\frac{k!}{(2k-j)!}\frac{r^{j-k}}{2^{j-k}}$$

and in particular

$$\tilde{\mathbf{B}}_r(2n,n)=\binom{2n}{n}\sum_{j=0}^{2n}\binom{n}{j-n}\frac{n!}{(2n-j)!}\frac{r^{j-n}}{2^{j-n}}.$$

Finally,

$$\tilde{\mathbf{B}}_r(n,k)=\tilde{\mathbf{B}}_r(n-1,k-1)+\tilde{\mathbf{B}}_r(n-1,k)+r(n-1)\tilde{\mathbf{B}}_r(n-2,k-1).$$

The foregoing has shown that the triangles $\mathbf{B}_r$, and more generally $\tilde{\mathbf{B}}_r$, defined in terms of exponential Riordan arrays, are worthy of further study. Many of the sequences linked to them have significant combinatorial interpretations. $\mathbf{B}_{\frac{1}{2}}$ as documented in A100862 by Deutsch has a clear combinatorial meaning. This leaves us with the challenge of finding combinatorial interpretations for the general arrays $\tilde{\mathbf{B}}_r$, $r\in\mathbf{Z}$.

11.6 A family of generalized Narayana triangles

We can use the mechanism of generalized factorials to develop a family of generalized Narayana triangles in a manner similar to the foregoing. We recall that $\tilde{\mathbf{N}}=\mathbf{P}_{\binom{n+1}{2}}$. In other words,

$$\tilde{N}_{n,k}=[k\le n]\frac{a!(n)}{a!(k)a!(n-k)}$$

where $a(n)=\binom{n+1}{2}$. We now form the matrix $\tilde{\mathbf{N}}^s$ with general term

$$N^s_{n,k}=[k\le n]\frac{a!(k)}{a!(n-k)a!(2k-n)}.$$

This matrix therefore begins

$$\tilde{\mathbf{N}}^s=\begin{pmatrix}1&0&0&0&0&0&\dots\\0&1&0&0&0&0&\dots\\0&1&1&0&0&0&\dots\\0&0&3&1&0&0&\dots\\0&0&1&6&1&0&\dots\\0&0&0&6&10&1&\dots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Then the product $\tilde{\mathbf{N}}\tilde{\mathbf{N}}^s$ is again a Pascal-like matrix, which begins

$$\begin{pmatrix}1&0&0&0&0&0&\dots\\1&1&0&0&0&0&\dots\\1&6&1&0&0&0&\dots\\1&24&24&1&0&0&\dots\\1&70&260&70&1&0&\dots\\1&165&1850&1850&165&1&\dots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

In general, we have the result

Proposition 266. *Let $\tilde{\mathbf{N}}^s(r)$ be the matrix with general element*

$$N^s_{n,k}(r) = [k \le n]\frac{a!(k)}{a!(n-k)a!(2k-n)}r^{n-k}.$$

Then the product

$$\tilde{\mathbf{N}}\tilde{\mathbf{N}}^s(r)$$

is a Pascal-like triangle.

Thus we have another one-parameter family of Pascal-like triangles, for which the case $r = 0$ is the Narayana triangle $\tilde{\mathbf{N}}$. The general term of the product matrix is given by

$$\sum_{j=0}^{n}[j \le n]\frac{a!(n)}{a!(j)a!(n-j)}[k \le j]\frac{a!(k)r^{j-k}}{a!(j-k)a!(2k-j)}$$

where $a(n) = \binom{n+1}{2}$. We can in fact show that this is equal to

$$a!(n)\sum_{j=0}^{k}\frac{r^j}{a!(j)a!(k-j)a!(n-k-j)}.$$

Chapter 12

The Hankel transform of integer sequences

In recent years, the Hankel transform of integer sequences has been the centre of much attention [61, 78, 80, 85, 121, 139, 163, 188, 184, 187, 182, 186, 183, 181, 207]. Although the transform is easy to define, by means of special determinants, it is possible to describe its effect on specific sequences by means of a closed formula only in a small number of cases. Even where such formulas are known, it is not normally easy to relate these formulas to a clear combinatorial interpretation. Techniques used to elucidate the nature of many Hankel transforms rely heavily on the theory of determinants, the theory orthogonal polynomials, measure theory, the theory of continued fractions, the theory of plane partitions and lattice paths ([100, 223]). Hankel transforms have appeared in some seminal works e.g. [160].

12.1 The Hankel transform

The *Hankel transform* of a given sequence $A = \{a_0, a_1, a_2, ...\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ where $h_n = |a_{i+j}|_{i,j=0}^{n}$, i.e

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \rightarrow \quad h = \{h_n\}_{n\in\mathbb{N}_0} : \quad h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix} \tag{12.1}$$

The Hankel transform of a sequence a_n and its binomial transform are equal.

It is known (for example, see [132, 227]) that the Hankel determinant h_n of order n of the sequence $(a_n)_{n\geq 0}$ equals

$$h_n = a_0^{n+1}\beta_1^n\beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n \ , \tag{12.2}$$

where $(\beta_n)_{n\geq 1}$ is the sequence given by:

$$\mathcal{G}(x) = \sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1+\alpha_0 x - \cfrac{\beta_1 x^2}{1+\alpha_1 x - \cfrac{\beta_2 x^2}{1+\alpha_2 x - \cdots}}} \ . \tag{12.3}$$

The sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ are the coefficients in the recurrence relation

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x) \ , \tag{12.4}$$

where $(P_n(x))_{n\geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional $\mathcal{L}$ determined by

$$a_n = \mathcal{L}[x^n] \quad (n = 0, 1, 2, \ldots) \ . \tag{12.5}$$

In some cases, there exists a weight function $w(x)$ such that the functional $\mathcal{L}$ can be expressed by

$$\mathcal{L}[f] = \int_{\mathbb{R}} f(x)\ w(x)\ dx \qquad \big(f(x) \in C(\mathbb{R});\ w(x) \geq 0\big) \ . \tag{12.6}$$

Thus we can associate to every weight $w(x)$ two sequences of coefficients, i.e.

$$w(x)\ \mapsto \{\alpha_n, \beta_n\}_{n\in\mathbb{N}_0} \ , \tag{12.7}$$

by

$$\alpha_n = \frac{\mathcal{L}[x\ P_n^2(x)]}{\mathcal{L}[P_n^2(x)]} \ , \qquad \beta_n = \frac{\mathcal{L}[P_n^2(x)]}{\mathcal{L}[P_{n-1}^2(x)]} \qquad (n \in \mathbb{N}_0) \ . \tag{12.8}$$

For a family of monic orthogonal polynomials $(P_n)_{n\geq 0}$ we can write

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k,$$

where $a_{n,n} = 1$. Then the coefficient array $(a_{n,k})_{n,k\geq 0}$ forms a lower-triangular matrix.

12.2 Examples of the Hankel transform of an integer sequence

Example 267. We consider generalized Fibonacci sequences of the form

$$a_n = ra_{n-1} + sa_{n-2}$$

for given integers r and s, where $a_0 = 1$, $a_1 = 1$. The case $r = s = 1$ is the case of the Fibonacci numbers $F(n+1)$. The order of this recurrence is clearly 2. If we denote by H_j^n the j-th column of the determinant h_n, then we see that

$$H_j^n = rH_{j-1}^n + sH_{j-2}^n$$

corresponding to the recurrence that determines the sequence a_n. Thus after $n > 1$, there is a linear dependence between columns of the determinants and hence their value is 0. We find that the Hankel transform of a_n is equal to the sequence

$$1, r+s-1, 0, 0, 0, \ldots$$

$$h_0 = \begin{vmatrix} 1 \end{vmatrix} = 1, \qquad h_1 = \begin{vmatrix} 1 & 1 \\ 1 & r+s \end{vmatrix} = r+s-1.$$

Since many pairs (r, s) have sum $r+s$ we see that many sequences can have the same Hankel transform. Thus the Hankel transform is not unique and therefore is not invertible.

More generally, if

$$a_n = ra_{n-1} + sa_{n-2}, \qquad a_0 = t_0, \qquad a_1 = t_1,$$

then the Hankel transform of a_n is given by

$$t_0, rt_0t_1 + st_0^2 - t_1^2, 0, 0, 0, \ldots$$

Similarly, if

$$a_n = ra_{n-1} + sa_{n-2} + ta_{n-3}, \qquad a_0 = 1, \qquad a_1 = 1, \qquad a_2 = 1,$$

then the Hankel transform of a_n is given by

$$1, 0, 2(1 - rs - rt - st) - (r-1)^2 - (s-1)^2 - (t-1)^2, 0, 0, 0, \ldots$$

Example 268. The Catalan numbers have many remarkable properties. Their Hankel transform is no exception to this. We can in fact characterize the Catalan numbers as the unique integer sequence C_n such that both C_n and C_{n+1} have the all 1's sequence $1, 1, 1, \ldots$ as Hankel transforms.

$$|1| = 1, \qquad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \qquad \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{vmatrix} = 1, \qquad \ldots$$

and

$$|1| = 1, \qquad \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1, \qquad \begin{vmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{vmatrix} = 1, \qquad \ldots$$

Many proofs of this result exist. For instance, to show the necessity, we can cite the following result [100] :

$$\mid C_{i+\alpha_j} \mid_0^n = \prod_{0 \le i < j \le n} (\alpha_j - \alpha_i) \prod_{j=0}^{n} \frac{(2\alpha_j)!}{\alpha_j!(n+\alpha_j+1)!} \prod_{i=0}^{n} \frac{(2i+1)!}{i!}.$$

Allowing $\alpha_j = j$ and $\alpha_j = j+1$ yields the sequence $1, 1, 1, \ldots$ in both cases.

In order to illustrate techniques that will be employed in later chapters, we provide two different proofs of the fact that the Hankel transform of C_n is $1, 1, 1, \ldots$.

Proof 1. We have

$$C_n = \frac{1}{2\pi} \int_0^4 x^n \frac{\sqrt{x(4-x)}}{x} dx.$$

Thus the measure for which the Catalan numbers are moments is given by

$$\mu(x) = w(x)dx = \frac{1}{2\pi} \frac{\sqrt{x(4-x)}}{x} dx = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} dx.$$

Making the change of variable $x = 2 + 2t$ we obtain

$$\sqrt{\frac{4-x}{x}} = \sqrt{\frac{2-2t}{2+2t}} = \sqrt{\frac{1-t}{1+t}}$$

and $dx = 2dt$. Thus

$$\mu = 2\sqrt{\frac{1-t}{1+t}} dt,$$

or 2 times the measure for W_n, the Chebyshev polynomials of the fourth kind. For these polynomials, we have [99]

$$\beta_0 = \pi, \qquad \beta_n = \frac{1}{4}, \qquad n \geq 1.$$

Applying (i) and (ii) of the following lemma [99] now allows us to conclude that $h_n = 1$.

Lemma 269. *Let*

$$w(x) \ \mapsto \ \{\alpha_n, \ \beta_n\}_{n \in \mathbb{N}_0}, \qquad \tilde{w}(x) \ \mapsto \ \{\tilde{\alpha}_n, \quad \tilde{\beta}_n\}_{n \in \mathbb{N}_0} \ . \tag{12.9}$$

Then

$$\text{(i)} \qquad \tilde{w}(x) = Cw(x) \quad \Rightarrow \quad \{\tilde{\alpha}_n = \alpha_n, \quad \tilde{\beta}_0 = C\beta_0, \quad \tilde{\beta}_n = \beta_n \ (n \in \mathbb{N})\} \ ; \tag{12.10}$$

$$\text{(ii)} \qquad \tilde{w}(x) = w(ax+b) \quad \Rightarrow \quad \{\tilde{\alpha}_n = \frac{\alpha_n - b}{a}, \quad \tilde{\beta}_0 = \frac{\beta_0}{a}, \quad \tilde{\beta}_n = \frac{\beta_n}{a^2} \ (n \in \mathbb{N})\} \ ; \tag{12.11}$$

(iii) If

$$w_c(x) = \frac{\tilde{w}(x)}{x-c} \qquad (c \notin supp(\tilde{w})) \ , \tag{12.12}$$

then

$$\begin{aligned} \alpha_{c,0} &= \tilde{\alpha}_0 + r_0 \ , & \alpha_{c,k} &= \tilde{\alpha}_k + r_k - r_{k-1}, \\ \beta_{c,0} &= -r_{-1}, & \beta_{c,k} &= \tilde{\beta}_{k-1}\frac{r_{k-1}}{r_{k-2}} \qquad (k \in \mathbb{N}) \ , \end{aligned} \tag{12.13}$$

where

$$r_{-1} = -\int_{\mathbb{R}} w_c(x) \ dx, \qquad r_n = c - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots) \ . \tag{12.14}$$

Proof 2. We use the following result from [121]

Theorem 270. *Consider an o.g.f.* $\Phi(x) = \frac{1}{1-\Psi(x)}$ *of a sequence* (a_n)*, with* $\Psi(0) = 0$ *and suppose that*

$$\xi(x) = \frac{\Psi(x)}{x} - \Psi'(0)$$

satisfies

$$\xi(x) = x(\lambda + \mu\xi(x) + \nu\xi^2(x)).$$

Then the Hankel transform (h_n) *of* (a_n) *is given by*

$$h_n = \lambda^{n(n+1)/2}\nu^{n(n-1)/2}.$$

Thus we let $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the o.g.f. of the Catalan numbers, and let

$$\Phi(x) = c(x) = \frac{1-\sqrt{1-4x}}{2x},$$

and

$$\Psi(x) = 1 - \frac{1}{\Phi(x)} = \frac{1-\sqrt{1-4x}}{2}.$$

We note that $\Psi'(0) = 1$ and so we can define

$$\xi(x) = \frac{\Psi(x)}{x} - \Psi'(0) = \frac{1-2x-\sqrt{1-4x}}{2x}.$$

We now seek λ, μ and ν such that if

$$g(x) = \lambda + \mu\xi(x) + \nu\xi^2(x)$$

then we have

$$g(x) = \frac{\xi(x)}{x}.$$

In this case, we find that

$$\lambda = 1, \qquad \mu = 2, \qquad \nu = 1.$$

Thus the Hankel transform of C_n is given by

$$\lambda^{n(n+1)/2}\nu^{n(n-1)/2} = 1.$$

Now let

$$\Phi(x) = \frac{1-2x-\sqrt{1-4x}}{2x^2}$$

be the o.g.f. of C_{n+1}. Then

$$\Psi(x) = 1 - \frac{1}{\Phi(x)} = \frac{1+2x-\sqrt{1-4x}}{2}$$

with $\Psi'(0) = 2$. Thus we can define

$$\xi(x) = \frac{\Psi(x)}{x} - \Psi'(0) = \frac{1 - 2x - \sqrt{1-4x}}{2x}.$$

As in the previous case, we find that for

$$g(x) = \lambda + \mu\xi(x) + \nu\xi^2(x)$$

with

$$g(x) = \frac{\xi(x)}{x},$$

we have

$$\lambda = 1, \qquad \mu = 2, \qquad \nu = 1.$$

Hence again we find that the Hankel transform of C_{n+1} is the all 1's sequence.

12.3 A family of Hankel transforms defined by the Catalan numbers

We now study a family of Hankel transforms that give rise to sequences that have been much studied in the literature. This family will be defined by the Hankel transforms of the columns of the sequence array of the Catalan numbers. Thus we consider the array $(c(x), x))$ with general term $T_{n,k} := C_{n-k}[k \le n]$. We define the array with general term $H_{n,k} = |T_{i+j,k}|_{i,j=0}^n$. Thus the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 5 & 2 & 1 & 1 & 0 & 0 & \dots \\ 14 & 5 & 2 & 1 & 1 & 0 & \dots \\ 42 & 14 & 5 & 2 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is mapped onto the matrix of Hankel transforms

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -2 & -1 & 0 & 0 & 0 & \dots \\ 1 & -3 & -5 & 1 & 0 & 0 & \dots \\ 1 & -4 & -14 & 14 & 1 & 0 & \dots \\ 1 & -5 & -30 & 84 & 42 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Multiplying by $(-1)^{\binom{k+1}{2}}$, we obtain the positive matrix

$$((-1)^{\binom{k+1}{2}}H_{n,k}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 3 & 5 & 1 & 0 & 0 & \dots \\ 1 & 4 & 14 & 14 & 1 & 0 & \dots \\ 1 & 5 & 30 & 84 & 42 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The corresponding square array $((-1)^{\binom{k+1}{2}}H_{n+k,k})$ is given by

$$((-1)^{\binom{k+1}{2}}H_{n+k,k}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 5 & 14 & 42 & 132 & \dots \\ 1 & 3 & 14 & 84 & 594 & 4719 & \dots \\ 1 & 4 & 30 & 330 & 4719 & 81796 & \dots \\ 1 & 5 & 55 & 101 & 26026 & 884884 & \dots \\ 1 & 6 & 91 & 2548 & 111384 & 6852768 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This last array plays an important role in several theories:

1. It enumerates the number of Kekulé structures for certain prolate pentagons (special pentagon-shaped benzenoid hydrocarbons) [62].

2. It corresponds to the transpose of the matrix with general element

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{i+j+2k}{i+j} = \prod_{i=1}^{n}\frac{\binom{2i+2k}{i}}{\binom{2i}{i}}$$

which enumerates certain plane partitions [95].

3. This array can be embedded into the more general array

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 2 & 5 & 14 & 42 & \dots \\ 1 & 1 & 3 & 14 & 84 & 594 & \dots \\ 1 & 1 & 4 & 30 & 330 & 4719 & \dots \\ 1 & 1 & 5 & 55 & 101 & 26026 & \dots \\ 1 & 1 & 6 & 91 & 2548 & 111384 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where we conjecture that the o.g.f. of the n-th row is given by

$${}_{n+1}F_n(1, 1/2, 3/2, \dots, (2n-1)/2; n+1, n+2, \dots, 2n; 2^{2n}x).$$

The transpose of this array has general term

$$\prod_{i=1}^{n-1}\frac{\binom{2i+2k}{i}}{\binom{2i}{i}}.$$

12.4 Krattenthaler's results

It is useful to summarize the results of Krattenthaler on Hankel transforms of integer sequences. These can be derived from the interpretation of the Hankel transform that is based on a study of appropriate non-intersecting Motzkin paths, first found in [223].

Proposition 271. *[131, 132, 134] Let $(\mu_k)_{k\geq 0}$ be a sequence of numbers with generating function $\sum_{k=0}^{\infty}\mu_k x^k$ written in the form*

$$\sum_{k=0}^{\infty}\mu_k x^k = \cfrac{\mu_0}{1-a_0x-\cfrac{b_1x^2}{1-a_1x-\cfrac{b_2x^2}{1-a_2x-\cfrac{b_3x^2}{1-a_3x-\dots}}}}$$

Then

1. $$det(\mu_{k+j})_{0\leq i,j\leq n-1} = \mu_0^n b_1^{n-1}b_2^{n-2}\cdots b_{n-2}^2 b_{n-1}.$$

2. *If $(q_n)_{n\geq 0}$ is the sequence recursively defined by $q_0=1$, $q_1=-a_0$, and*

$$q_{n+1} = a_n q_n - b_n q_{n-1}, \tag{12.15}$$

then

$$det(\mu_{i+j+1})_{0\leq i,j\leq n-1} = \mu_0^n b_1^{n-1}b_2^{n-2}\cdots b_{n-2}^2 b_{n-1}q_n$$

and

$$det(\mu_{i+j+2})_{0\leq i,j\leq n-1} = \mu_0^n b_1^{n-1}b_2^{n-2}\cdots b_{n-2}^2 b_{n-1}\sum_{k=0}^{n} q_k^2 b_{k+1}\cdots b_{n-1}b_n.$$

3. *Letting $\mu_{-1}=0$, for $n\geq 2$ we have*

$$det(\mu_{i+j-1})_{0\leq i,j\leq n-1} = -\mu_0^2 b_0^{n-2} det(\tilde{\mu}_{i+j+1})_{0\leq i,j\leq n-3},$$

where the $\tilde{\mu}_k$'s are given by the generating function

$$\sum_{k=0}^{\infty}\tilde{\mu}_k x^k = \cfrac{\mu_0}{1-a_1x-\cfrac{b_2x^2}{1-a_2x-\cfrac{b_3x^2}{1-a_3x-\cfrac{b_4x^2}{1-a_4x-\dots}}}}.$$

Example 272. We illustrate (2) in the above with the sequence u_n given by $1, 1, 2, 6, 21, 79, 311, \ldots$, A033321, the binomial transform of the Fine numbers A000957. The g.f. of the sequence is given by

$$g(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - 3x - \cfrac{x^2}{1 - 3x - \cfrac{x^2}{1 - 3x - \dots}}}}$$

or $\frac{2}{1+x+\sqrt{1-6x+5x^2}} = \frac{1+x-\sqrt{1-6x+5x^2}}{2x(2-x)}$. It is given by the first column of the Riordan array

$$\left(\frac{1+2x}{1+3x+x^2}, \frac{x}{1+3x+x^2}\right)^{-1} = \left(\frac{1+x-\sqrt{1-6x+5x^2}}{2x(2-x)}, \frac{1-3x-\sqrt{1-6x+5x^2}}{2x}\right),$$

which has production array

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 3 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We see immediately that $h_n = 1$. We now wish to calculate the Hankel transform of u_{n+1}. Thus we must find q_n such that $q_0 = 1$, $q_1 = -a_0$, and

$$q_{n+1} = a_n q_n - b_n q_{n-1},$$

where $a_n = 1$ and $b_n = 3 - 2 \cdot 0^n$

We find that $q(n)$ is the sequence $F(2n-1)$ or $1, 1, 2, 5, 13, 34, 89, \dots$. Thus the Hankel transform of u_{n+1} is $F(2n+1)$. Now the Riordan array $\left(\frac{1+2x}{1+3x+x^2}, \frac{x}{1+3x+x^2}\right)$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & -4 & 1 & 0 & 0 & 0 & \dots \\ -5 & 13 & -7 & 1 & 0 & 0 & \dots \\ 13 & -40 & 33 & -10 & 1 & 0 & \dots \\ -34 & 120 & -132 & 62 & -13 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Inspecting equations 2.1 and 12.15, we see that we in fact have $q_n = (-1)^n t_n$, where t_n is the sequence in the first column of the coefficient array for the orthogonal polynomials associated to u_n. In this case, this is the Riordan array $\left(\frac{1+2x}{1+3x+x^2}, \frac{x}{1+3x+x^2}\right)$. We thus have the following result: the Hankel transform of the sequence u_{n+1}, where u_n is the binomial transform of the Fine numbers, is the sequence $F(2n+1)$ with g.f. $\frac{1-x}{1-3x+x^2}$.

It is instructive to carry out the same analysis for the Fine numbers. We can easily derive the following: The Fine numbers are defined as the first column of the Riordan array

$$\left(\frac{1+2x}{(1+x)^2}, \frac{1}{(1+x)^2}\right)^{-1} = \left(\frac{1}{1-(xc(x))^2}, c(x)-1\right),$$

which has production array

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 2 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The array $\left(\frac{1+2x}{(1+x)^2}, \frac{1}{(1+x)^2}\right)$ begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & -2 & 1 & 0 & 0 & 0 & \dots \\ 2 & 2 & -4 & 1 & 0 & 0 & \dots \\ -3 & 0 & 9 & -6 & 1 & 0 & \dots \\ 4 & -5 & -14 & 20 & -8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We deduce that the Hankel transform of the Fine numbers is $h_n = 1$, while that of the once shifted Fine sequence is $-n$.

Chapter 13

Row sum and central coefficient sequences of Pascal triangles defined by exponential Riordan arrays

In this chapter, we study sequences associated to two closely linked families of Pascal-like matrices. We derive expressions for the Hankel transform of the row sums of one of the families, and we characterize sequences of central coefficients in terms of the associated Laguerre polynomials. Links to the Narayana numbers are made explicit.

In Chapter 11 (and see [17]) we studied a family of Pascal-like matrices $\mathbf{B}_r$, and gave a characterization of their central coefficients and associated analogues of the Catalan numbers. It was indicated that a family $\tilde{\mathbf{B}}$ of matrices was the more fundamental family further. We now investigate aspects of these two families. In doing so, we introduce a new family of Pascal-like triangles, and study properties of this new family. This allows us to re-interpret and extend some results in Chapter 11.

We use the vehicle of exponential Riordan arrays to give a unifying theme to methods employed in this chapter.

The Laguerre and Hermite polynomials will be seen to play an important role in this chapter. We follow the notation of Chapter 11.

The associated Laguerre polynomials [241] are defined by

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!}\binom{n+\alpha}{n-k}(-x)^k.$$

Their generating function is

$$\frac{e^{\frac{-xz}{1-z}}}{(1-z)^{\alpha+1}}.$$

The Laguerre polynomials are given by $L_n(x) = L_n^{(0)}(x)$. The associated Laguerre polynomials are closed linked to the exponential Riordan array

$$\mathbf{Lag}^{(\alpha)}[t] = \left[\frac{1}{(1-tx)^{\alpha+1}}, \frac{x}{1-tx}\right]$$

where we have followed the notation of Chapter 11. The general term of this matrix is

$$Lag^{(\alpha)}[t](n,k) = \frac{n!}{k!}\binom{n+\alpha}{n-k}t^{n-k}$$

The Hermite polynomials $H_n(x)$ [238] are defined by

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

They obey $H_n(-x) = (-1)^n H_n(x)$ and can be defined by the recurrence

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{13.1}$$

They have a generating function given by

$$e^{2tx-x^2} = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!}x^n.$$

These polynomials are closely related to the generalized exponential Riordan array

$$[e^{-x^2}, 2x],$$

which is A060821.

We shall be interested in the Hankel transform of certain sequences in this chapter.

Example 273. A well-known Hankel transform [183] is that of the Bell numbers B_n (see Example 33), defined by

$$B_n = \sum_{k=0}^{n} S(n,k)$$

where $S(n,k)$ represents the Stirling numbers of the second kind, elements of the matrix

$$\mathbf{S} = [1, e^x - 1].$$

This is the sequence $1, 2, 5, 15, 52\ldots$, A000110. Its Hankel transform is given by

$$\prod_{k=1}^{n} k!$$

One way of making this explicit is to calculate the LDL^t decomposition of the Hankel matrix with general term B_{i+j}. We let $\mathbf{D} = \text{diag}(n!)$ be the diagonal matrix with diagonal elements $1, 1, 2, 6, 24, \ldots$. Then we find that

$$(B_{i+j})_{i,j\geq 0} = \mathbf{S}\cdot\mathbf{B}\cdot\mathbf{D}\cdot\mathbf{B}^t\cdot\mathbf{S}^t.$$

Since $\mathbf{S}$ and $\mathbf{B}$ are lower-triangular with 1's on the diagonal, we see that

$$\det(B_{i+j})_{0\leq i,j\leq n} = \prod_{k=0}^{n} k!$$

as required.

The internal matrix $\mathbf{B}\cdot\mathbf{D}\cdot\mathbf{B}^t$ has general term $\sum_j \binom{n}{j}\binom{k}{j}j!$. We have

$$\mathbf{B}\cdot\mathbf{D}\cdot\mathbf{B}^t = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ 1 & 3 & 7 & 13 & 21 & 31 & \dots \\ 1 & 4 & 13 & 34 & 73 & 136 & \dots \\ 1 & 5 & 21 & 73 & 209 & 501 & \dots \\ 1 & 6 & 31 & 136 & 501 & 1546 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is A088699, with e.g.f. $\frac{e^x}{1-y-xy}$. The lower triangular matrix associated to this infinite square matrix thus has general term

$$T(n,k) = [k \le n]\sum_{j=0}^{n}\binom{n-k}{j}\binom{k}{j}j!$$

where we have used the Iverson bracket notation [106], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false. This matrix is directly related to matrices in which we will be interested in the next section.

13.1 The family $\tilde{\mathbf{B}}_r$ of Pascal-like matrices

Following Chapter 11 (see also [17]), we define the family $\tilde{\mathbf{B}}_r$ of Pascal-like matrices by

$$\tilde{\mathbf{B}}_r = \left[e^x, x\left(1+\frac{r}{2}x\right)\right].$$

The general term $\tilde{B}_r(n,k)$ of this matrix is seen to be

$$\begin{aligned} \tilde{B}_r(n,k) &= \frac{n!}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{r^j}{2^j(n-k-j)!} \\ &= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\frac{(n-j)!r^j}{2^j(n-k-j)!} \\ &= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\binom{n-k}{j}j!\left(\frac{r}{2}\right)^j \\ &= \binom{n}{k}\tilde{T}_r(n,k) \end{aligned}$$

where

$$\tilde{T}_r(n,k) = \sum_{j=0}^{k}\binom{n-k}{j}\binom{k}{j}j!\left(\frac{r}{2}\right)^j.$$

Note that $\tilde{T}_1(n,k)$ is not an integer matrix.

We are interested in the row sums of $\tilde{\mathbf{B}}_r$. By the theory of exponential Riordan arrays, these row sums have e.g.f.

$$e^x e^{x(1+\frac{r}{2}x)} = e^{2x+\frac{r}{2}x^2}.$$

By the invariance of the Hankel transform under the binomial transform, this means that the Hankel transform of the row sums of $\tilde{\mathbf{B}}_r$ is the same as the Hankel transform of the sequence with e.g.f. $e^{\frac{r}{2}x^2}$. Now we have

$$\begin{aligned} n![x^n]e^{bx^2} &= n![x^n]\sum_{k=0}^{\infty}\frac{b^k}{k!}x^{2k} \\ &= n!\frac{b^{\frac{n}{2}}}{(\frac{n}{2})!}\frac{1+(-1)^n}{2}, \end{aligned}$$

hence the sequence with e.g.f. $e^{\frac{r}{2}x^2}$ has n-th term

$$n!\frac{(\frac{r}{2})^{\frac{n}{2}}}{(\frac{n}{2})!}\frac{1+(-1)^n}{2}.$$

Rather than working with this sequence directly, we use a result of Radoux, [186, 181, 183], namely that the Hankel transform of the sequence of involutions A000085 with e.g.f. $e^{x+x^2/2}$ is equal to $\prod_{k=1}^n k!$. An easy modification of the proof method in [183], or an appeal to the multilinearity of the determinant function, shows that the Hankel transform of the sequence with e.g.f. $e^{x+\frac{r}{2}x^2}$ is given by

$$\prod_{k=1}^n r^k k! = r^{\binom{n+1}{2}}\prod_{k=1}^n k!$$

Thus we obtain

Proposition 274. *The Hankel transform of the row sums of the matrix $\tilde{\mathbf{B}}_r$ is the sequence*

$$\prod_{k=1}^n r^k k! = r^{\binom{n+1}{2}}\prod_{k=1}^n k!$$

We can also use this to extend a result of [17] (see Proposition 252 of Chapter 11).

Proposition 275. *The sequence with e.g.f. $e^{ax+\frac{b}{2}x^2}$ has general term u_n given by*

$$u_n = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\frac{(2k)!}{2^k k!}a^{n-2k}b^k = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\frac{C_k}{2^k}(k+1)!a^{n-2k}b^k$$

and Hankel transform given by

$$\prod_{k=1}^n b^k k!$$

We finish this section by relating this row sum to the Hermite polynomials. We have

$$\begin{aligned} e^{2x+\frac{r}{2}x^2} &= e^{2(-i\sqrt{\frac{2}{r}})(i\sqrt{\frac{r}{2}}x)-(i\sqrt{\frac{r}{2}}x)^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} H_n\left(-i\sqrt{\frac{2}{r}}\right)\left(i\sqrt{\frac{r}{2}}x\right)^n \\ &= \sum_{n=0}^{\infty} H_n\left(-i\sqrt{\frac{2}{r}}\right)\left(i\sqrt{\frac{r}{2}}\right)^n \frac{x^n}{n!}. \end{aligned}$$

We conclude that the row sums of $\tilde{\mathbf{B}}_r$ are given by

$$\sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{r^j}{2^j(n-k-j)!} = H_n\left(-i\sqrt{\frac{2}{r}}\right)\left(i\sqrt{\frac{r}{2}}\right)^n.$$

We can of course reverse this identity to solve for $H_n(x)$ in terms of $\tilde{B}_r(n,k)$. Writing $T_{n,k}(r) = \tilde{B}_r(n,k)$ where now r can take on complex values, we obtain

$$H_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j x^{n-2j}}{(n-k-j)!} = \sum_{k=0}^{n} x^n T_{n,k}\left(-\frac{2}{x^2}\right). \tag{13.2}$$

This now allows us to define, for an integer m, the *generalized Hermite polynomials*

$$H_n^{(m)}(x) = \sum_{k=0}^{n} x^n T_{n,k}\left(-\frac{m}{x^2}\right). \tag{13.3}$$

13.2 Central sequences related to the family $\mathbf{T}_r$

In this section, we will use the notation $\mathbf{B}_r$ to represent the matrix $[e^x, x(1+rx)]$. Thus we have, for instance, $\mathbf{B}_{\frac{1}{2}} = \tilde{\mathbf{B}}_1$. It is evident from the last section that the general term $B_r(n,k)$ of the matrix $\mathbf{B}_r$ is given by

$$B_r(n,k) = \binom{n}{k} T_r(n,k)$$

where

$$T_r(n,k) = \sum_{j=0}^{k} \binom{n-k}{j}\binom{k}{j} j! r^j.$$

We therefor define the matrix $\mathbf{T}_r$ to be the matrix with general term $[k \le n] \sum_{j=0}^{k} \binom{n-k}{j}\binom{k}{j} j! r^j$. We note that

$$T_r(n,k) = \sum_{j=0}^{k} \binom{n-k}{k-j} \frac{k!}{j!} r^j.$$

We have, for instance,

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & 0 & \dots \\ 1 & 4 & 7 & 4 & 1 & 0 & \dots \\ 1 & 5 & 13 & 13 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The row sums of this matrix are given by A081124, the binomial transform of $\lfloor \frac{n}{2} \rfloor !$. Similarly,

$$\mathbf{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 5 & 5 & 1 & 0 & 0 & \dots \\ 1 & 7 & 17 & 7 & 1 & 0 & \dots \\ 1 & 9 & 37 & 37 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We note that $\mathbf{T}_0$ is the partial sum matrix, with 1's in all the non-zero locations.

We now turn our attention to the central coefficients of $\mathbf{T}_r$. Thus we have

$$\begin{aligned} T_r(2n, n) &= \sum_{j=0}^{n} \binom{2n-n}{j} \binom{n}{j} j! r^j \\ &= \sum_{j=0}^{n} \binom{n}{j}^2 j! r^j \end{aligned}$$

Alternatively,

$$\begin{aligned} T_r(2n, n) &= \sum_{j=0}^{n} \binom{n}{n-j} \frac{n!}{j!} r^j \\ &= \sum_{j=0}^{n} \binom{n}{j} \frac{n!}{j!} r^j. \end{aligned}$$

Clearly, we have

$$T_r(2n, n+1) = \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{n+1}{j} j! r^j.$$

We define *the generalized Catalan numbers associated to* $\mathbf{T}_r$ to be the numbers

$$C_r^T(n) = T_r(2n, n) - T_r(2n, n+1)$$

in analogy to the usual Catalan numbers which are equal to $\binom{2n}{n} - \binom{2n}{n+1}$. Then we have

$$C_r^T(n) = \sum_{j=0}^{n} \binom{n}{j}^2 j! r^j - \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{n+1}{j} j! r^j.$$

It is easy to characterize the central coefficients $T_r(2n, n)$.

Proposition 276. *$T_r(2n, n)$ are the row sums of the matrix* $\mathbf{Lag}[r]$.

Proof. We have
$$\mathbf{Lag}[r] = \left[\frac{1}{1-rx}, \frac{x}{1-rx}\right].$$
This matrix has row sums with e.g.f. $\frac{1}{1-rx}e^{\frac{x}{1-rx}}$. Expanding this expression, we find the general term to be

$$\begin{aligned}
n![x^n]\frac{1}{1-rx}e^{\frac{x}{1-rx}} &= n!\sum_{j=0}^{n}\binom{n}{j}\frac{r^j}{(n-j)!} \\
&= \sum_{j=0}^{n}\binom{n}{j}\frac{n!}{(n-j)!}r^j \\
&= \sum_{j=0}^{n}\binom{n}{j}^2 j!r^j
\end{aligned}$$

From the above, this is precisely $T_r(2n, n)$. □

Corollary 277. $T_r(2n, n) = n!r^n L_n(-1/r)$.

Proof. We have

$$\begin{aligned}
T_r(2n, n) &= \sum_{j=0}^{n}\binom{n}{j}\frac{n!}{(n-j)!}r^j \\
&= \sum_{j=0}^{n}\binom{n}{n-j}\frac{n!}{j!}r^{n-j} \\
&= r^n\sum_{j=0}^{n}\binom{n}{j}\frac{n!}{j!}\left(-\left(-\frac{1}{r}\right)\right)^j \\
&= n!r^n L_n\left(-\frac{1}{r}\right).
\end{aligned}$$

□

We now wish to study $T_r(2n, n+1) = T_r(2n, n-1)$. To this end, we let $a_n = T_r(2n, n+1)$

and examine the shifted sequence a_{n+1} first. We have

$$\begin{aligned} a_{n+1} &= \sum_{j=0}^{n} \binom{n}{j}\binom{n+2}{j} j! r^j \\ &= \sum_{j=0}^{n} \binom{n+2}{j} \frac{n!}{(n-j)!} r^j \\ &= \sum_{j=0}^{n} \binom{n+2}{n-j} \frac{n!}{j!} r^{n-j} \\ &= n! r^n L_n^{(2)}\left(-\frac{1}{r}\right). \end{aligned}$$

Thus we obtain

Proposition 278. *The generalized Catalan numbers $C_r^T(n)$ are given by $C_r^T(0) = 1$, and*

$$C_r^T(n) = n! r^n L_n\left(-\frac{1}{r}\right) - (n-1)! r^{n-1} L_{n-1}^{(2)}\left(-\frac{1}{r}\right), \qquad n > 0.$$

We can also characterize these numbers in term of the Narayana numbers [212], [213]

$$N(n,k) = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}.$$

This follows from the fact - shown after the next proposition - that their e.g.f. is given by

$$r e^{\frac{x}{1-rx}} - (r-1).$$

Proposition 279.

$$C_r^T(n) = \sum_{k=0}^{n} N(n,k) r^{k+1}(k+1)! - (r-1)0^n.$$

Proof. We have

$$\begin{aligned} n![x^n] e^{\frac{x}{1-rx}} &= n![x^n] \sum_{i=0}^{\infty} \frac{1}{i!} \frac{x^i}{(1-rx)^i} \\ &= n![x^n] \sum_{i=0}^{n} \frac{x^i}{i!} \sum_{k=0} \binom{i+k-1}{k} r^k x^k \\ &= n! \sum_{k=0}^{n-1} \frac{1}{(n-k)!} \binom{n-1}{k} r^k \\ &= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \binom{n-1}{k} r^k \\ &= \sum_{k=0}^{n} \binom{n}{k}\binom{n-1}{k} k! r^k \\ &= \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k} (k+1)! r^k. \end{aligned}$$

Thus the general term in the expansion of $re^{\frac{x}{1-rx}} - (r-1)$ is given by

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}(k+1)!r^{k+1} - (r-1)0^n.$$

$\square$

We note that the e.g.f. of $T_r(2n, n+1)$ is given by $\frac{e^{\frac{x}{1-rx}}(1-r+r^2x)}{1-rx} + (r-1)$. This is essentially the statement that the e.g.f. of $a_{n+1} = T_r(2n+2, n+2)$ is given by

$$\frac{e^{\frac{x}{1-rx}}}{(1-rx)^3},$$

which follows immediately from the fact that

$$a_{n+1} = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n+2}{n-k}r^{n-k}.$$

In other words, a_{n+1} represents the row sums of the matrix

$$\mathbf{Lag}^{(2)}[r] = \left[\frac{1}{(1-rx)^3}, \frac{x}{1-rx}\right].$$

Thus the e.g.f. of $C_r^T(n)$ is given by

$$\frac{1}{1-rx}e^{\frac{x}{1-x}} - \frac{e^{\frac{x}{1-rx}}(1-r+r^2x)}{1-rx} - (r-1),$$

which is $re^{\frac{x}{1-rx}} - (r-1)$.

13.3 Central coefficient sequences of the family $\mathbf{B}_r$

The results of the last section now allow us to re-examine and extend some results of Chapter 11 (see also [17]) concerning the central coefficients of $\mathbf{B}_r$. We have

$$\begin{aligned} B_r(2n, n) &= \binom{2n}{n}T_r(2n, n) \\ &= \binom{2n}{n}r^n n! L_n\left(-\frac{1}{r}\right) \\ &= \frac{(2n)!}{n!}r^n L_n\left(-\frac{1}{r}\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} B_r(2n, n+1) &= B_r(2n, n-1) \\ &= \binom{2n}{2n-1}r^{n-1}(n-1)!L_{n-1}^{(2)}\left(-\frac{1}{r}\right) \\ &= \frac{(2n)!}{(n+1)!}r^{n-1}L_{n-1}^{(2)}\left(-\frac{1}{r}\right). \end{aligned}$$

Hence we have

$$C_r^B(n) = B_r(2n,n) - B_r(2n,n+1) = \frac{(2n)!}{n!}\left[r^n L_n\left(-\frac{1}{r}\right) - \frac{1}{n+1}r^{n-1}L_{n-1}^{(2)}\left(-\frac{1}{r}\right)\right].$$

In Chapter 11 (see also [17]), we studied the ratio of the generalized Catalan numbers $C_r^B(n)$ and C_n, the Catalan numbers. Using the above expression, we obtain

$$\frac{C_r^B(n)}{C_n} = (n+1)!r^n L_n\left(-\frac{1}{r}\right) - n!L_{n-1}^{(2)}\left(-\frac{1}{r}\right).$$

In the notation of Chapter 10, this is equal to

$$\sum_{k=0}^{n} \tilde{N}(n,k)(k+1)!r^k$$

where $\tilde{N}(n,k) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k}$. We immediately obtain

Proposition 280.

$$C_r^B(n) = \frac{C_n C_r^T(n+1)}{r}, \qquad r \neq 0.$$

13.4 A note on the construction of $\mathbf{T}_r$

As noted in A108350, the method of construction of the matrices $\mathbf{T}_r$ is quite general. This becomes clear when we realize that it is the lower triangular version of the symmetric matrix

$$\mathbf{B} \cdot \mathbf{D} \cdot \mathbf{B}^t$$

where for $\mathbf{T}_r$, $\mathbf{D} = \mathrm{diag}(n!r^n)$. A108350 is similarly constructed with $\mathbf{D} = \mathrm{diag}(n+1 \bmod 2)$. For $\mathbf{D} = \mathrm{diag}(k^n)$ we get a series of "$(1,k,1)$-Pascal" matrices, with $k = 1$ giving Pascal's triangle A007318, and $k = 2$ giving the Delannoy triangle A008288. A086617 corresponds, for instance, to $\mathbf{D} = \mathrm{diag}(C_n)$.

Chapter 14

Generalized trinomial numbers, orthogonal polynomials and Hankel transforms [1]

14.1 Introduction

This chapter takes the generalized central trinomial numbers [171] as a vehicle to explore the links that exist between certain sequences of integers, orthogonal polynomials, Riordan arrays and Hankel transforms.
We recall that the central binomial coefficients $1, 2, 6, 20, 70, 252 \ldots$ with general term $\binom{2n}{n}$ are the sequence A000984. They have g.f. $\frac{1}{\sqrt{1-4x}}$. The aerated sequence $1, 0, 2, 0, 6, 0, 20, \ldots$ with g.f. $\frac{1}{\sqrt{1-4x^2}}$ has general term equal to $[x^n](1 + x^2)^n$. Similarly, the central trinomial coefficients $t_n = [x^n](1+x+x^2)^n$ which begin $1, 1, 3, 7, 19, 51, \ldots$ have g.f. equal to $\frac{1}{\sqrt{1-2x-3x^2}}$.

The study of integer sequences often involves looking at transformations that send one integer sequence into another one. For instance, we know that the *binomial transform* [230] of the sequence with general term a_n returns the sequence with general term b_n defined by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

This transformation is invertible, with inversion formula

$$a_n = \sum_{k=0} (-1)^{n-k} \binom{n}{k} b_k.$$

If we regard the sequence $(a_n)_{n\geq 0}$ as the column vector $(a_0, a_1, a_2, \ldots)^T$ then this transformation can be represented by the matrix $\mathbf{B}$ with general term $\binom{n}{k}$ (where we take the top

[1] This chapter reproduces the content of the conference paper "P. Barry, P. M. Rajkovic and M. D. Markovic, Generalized trinomial numbers, orthogonal polynomials and Hankel transforms, ALA2008, Novi Sad, Serbia." [23].

left term of the matrix to have index $(0,0)$) multiplying the vector on the left. For instance, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 6 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 7 \\ 19 \\ 51 \\ \vdots \end{pmatrix}.$$

That is, the central trinomial numbers are the binomial transform of the aerated central binomial coefficients. We note that the matrix $\mathbf{B}$ is in fact Pascal's triangle.
In particular, we have

$$\frac{1}{1-x}\frac{1}{\sqrt{1-4(\frac{x}{1-x})^2}} = \frac{1}{\sqrt{1-2x-3x^2}}.$$

14.2 The central trinomial coefficients, orthogonal polynomials and Hankel transform

In this section, we shall look at the specific example of the central trinomial coefficients to exhibit links between an integer sequence, Riordan arrays, orthogonal polynomials and the Hankel transform.

Thus we let t_n denote the general term of the sequence with g.f. $\frac{1}{\sqrt{1-2x-3x^2}}$. Many formulas are known for t_n, including

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \binom{n}{2k}\binom{2k}{k} \\ &= \sum_{k=0}^{n} \binom{n}{k}\binom{k}{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k}\binom{k}{k/2}(1+(-1)^k)/2 \\ &= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\binom{2k}{k}. \end{aligned}$$

These equations show that t_n is both the binomial transform of the aerated central binomial coefficients $\binom{n}{n/2}(1+(-1)^n)/2$ or $1, 0, 2, 0, 6, 0, 20, \ldots$ and the inverse binomial transform of the central binomial coefficients $\binom{2n}{n}$. It is easy to verify these algebraically by means of the Riordan array representation of $\mathbf{B} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ and the generating functions of $\binom{2n}{n}$ and its aeration. Thus we have

$$\frac{1}{1-x}\frac{1}{\sqrt{1-4(\frac{x}{1-x})^2}} = \frac{1}{\sqrt{1-2x-3x^2}}$$

while

$$\frac{1}{1+x}\frac{1}{\sqrt{1-4\frac{x}{1+x}}}=\frac{1}{\sqrt{1-2x-3x^2}}.$$

We now wish to represent the central trinomial numbers in moment form:

$$t_n=\int_{\mathbb{R}}x^n w(x)dx$$

for the appropriate weight function $w(x)$. Using the Sieltjes transform on the g.f. $\frac{1}{\sqrt{1-2x-3x^2}}$, we find that

$$t_n=\frac{1}{\pi}\int_{-1}^{3}x^n\frac{1}{\sqrt{-x^2+2x+3}}dx$$

and hence $w(x)=\frac{1}{\pi}\frac{1}{\sqrt{-x^2+2x+3}}\mathbf{1}_{[-1,3]}$.

We can now use this to calculate the sequences (α_n) and (β_n), and from these we can construct both the associated family of orthogonal polynomials $P_n(x)$, and the Hankel transform of t_n.

We start with the weight function $w_0(x)=\frac{1}{\sqrt{1-x^2}}$ of the Chebyshev polynomials of the first kind $T_n(x)$. For these polynomials, we have

$$\alpha_n^{(0)}=0,\qquad \beta_0^{(0)}=\pi,\qquad \beta_1^{(0)}=\frac{1}{2},\qquad \beta_n^{(0)}=\frac{1}{4},\quad (n>1).$$

Now

$$w_1(x)=\frac{1}{\sqrt{-x^2+2x+3}}=\frac{1}{2}\frac{1}{\sqrt{1-(\frac{x-1}{2})^2}}=\frac{1}{2}w_0(\frac{x-1}{2}).$$

Hence by Lemma 269 we have

$$\alpha_n^{(1)}=\frac{0+1/2}{1/2}=1,\qquad \beta_0^{(1)}=\frac{1}{2}2\pi=\pi,\qquad \beta_1^{(1)}=4\frac{1}{2}=2,\qquad \beta_n^{(1)}=4\frac{1}{4}=1,\quad (n>1).$$

Finally $w(x)=\frac{1}{\pi}w_1(x)$ and so

$$\alpha_n=1,\qquad \beta_0=\frac{1}{\pi}\pi=1,\qquad \beta_1=2,\qquad \beta_n=1,\quad (n>1).$$

We immediately see that the Hankel transform of t_n is given by $h_n=2^n$.

Also, the family of orthogonal polynomials $P_n(x)$ associated to the sequence t_n, which satisfy the recurrence

$$P_{n+1}(x)=(x-\alpha_n)P_n(x)-\beta_n P_{n-1}(x),\qquad P_{-1}=0,\qquad P_0=1,$$

can be calculated as follows:

$$\begin{aligned}
P_1(x) &= (x-1)P_0(x)-P_{-1}(x)=x-1;\\
P_2(x) &= (x-1)P_1(x)-2P_0(x)=(x-1)^2-2=x^2-2x-1;\\
P_3(x) &= (x-1)P_2(x)-P_1(x)=(x-1)(x^2-2x-1)-(x-1)=x^3-3x^2+2;\\
&\dots
\end{aligned}$$

This implies that the coefficient array for the polynomials $P_n(x)$ takes the form

$$(a_{n,k}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & -2 & 1 & 0 & 0 & 0 & \dots \\ 2 & 0 & -3 & 1 & 0 & 0 & \dots \\ -1 & 4 & 2 & -4 & 1 & 0 & \dots \\ -1 & -5 & 5 & 5 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In terms of Riordan arrays, we have

$$(a_{n,k}) = \left(\frac{1-x^2}{1+x+x^2}, \frac{x}{1+x+x^2}\right).$$

We note that

$$\left(\frac{1-x^2}{1+x+x^2}, \frac{x}{1+x+x^2}\right) = \left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)\left(\frac{1}{1+x}, \frac{x}{1+x}\right),$$

where the first array on the RHS is closely associated to the Chebyshev polynomials of the first kind (it is the coefficient array for $2T_n(x/2)$). We can deduce from the last equation that

$$a_{n,k} = \sum_{j=0}^{n} \frac{2n+0^{n+j}}{n+j+0^{n+j}} \binom{\frac{n+j}{2}}{\frac{n-j}{2}} (-1)^{(n-j)/2} \frac{(1+(-1)^{n-j})}{2} (-1)^{j-k} \binom{j}{k}.$$

Equivalently, we have

$$P_n(x) = 2T_n((x-1)/2).$$

We have the following equality of Riordan arrays

$$\left(\frac{1-x^2}{1+x+x^2}, \frac{x}{1+x+x^2}\right)^{-1} = \left(\frac{1}{\sqrt{1-2x-3x^2}}, \frac{1-x-\sqrt{1-2x-3x^2}}{2x}\right)$$

which shows an explicit link to the numbers t_n, which appear as the first column of the inverse. Writing

$$\mathbf{L} = \left(\frac{1-x^2}{1+x+x^2}, \frac{x}{1+x+x^2}\right)^{-1}$$

we obtain the following factorization of the (infinite) Hankel matrix $\mathbf{H} = (t_{i+j})_{i,j\geq}$:

$$\mathbf{H} = \mathbf{L} \cdot \mathbf{D} \cdot \mathbf{L}^T$$

where $\mathbf{D}$ is the diagonal matrix with entries $1, 2, 2, 2, \ldots$. In detail, we have

$$\begin{pmatrix} 1 & 1 & 3 & 7 & 19 & 51 & \dots \\ 1 & 3 & 7 & 19 & 51 & 141 & \dots \\ 3 & 7 & 19 & 51 & 141 & 393 & \dots \\ 7 & 19 & 51 & 141 & 393 & 1107 & \dots \\ 19 & 51 & 141 & 393 & 1107 & 3139 & \dots \\ 51 & 141 & 393 & 1107 & 3139 & 8953 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 2 & 1 & 0 & 0 & 0 & \dots \\ 7 & 6 & 3 & 1 & 0 & 0 & \dots \\ 19 & 16 & 10 & 4 & 1 & 0 & \dots \\ 51 & 45 & 30 & 15 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 & 7 & 19 & 51 & \dots \\ 0 & 1 & 2 & 6 & 16 & 45 & \dots \\ 0 & 0 & 1 & 3 & 10 & 30 & \dots \\ 0 & 0 & 0 & 1 & 4 & 15 & \dots \\ 0 & 0 & 0 & 0 & 1 & 5 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that as expected the production matrix of $\mathbf{L}$ is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

14.3 Generalized central trinomial coefficients, orthogonal polynomials and Hankel transforms

In this section, we turn our attention to the general case of the central coefficients of the expression $(1+\alpha x+\beta x^2)^n$. Following [171], we call these numbers *generalized central trinomial coefficients*, with integer parameters α and β. We will use the notation $t_n(\alpha,\beta)$ when it is necessary to specify the dependence on α and β. Thus

$$t_n(\alpha,\beta)=[x^n](1+\alpha x+\beta x^2)^n.$$

We have

$$t_n(\alpha,\beta)=\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}\binom{2k}{k}\alpha^{n-2k}\beta^k.$$

The generating function for $t_n(\alpha,\beta)$ is given by

$$\frac{1}{\sqrt{1-2\alpha x+(\alpha^2-4\beta)x^2}}.$$

This can be obtained through an application of the Lagrange inversion formula (see Example 12). Applying the Stieltjes transform, we find the moment representation

$$\begin{aligned} t_n(\alpha,\beta) &= \frac{1}{\pi}\int_{\alpha-2\sqrt{\beta}}^{\alpha+2\sqrt{\beta}} x^n \frac{1}{\sqrt{(4\beta-\alpha^2)+2\alpha x-x^2}}dx \\ &= \frac{1}{\pi}\int_{\alpha-2\sqrt{\beta}}^{\alpha+2\sqrt{\beta}} x^n \frac{1}{\sqrt{4\beta-(x-\alpha)^2}}dx. \end{aligned}$$

Proposition 281. *The Hankel transform of $t_n(\alpha,\beta)$ is given by $h_n = 2^n\beta^{\binom{n+1}{2}}$.*

Proof. We have

$$w(x) = \frac{1}{\pi}\frac{1}{\sqrt{4\beta-(x-\alpha)^2}}\mathbf{1}_{[\alpha-2\sqrt{\beta},\alpha+2\sqrt{\beta}]}.$$

We start with the weight function $w_0(x) = \frac{1}{\sqrt{1-x^2}}$ of the Chebyshev polynomials of the first kind $T_n(x)$. For these polynomials, we have

$$\alpha_n^{(0)} = 0, \qquad \beta_0^{(0)} = \pi, \qquad \beta_1^{(0)} = \frac{1}{2}, \qquad \beta_n^{(0)} = \frac{1}{4}, \quad (n > 1).$$

Now

$$w_1(x) = \frac{1}{\sqrt{4\beta-(x-\alpha)^2}} = \frac{1}{2\sqrt{\beta}}\frac{1}{\sqrt{1-(\frac{x-\alpha}{2\sqrt{\beta}})^2}} = \frac{1}{2\sqrt{\beta}}w_0(\frac{x-\alpha}{2\sqrt{\beta}}).$$

Hence by Lemma 269 we have

$$\alpha_n^{(1)} = \frac{0+\alpha/2\sqrt{\beta}}{1/2\sqrt{\beta}} = \alpha, \qquad \beta_0^{(1)} = \frac{1}{2\sqrt{\beta}}2\sqrt{\beta}\pi = \pi,$$

$$\beta_1^{(1)} = 4\beta\frac{1}{2} = 2\beta, \qquad \beta_n^{(1)} = 4\beta\frac{1}{4} = \beta, \quad (n > 1).$$

Finally $w(x) = \frac{1}{\pi}w_1(x)$ and so

$$\alpha_n = \alpha, \qquad \beta_0 = \frac{1}{\pi}\pi = 1, \qquad \beta_1 = 2\beta, \qquad \beta_n = \beta, \quad (n > 1).$$

Hence $h_n = 2\beta^{\binom{n+1}{2}}$ as required. □

The family of orthogonal polynomials $P_n(x)$ associated to the sequence t_n, which satisfy the recurrence

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x), \qquad P_{-1} = 0, \qquad P_0 = 1,$$

can be calculated as follows:

$$\begin{aligned}
P_1(x) &= (x-\alpha)P_0(x) - P_{-1}(x) = x-\alpha;\\
P_2(x) &= (x-\alpha)P_1(x) - 2\beta P_0(x) = (x-\alpha)^2 - 2\beta = x^2 - 2\alpha x + \alpha^2 - 2\beta;\\
P_3(x) &= (x-\alpha)P_2(x) - \beta P_1(x) = x^3 - 3\alpha x^2 + (3\alpha^2 - 3\beta)x + \alpha(3\beta-\alpha^2);\\
&\dots
\end{aligned}$$

This implies that the coefficient array for the polynomials $P_n(x;\alpha,\beta)$ takes the form

$$(a_{n,k}) = \left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \dots\\
-\alpha & 1 & 0 & 0 & 0 & 0 & \dots\\
\alpha^2-2\beta & -2\alpha & 1 & 0 & 0 & 0 & \dots\\
\alpha(3\beta-\alpha^2) & 3(\alpha^2-\beta) & -3\alpha & 1 & 0 & 0 & \dots\\
\alpha^4-4\alpha^2\beta+2\beta^2 & 4\alpha(2\beta-\alpha^2) & 2(3\alpha^2-2\beta) & -4\alpha & 1 & 0 & \dots\\
-\alpha(\alpha^4-5\alpha^2\beta+5\beta^2) & 5(\alpha^4-3\alpha^2\beta+\beta^2) & 5\alpha(3\beta-2\alpha^2) & 5(2\alpha^2-\beta) & -5\alpha & 1 & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

That is,

$$(a_{n,k}) = \left(\frac{1-\beta x^2}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right).$$

We now note that

$$\left(\frac{1-\beta x^2}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right) = \left(\frac{1-\beta x^2}{1+\beta x^2}, \frac{x}{1+\beta x^2}\right)\left(\frac{1}{1+\alpha x}, \frac{x}{1+\alpha x}\right).$$

We deduce that the family of orthogonal polynomials $(P_n(x;\alpha,\beta))_{n\ge 0}$ associated to the generalized trinomial numbers is related to the Chebyshev polynomials of the first kind T_n as follows:

$$P_n(x;\alpha,\beta) = 2\beta^n T_n\left(\frac{x-\alpha}{2\beta}\right).$$

The production array of $\mathbf{L} = (a_{n,k})^{-1}$ is given by

$$\left(\begin{array}{ccccccc}
\alpha & 1 & 0 & 0 & 0 & 0 & \ldots \\
2\beta & \alpha & 1 & 0 & 0 & 0 & \ldots \\
0 & \beta & \alpha & 1 & 0 & 0 & \ldots \\
0 & 0 & \beta & \alpha & 1 & 0 & \ldots \\
0 & 0 & 0 & \beta & \alpha & 1 & \ldots \\
0 & 0 & 0 & 0 & \beta & \alpha & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

14.4 A conjecture

The simple expression obtained for the Hankel transform of the expression $t_n(\alpha,\beta)$ might lead one to conclude that the sequence

$$r_n(\alpha,\beta) = [x^{n-1}](1+\alpha x+\beta x^2)^n$$

should also have a relatively simple expression. This sequence has $r_0 = 0$. We can conjecture the following format for the Hankel transform of the sequence r_{n+1}:

Conjecture 282. *If*

$$r_n = [x^{n-1}](1+\alpha x+\beta x^2)^n,$$

then the Hankel transform of $r_{n+1}(\alpha,\beta)$ is given by

$$\beta^{\binom{n}{2}}[x^n]\frac{1-(\alpha^2-3\beta)x+\beta^2x^2-\beta^3x^3}{1+\beta(\alpha^2-2\beta)x^2+\beta^4x^4}.$$

Example 283. The sequence $r_n(-1,-1)$ with general term $[x^{n-1}](1-x-x^2)^n$ begins

$$0, 1, -2, 0, 8, -15, -6, 77, \ldots$$

This sequence has Hankel transform $0, -1, 0, 1, 0, -1, \ldots$ with generating function $\frac{-x}{1+x^2}$. The Hankel transform of $r_{n+1}(-1,-1)$ is the sequence starting

$$1, -4, -4, 11, 11, -29, -29, \ldots,$$

with general term

$$(-1)^{\binom{n}{2}}[x^n]\frac{1-4x+x^2+x^3}{1-3x^2+x^4}.$$

Thus based on the conjecture, the generating function of the Hankel transform of $r_{n+1}(-1,-1)$ is

$$\frac{1-4x-x^2-x^3}{1+3x^2+x^4}.$$

14.5 On the row sums of $\mathbf{L}(\alpha, \beta) = (a_{n,k})^{-1}$

In this section, we shall be interested in the row sums of the matrix $\mathbf{L}$ where

$$\begin{aligned}
\mathbf{L}(\alpha,\beta) &= \left(\frac{1-\beta x^2}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right)^{-1} \\
&= \left(\frac{1}{\sqrt{1-2\alpha x+(\alpha^2-4\beta)x^2}}, \frac{1-\alpha x-\sqrt{1-2\alpha x+(\alpha^2-4\beta)x^2}}{2\beta x}\right).
\end{aligned}$$

We recall that the row sums of the Riordan array $(g(x), f(x))$ have generating function $g(x)/(1-f(x))$. Applying this in our case, and simplifying, we obtain the following generating function for the row sums:

$$s(x;\alpha,\beta) = \frac{1}{2}\frac{1}{1-(\alpha+\beta+1)x} + \frac{1}{2}\frac{1-(\alpha+2\beta)x}{1-(\alpha+\beta+1)x}\frac{1}{\sqrt{1-2\alpha x+(\alpha^2-4\beta)x^2}}.$$

Now $\frac{1-(\alpha+2\beta)x}{1-(\alpha+\beta+1)x}$ is the generating function of the sequence with general term

$$(1-\beta)(\alpha+\beta+1)^{n-1} + \frac{\alpha+2\beta}{1+\alpha+\beta}\cdot 0^n. \tag{14.1}$$

Thus the row sums are the mean of the function $(\alpha+\beta+1)^n$ and the convolution of the function above (14.1) and $t_n(\alpha,\beta)$.

We can characterize these sums in another way, be first recalling that

$$\left(\frac{1-\beta x^2}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right) = \left(\frac{1-\beta x^2}{1+\beta x^2}, \frac{x}{1+\beta x^2}\right)\left(\frac{1}{1+\alpha x}, \frac{x}{1+\alpha x}\right).$$

Hence

$$\begin{aligned}
\mathbf{L} &= \left(\frac{1-\beta x^2}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right)^{-1}\\
&= \left(\frac{1}{1+\alpha x}, \frac{x}{1+\alpha x}\right)^{-1}\left(\frac{1-\beta x^2}{1+\beta x^2}, \frac{x}{1+\beta x^2}\right)^{-1}\\
&= \left(\frac{1}{1-\alpha x}, \frac{x}{1-\alpha x}\right)\left(\frac{1}{\sqrt{1-4\beta x^2}}, \frac{1-\sqrt{1-4\beta x^2}}{2\beta x}\right)\\
&= \left(\frac{1}{1-\alpha x}, \frac{x}{1-\alpha x}\right)\left(\frac{1}{\sqrt{1-4\beta x^2}}, xc(\beta x^2)\right)
\end{aligned}$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the g.f. of the Catalan numbers $C_n = \binom{2n}{n}/(n+1)$, A000108. Hence the row sums of $\mathbf{L}$ are given by the α-th binomial transform of the row sums of the Riordan array

$$\left(\frac{1}{\sqrt{1-4\beta x^2}}, xc(\beta x^2)\right).$$

These latter row sums have generating function

$$s(x;0,\beta) = \frac{1}{2}\cdot\frac{1}{1-(\beta+1)x} + \frac{1}{2}\cdot\frac{1-2\beta x}{1-(\beta+1)x}\cdot\frac{1}{\sqrt{1-4\beta x^2}}.$$

We now wish to calculate the Hankel transform of the row sums of $\mathbf{L}(\alpha,\beta)$. By the binomial invariance property of the Hankel transform, it suffices to calculate that of $\mathbf{L}(0,\beta)$. Thus the Hankel transform is independent of α.

It is clear that the general element of the sum, $s_n(0,\beta)$, is given by

$$s_n(0,\beta) = \frac{1}{2\pi}\int_{-2\sqrt{\beta}}^{2\sqrt{\beta}} x^n \frac{2\beta - x}{1+\beta-x}\frac{1}{\sqrt{4\beta-x^2}}dx.$$

The following may now be conjectured.

Conjecture 284. *The Hankel transform $h_n(\alpha,\beta) = h_n(\beta)$ of the row sums of $\mathbf{L}(\alpha,\beta)$ is given by*

$$h_n = \beta^{\lceil \frac{n^2}{2}\rceil - 0^n} u_n$$

where u_n is the n-th term of the sequence with generating function

$$\frac{(1+x)^2}{1-(4\beta-2)x^2+x^4}.$$

14.6 Pascal-like triangles

In Chapter 10, we studied a family of Pascal-like triangles, parameterized by r, whose general term was given by

$$T_{n,k}^{(r)} = \sum_{j=0} \binom{k}{j}\binom{n-k}{j} r^j.$$

We note that the central terms of this matrix are given by

$$T_{2n,n}^{(r)} = \sum_{j=0}^{n} \binom{n}{j}^2 r^j.$$

We now note that we have the identity

$$T_{2n,n}^{(r)} = \sum_{j=0}^{n} \binom{n}{j}^2 r^j = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} r^k (r+1)^{n-2k}.$$

We now associate these observations to our above results by means of the following proposition.

Proposition 285.

$$[x^n](1+ax+bx^2)^n = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} a^k b^{n-2k}.$$

Proof. We have

$$\begin{aligned}
[x^n](1+ax+bx^2)^n &= [x^n]\sum_{j=0}^{n} \binom{k}{j} x^j (a+bx)^j \\
&= [x^n]\sum_{j=0}^{n} \binom{k}{j} x^j \sum_{i=0}^{j} \binom{j}{i} x^i a^i b^{j-i} \\
&= [x^n]\sum_{j=0}^{n}\sum_{i=0}^{j} \binom{k}{j}\binom{j}{i} a^i b^{j-i} x^{i+j} \\
&= \sum_{i=0}^{n} \binom{n}{n-i}\binom{n-i}{i} a^i b^{n-2i} \\
&= \sum_{i=0}^{n} \binom{n}{i}\binom{n-i}{i} a^i b^{n-2i}.
\end{aligned}$$

□

Corollary 286.

$$T_{2n,n}^{(r)} = [x^n](1+(r+1)x+rx^2)^n.$$

Corollary 287. *The Hankel transform of* $T_{2n,n}^{(r)}$ *is equal to* $2^n r^{\binom{n+1}{2}}$.

It is instructive to relate this result to the LDL^T decomposition of the Hankel matrix $H(r)$ of $T_{2n,n}^{(r)}$. We take the example of $r = 2$. In this case,

$$H(2) = \begin{pmatrix} 1 & 3 & 13 & 63 & \ldots \\ 3 & 13 & 63 & 321 & \ldots \\ 13 & 63 & 321 & 1683 & \ldots \\ 63 & 321 & 1683 & 8989 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then

$$\begin{aligned} H(2) &= L(2)D(2)L(2)^T \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 3 & 1 & 0 & 0 & \ldots \\ 13 & 6 & 1 & 0 & \ldots \\ 63 & 33 & 9 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 0 & 0 & \ldots \\ 0 & 0 & 8 & 0 & \ldots \\ 0 & 0 & 0 & 16 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 3 & 13 & 63 & \ldots \\ 0 & 1 & 6 & 33 & \ldots \\ 0 & 0 & 1 & 9 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

Hence the Hankel transform of $T(2n, n, 2)$ is equal to the sequence with general term

$$\prod_{k=0}^{n} (2.2^k - 0^k) = 2^n 2^{\binom{n+1}{2}}.$$

$L(2)$ is in fact the Riordan array

$$\left(\frac{1}{\sqrt{1 - 6x + x^2}}, \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x} \right)$$

or

$$\left(\frac{1 - 2x^2}{1 + 3x + 2x^2}, \frac{x}{1 + 3x + 2x^2} \right)^{-1}.$$

In general, we can show that $H(r) = L(r)D(r)L(r)^T$ where $L(r)$ is the Riordan array

$$\left(\frac{1}{\sqrt{1 - 2(r+1)x + (r-1)^2x^2}}, \frac{1 - (r+1)x - \sqrt{1 - 2(r+1)x + (r-1)^2x^2}}{2rx} \right)$$

and $D(r)$ is the diagonal matrix with n-th term $2.r^n - 0^n$. Hence the Hankel transform of $T(2n, n, r)$ is given by

$$\prod_{k=0}^{n} (2.r^k - 0^k) = 2^n r^{\binom{n+1}{2}}.$$

We note that the Riordan array $L(r)$

$$\left(\frac{1}{\sqrt{1 - 2(r+1)x + (r-1)^2x^2}}, \frac{1 - (r+1)x - \sqrt{1 - 2(r+1)x + (r-1)^2x^2}}{2rx} \right)$$

is the inverse of the Riordan array

$$\left(\frac{1-rx^2}{1+(r+1)x+rx^2}, \frac{x}{1+(r+1)x+rx^2}\right).$$

Its general term is given by

$$\sum_{j=0}^{n}\binom{n}{j}\binom{n}{j-k}r^{j-k}=\sum_{j=0}^{n}\binom{n}{j}\binom{j}{n-k-j}r^{n-k-j}(r+1)^{2j-(n-k)}.$$

Its k-th column has exponential generating function given by

$$e^{(r+1)x}I_k(2\sqrt{r}x)/\sqrt{r}^k.$$

Corollary 288. *The sequences with e.g.f. $I_0(2\sqrt{r}x)$ have Hankel transforms given by $2^n r^{\binom{n+1}{2}}$.*

Proof. By [17] or otherwise, we know that the sequences $T(2n,n,r)$ have e.g.f.

$$e^{(r+1)x}I_0(2\sqrt{r}x).$$

By the above proposition and the binomial invariance property of the Hankel transform [139], $\mathbf{B}^{-r-1}T(2n,n,r)$ has the desired Hankel transform. But $\mathbf{B}^{-r-1}T(2n,n,r)$ has e.g.f. given by

$$e^{-(r+1)x}e^{(r+1)x}I_0(2\sqrt{r}x)=I_0(2\sqrt{r}x).$$

□

We have in fact the following general result :

Proposition 289.

$$[x^{n-k}](1+ax+bx^2)^n=\sum_{i=0}^{n}\binom{n}{n-k-i}\binom{n-k-i}{i}a^{n-2k-i}b^i$$

is the general term of the Riordan array

$$\left(\frac{1-bx^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}=\left(\frac{1}{\sqrt{1-2ax+x^2(a^2-4b)}}, \frac{1-ax-\sqrt{1-2ax+x^2(a^2-4b)}}{2bx}\right).$$

14.7 Hankel transform of generalized Catalan numbers

Following [17], we denote by $c(n;r)$ the sequence of numbers

$$c(n;r)=T(2n,n,r)-T(2n,n+1,r).$$

For instance, $c(n;1)=C_n$, the sequence of Catalan numbers. We have

Proposition 290. *The Hankel transform of $c(n;r)$ is $r^{\binom{n+1}{2}}$.*

Proof. Again, we use the LDL^T decomposition of the associated Hankel matrices. For instance, when $r = 3$, we obtain

$$H(3) = \begin{pmatrix} 1 & 3 & 12 & 57 & \dots \\ 3 & 12 & 57 & 300 & \dots \\ 12 & 57 & 300 & 1686 & \dots \\ 57 & 300 & 1686 & 9912 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then

$$\begin{aligned} H(3) &= L(3)D(3)L(3)^T \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 3 & 1 & 0 & 0 & \dots \\ 12 & 7 & 1 & 0 & \dots \\ 57 & 43 & 11 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 0 & 0 & \dots \\ 0 & 0 & 9 & 0 & \dots \\ 0 & 0 & 0 & 27 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 3 & 12 & 57 & \dots \\ 0 & 1 & 7 & 43 & \dots \\ 0 & 0 & 1 & 11 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

Hence the Hankel transform of $c(n;3)$ is

$$\prod_{k=0}^{n} 3^k = 3^{\binom{n+1}{2}}.$$

In this case, $L(3)$ is the Riordan array

$$\left(\frac{1}{1+3x}, \frac{x}{1+4x+3x^2}\right)^{-1}.$$

In general, we can show that $H(r) = L(r)D(r)L(r)^T$ where

$$L(r) = \left(\frac{1}{1+rx}, \frac{x}{1+(r+1)x+rx^2}\right)^{-1}$$

and $D(r)$ has n-th term r^n. Hence the Hankel transform of $c(n;r)$ is given by

$$\prod_{k=0}^{n} r^k = r^{\binom{n+1}{2}}.$$

□

We finish this section with some notes concerning production matrices as found, for instance, in [75]. It is well known that the production matrix $P(1)$ for the Catalan numbers $C_n = c(n,1)$ is given by

$$P(1) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Following [75], we can associate a Riordan array $A_P(1)$ to $P(1)$ as follows. The second column of P has generating function $\frac{1}{1-x}$. Solving the equation

$$u = \frac{1}{1-xu}$$

we obtain $u(x) = \frac{1-\sqrt{1-4x}}{2x} = c(x)$. Since the first column is all 0's, this means that $A_P(1)$ is the Riordan array $(1, xc(x))$. This is the inverse of $(1, x(1-x))$. We have

$$A_P(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 2 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Multiplying on the right by B, the binomial matrix, we obtain

$$A_P(1)B = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 2 & 3 & 1 & 0 & \dots \\ 5 & 9 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = L(1)$$

which is the Riordan array

$$\left(\frac{1}{1-x}, xc(x)^2\right) = \left(\frac{1}{1+x}, \frac{1}{1+2x+x^2}\right)^{-1}.$$

Similarly the production matrix for the $c(n;2)$, or the large Schröder numbers, is given by

$$P(2) = \begin{pmatrix} 0 & 2 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & \dots \\ 0 & 1 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Here, the generating function for the second column is $\frac{2-x}{1-x}$. Now solving

$$u = \frac{2-xu}{1-xu}$$

which gives $u = \frac{1+x-\sqrt{1-6x+x^2}}{2x}$. Hence in this case, $A_P(2)$ is the Riordan array $\left(1, \frac{1+x-\sqrt{1-6x+x^2}}{2}\right)$. That is,

$$A_P(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 2 & 4 & 0 & \dots \\ 0 & 6 & 8 & 8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(1, \frac{x(1-x)}{2-x}\right)^{-1}.$$

The row sums of this matrix are $1, 2, 6, 22, 90, \ldots$ as expected. Multiplying $A_P(2)$ on the right by the binomial matrix B, we obtain

$$A_P(2)B = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 0 & 0 & \ldots \\ 6 & 10 & 4 & 0 & \ldots \\ 22 & 46 & 32 & 8 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the array

$$\left(\frac{1-x-\sqrt{1-6x+x^2}}{2x}, \frac{1-3x-\sqrt{1-6x+x^2}}{2x}\right).$$

Finally

$$A_P B \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & \frac{1}{2} & 0 & 0 & \ldots \\ 0 & 0 & \frac{1}{4} & 0 & \ldots \\ 0 & 0 & 0 & \frac{1}{8} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = A_P B(1, \frac{x}{2}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & \ldots \\ 6 & 5 & 1 & 0 & \ldots \\ 22 & 23 & 8 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = L(2)$$

which is

$$\left(\frac{1-x-\sqrt{1-6x+x^2}}{2x}, \frac{1-3x-\sqrt{1-6x+x^2}}{4x}\right)$$

or

$$L(2) = \left(\frac{1}{1+2x}, \frac{x}{1+3x+2x^2}\right)^{-1}.$$

We can generalize these results to give the following proposition.

Proposition 291. *The production matrix for the generalized Catalan sequence $c(n;r)$ is given by*

$$P(r) = \begin{pmatrix} 0 & r & 0 & 0 & \ldots \\ 0 & 1 & r & 0 & \ldots \\ 0 & 1 & 1 & r & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The associated matrix $A_P(r)$ is given by

$$A_P(r) = \left(1, \frac{x(1-x)}{r-(r-1)x}\right)^{-1} = \left(1, \frac{1+(r-1)x-\sqrt{1-2(r+1)x+(r-1)^2x^2}}{2}\right).$$

The matrix $L(r)$ in the decomposition $L(r)D(r)L(r)^T$ of the Hankel matrix $H(r)$ for $c(n;r)$, which is equal to $A_P(r)B(1, x/r)$, is given by

$$L(r) = \left(\frac{1-(r-1)x-\sqrt{1-2(r+1)x+(r-1)^2x^2}}{2x}, \frac{1-(r+1)x-\sqrt{1-2(r+1)x+(r-1)^2x^2}}{2rx}\right).$$

We have

$$L(r) = \left(\frac{1}{1+rx}, \frac{x}{1+(r+1)x+rx^2}\right)^{-1}.$$

We note that the elements of $L(r)^{-1}$ are in fact the coefficients of the orthogonal polynomials associated to $H(r)$.

Proposition 292. *The elements of the rows of the Riordan array* $\left(\frac{1}{1+rx}, \frac{x}{1+(r+1)x+rx^2}\right)$ *are the coefficients of the orthogonal polynomials associated to the Hankel matrix determined by the generalized Catalan numbers* $c(n;r)$*.*

14.8 Hankel transform of the sum of consecutive generalized Catalan numbers

We now look at the Hankel transform of the sum of two consecutive generalized Catalan numbers. That is, we study the Hankel transform of $c(n;r) + c(n+1;r)$. For the case $r = 1$ (the ordinary Catalan numbers) this was dealt with in [61], while the general case was studied in [188]. We use the methods developed above to gain greater insight. We start with the case $r = 1$. For this, the Hankel matrix for $C_n + C_{n+1}$ is given by

$$H = \begin{pmatrix} 2 & 3 & 7 & 19 & \ldots \\ 3 & 7 & 19 & 56 & \ldots \\ 7 & 19 & 56 & 174 & \ldots \\ 19 & 56 & 174 & 561 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proceeding to the LDL^T decomposition, we get

$$\begin{aligned} H &= LDL^T \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ \frac{3}{2} & 1 & 0 & 0 & \ldots \\ \frac{7}{2} & \frac{17}{5} & 1 & 0 & \ldots \\ \frac{19}{2} & 11 & \frac{70}{13} & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 & \ldots \\ 0 & \frac{5}{2} & 0 & 0 & \ldots \\ 0 & 0 & \frac{13}{5} & 0 & \ldots \\ 0 & 0 & 0 & \frac{34}{13} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{2} & \frac{7}{2} & \frac{19}{2} & \ldots \\ 0 & 1 & \frac{17}{5} & 11 & \ldots \\ 0 & 0 & 1 & \frac{70}{13} & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

This indicates that the Hankel transform of $C_n + C_{n+1}$ is given by

$$\prod_{k=0}^{n} \frac{F(2k+3)}{F(2k+1)} = F(2n+3).$$

This is in agreement with [61]. We note that in this case, L^{-1} takes the form

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -\frac{3}{2} & 1 & 0 & 0 & \ldots \\ \frac{8}{5} & -\frac{17}{5} & 1 & 0 & \ldots \\ -\frac{21}{13} & \frac{95}{13} & -\frac{70}{13} & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & \frac{1}{2} & 0 & 0 & \ldots \\ 0 & 0 & \frac{1}{5} & 0 & \ldots \\ 0 & 0 & 0 & \frac{1}{13} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -3 & 2 & 0 & 0 & \ldots \\ 8 & -17 & 5 & 0 & \ldots \\ -21 & 95 & -70 & 13 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where we see the sequences $F(2n+1)$ and $(-1)^n F(2n+2)$ in evidence.

Now looking at the case $r=2$, we get

$$H = \begin{pmatrix} 3 & 8 & 28 & 112 & \dots \\ 8 & 28 & 112 & 484 & \dots \\ 28 & 112 & 484 & 2200 & \dots \\ 112 & 484 & 2200 & 10364 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proceeding to the LDL^T decomposition, we obtain

$$\begin{aligned} H &= LDL^T \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{8}{3} & 1 & 0 & 0 & \dots \\ \frac{28}{3} & \frac{28}{5} & 1 & 0 & \dots \\ \frac{112}{3} & \frac{139}{5} & \frac{146}{17} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 & \dots \\ 0 & \frac{20}{3} & 0 & 0 & \dots \\ 0 & 0 & \frac{272}{20} & 0 & \dots \\ 0 & 0 & 0 & \frac{7424}{272} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & \frac{8}{2} & \frac{28}{3} & \frac{112}{3} & \dots \\ 0 & 1 & \frac{28}{5} & \frac{139}{5} & \dots \\ 0 & 0 & 1 & \frac{146}{17} & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

Thus the Hankel transform of $c(n;2)+c(n+1;2)$ is $3, 20, 272, 7424 \ldots$. This is in agreement with [188]. We note that different factorizations of L^{-1} can lead to different formulas for $h_n(2)$, the Hankel transform of $c(n;2)+c(n+1;2)$. For instance, we can show that

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -\frac{8}{3} & 1 & 0 & 0 & \dots \\ \frac{28}{5} & -\frac{28}{5} & 1 & 0 & \dots \\ -\frac{192}{17} & \frac{345}{17} & -\frac{146}{17} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{3} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{5} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{17} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -8 & 3 & 0 & 0 & \dots \\ 28 & -28 & 5 & 0 & \dots \\ -192 & 345 & -146 & 17 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We note that the diagonal elements of the last matrix correspond to the sequence $a(n)$ of terms $1, 3, 5, 17, 29, 99, \ldots$ with generating function

$$\frac{1+3x-x^2-x^3}{1-6x^2+x^4}.$$

This is A079496. It is the interleaving of bisections of the Pell numbers A000129 and their associated numbers A001333. We have

$$\begin{aligned} a(n) &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1}{2k} 2^{n+1-k-\lfloor \frac{n+2}{2} \rfloor} \\ &= -(\sqrt{2}-1)^n \left(\left(\frac{\sqrt{2}}{8} - \frac{1}{4} \right) (-1)^n - \frac{\sqrt{2}}{8} - \frac{1}{4} \right) - (\sqrt{2}+1)^n \left(\left(\frac{\sqrt{2}}{8} - \frac{1}{4} \right) (-1)^n - \frac{\sqrt{2}}{8} - 1/4 \right) \end{aligned}$$

Multiplying $a(n)$ by $4^{\lfloor\frac{(n+1)^2}{4}\rfloor}$, we obtain $1, 3, 20, 272, 7424, \ldots$. Hence

$$\begin{aligned}
1, 3, 20, 272, \ldots &= 4^{\lfloor\frac{(n+1)^2}{4}\rfloor} \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \binom{n+1}{2k} 2^{n+1-k-\lfloor\frac{n+2}{2}\rfloor} \\
&= 4^{\lfloor\frac{(n+1)^2}{4}\rfloor} 2^{n+1-\lfloor\frac{n+2}{2}\rfloor} \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \binom{n+1}{2k} 2^{-k} \\
&= 2^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \binom{n+1}{2k} 2^{-k}
\end{aligned}$$

That is, the Hankel transform $h_n(2)$ of $c(n; 2) + c(n + 1; 2)$ is given by

$$h_n(2) = 2^{\binom{n+2}{2}} \sum_{k=0}^{\lfloor\frac{n+2}{2}\rfloor} \binom{n+2}{2k} 2^{-k}.$$

For our purposes, the following factorization of L^{-1} is more convenient.

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -\frac{8}{3} & 1 & 0 & 0 & \ldots \\ \frac{56}{10} & -\frac{56}{10} & 1 & 0 & \ldots \\ -\frac{384}{34} & \frac{690}{34} & -\frac{292}{34} & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & \frac{1}{3} & 0 & 0 & \ldots \\ 0 & 0 & \frac{1}{10} & 0 & \ldots \\ 0 & 0 & 0 & \frac{1}{34} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -8 & 3 & 0 & 0 & \ldots \\ 56 & -56 & 10 & 0 & \ldots \\ -384 & 690 & -292 & 34 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We now note that the sequence $\frac{h_n(2)}{2^{\binom{n+1}{2}}}$ is the sequence $b_2(n + 1)$, where $b_2(n)$ is the sequence $1, 3, 10, 34, 116, \ldots$ with generating function $\frac{1-x}{1-4x+2x^2}$ and general term

$$b_2(n) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-k}{k} (-2)^k 4^{n-2k} - \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-k-1}{k} (-2)^k 4^{n-2k-1}.$$

Hence

$$h_n(2) = 2^{\binom{n+1}{2}} b_2(n + 1).$$

Noting that $b_2(n)$ is the binomial transform of the Pell A000129$(n + 1)$ numbers whose generating function is $\frac{1}{1-2x-x^2}$, we have the following alternative expressions for $b_2(n)$:

$$\begin{aligned}
b_2(n) &= \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{j}{k-j} 2^{2j-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{\lfloor\frac{k}{2}\rfloor} \binom{k-j}{j} 2^{k-2j}.
\end{aligned}$$

For $r=3$, we have

$$H = \begin{pmatrix} 4 & 15 & 69 & 357 & \ldots \\ 15 & 69 & 357 & 1986 & \ldots \\ 69 & 357 & 1986 & 11598 & \ldots \\ 357 & 1986 & 11598 & 70125 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We find that

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & \frac{1}{4} & 0 & 0 & \ldots \\ 0 & 0 & \frac{1}{17} & 0 & \ldots \\ 0 & 0 & 0 & \frac{1}{73} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -15 & 4 & 0 & 0 & \ldots \\ 198 & -131 & 17 & 0 & \ldots \\ -2565 & 2875 & -854 & 73 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the sequence $b_3(n)$ or $1, 4, 17, 73, 314, \ldots$ has generating function $\frac{1-x}{1-5x+3x^2}$ and

$$\begin{aligned} b_3(n) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-3)^k 5^{n-2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} (-3)^k 5^{n-2k-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{j}{k-j} 3^{2j-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} 3^{k-2j}. \end{aligned}$$

Then $3^{\binom{n}{2}} b_3(n)$ is the sequence $1, 4, 51, 1971, 228906, \ldots$. In other words, we have

$$h_n(3) = 3^{\binom{n+1}{2}} b_3(n+1).$$

We now note that $F(2n+1)$ has generating function $\frac{1-x}{1-3x+x^2}$ with

$$F(2n+1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-1)^k 3^{n-2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} (-1)^k 3^{n-2k-1}.$$

We can generalize this result as follows.

Proposition 293. *Let $h_n(r)$ be the Hankel transform of the sum of the consecutive generalized Catalan numbers $c(n;r) + c(n+1;r)$. Then*

$$h_n(r) = r^{\binom{n+1}{2}} \left(\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k} (-r)^k (r+2)^{n-2k+1} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-r)^k (r+2)^{n-2k} \right).$$

In other words, $h_n(r)$ is the product of $r^{\binom{n+1}{2}}$ and the $(n+1)$-st term of the sequence with generating function $\frac{1-x}{1-(r+2)x+rx^2}$. Equivalently,

$$\begin{aligned}
h_n(r) &= r^{\binom{n+1}{2}}(\sum_{k=0}^{n+1}\binom{k}{n-k+1}(r+2)^{2k-n-1}(-r)^{n-k+1}-\sum_{k=0}^{n}\binom{k}{n-k}(r+2)^{2k-n}(-r)^{n-k}) \\
&= r^{\binom{n+1}{2}}\sum_{k=0}^{n+1}\binom{n+1}{k}\sum_{j=0}^{k}\binom{j}{k-j}r^{2j-k} \\
&= r^{\binom{n+1}{2}}\sum_{k=0}^{n+1}\binom{n+1}{k}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k-j}{j}r^{k-2j}.
\end{aligned}$$

The two last expressions are a result of the fact that $\frac{1-x}{1-(r+2)x+rx^2}$ is the binomial transform of $\frac{1}{1-rx-x^2}$.

Bibliography

[1] M. Aigner, Motzkin numbers, *Europ. J. Combinatorics* (1998) **19**, pp. 663–675.

[2] M. Aigner, Catalan-like numbers and determinants, *J. Combinatorial Theory*, Series A **87** (1999) pp. 33–51.

[3] M. Aigner, A characterization of the Bell numbers, *Discrete Mathematics*, **205** (1999) pp. 207 – 210.

[4] M. Aigner, Catalan and other numbers: a recurrent theme, in: Grapo, Senato, eds., *Algebraic Combinatorics and Computer Science*, Springer Italia, Milan, 2001, pp. 347–390.

[5] R. Askey, Duality for classical orthogonal polynomials, *Journal of Computational and Applied Mathematics*, **178** (2005) pp. 37–43

[6] D. Baccherini, D. Merlini, R. Sprugnoli, Binary words excluding a pattern and proper Riordan arrays, *Discrete Mathematics* **307** (2007) pp. 1021–1037.

[7] R. Bacher, Matrices related to the Pascal triangle, arXiv:math/0109013v2.

[8] C. Banderier, D. Merlini, Lattice paths with an infinite set of jumps, In: Proceedings of. the 14-th International Conference on Formal Power Series and Algebraic Combinatorics, .

[9] E. Barcucci, A. Frosini, S. Rimone, On directed-convex polyominoes in a rectangle, *Discrete Mathematics* **298** (2005) pp. 62–78.

[10] E. Barcucci, E. Pergola, R. Pinzani, S. Rinaldi, ECO method and hill-free generalized Motzkin paths, *Séminaire Lotharingien de Combinatoire* **46** (2001), Article B46b.

[11] P. Barry, Triangle Geometry and Jacobsthal Numbers, *Irish Math. Soc. Bulletin* **51** (2003), pp. 45–57.

[12] P. Barry, A modulo theorem for a class of linear recurrences, *Preprint* 2003.

[13] P. Barry, On k-regular graphs and related integer sequences, *Preprint* 2005.

[14] P. Barry, Chebyshev transformations on integer sequences, with applications to knot invariants and graphs, *Preprint*, 2005.

[15] P. Barry, A Catalan Transform and Related Transformations on Integer Sequences, *Journal of Integer Sequences*, Vol. 8 (2005), Article 05.4.5.

[16] P. Barry, On integer-sequence-based constructions of generalized Pascal triangles, *J. Integer Sequences*, **9** (2006), Article 06.2.4.

[17] P. Barry, On a family of generalized Pascal triangles defined by exponential Riordan arrays, *J. Integer Sequences*, **10** (2007), Article 07.3.5.

[18] P. Barry, Some observations on the Lah and Laguerre transforms of integer sequences, *J. Integer Sequences*, **10** (2007), Article 07.4.6.

[19] P. Barry, On Integer Sequences Associated with the Cyclic and Complete Graphs, *J. Integer Sequences*, **10** (2007) Article 07.4.8

[20] P. Barry, P. Fitzpatrick, On a One-parameter Family of Riordan arrays and the Weight Distribution of MDS Codes, *J. Integer Sequences*, **10** (2007) Article 07.9.8.

[21] P. Barry, Some conjectures on the ratio of Hankel transforms for sequences and series reversion, arXiv:math/0701483.

[22] P. Barry, A Note on Krawtchouk Polynomials and Riordan Arrays, *J. Integer Sequences*, **11** (2008), Article 08.2.2.

[23] P. Barry, P.M. Rajković, M.D. Petković, Generalized trinomial numbers, orthogonal polynomials and Hankel transforms, Conference paper, ALA2008, Novi Sad, Serbia

[24] L. Bartholdi, Counting Paths in Graphs, arXiv:math.CO/001261.

[25] M. Bayat, H. Teimoori, The linear algebra of the generalized Pascal functional matrix, *Linear Algebra and its Applications* **295** (1999) pp. 81–89.

[26] A. T. Benjamin, N. T. Cameron, J. J. Quinn and C. R. Yerger, Catalan Determinants - A Combinatorial Approach, in: *Applications of Fibonacci Numbers*, Volume 11, (William Webb, ed.), Kluwer Academic Publishers, 2008.

[27] C. Berg, Fibonacci numbers and orthogonal polynomials, arXiv:math/0609283v1.

[28] F. Bergeron, S. Plouffe, Computing the Generating Function of a Series given its First Few Terms, *Experiment. Math.*, textbf1, Issue 4 (1992), pp. 307–312.

[29] F. R. Bernhart, Catalan, Motzkin and Riordan numbers, *Discrete Mathematics* **204** (1999) pp. 73–112.

[30] D. Birmajer, J. B. Gil, Arithmetic in the Ring of Formal Power Series with Integer Coefficients, *American Math. Monthly* **115** No. 6, (2008) pp. 541–549.

[31] P. Blasiak, K. A. Penson and A. I. Solomon, The boson normal ordering problem and generalized numbers, *Annals of Combinatorics*, **7** (2003) 127–139. Available electronically at `http://arxiv.org/PS_cache/quant-ph/pdf/0212/0212072.pdf`, 2007.

[32] P. Blasiak, K. A. Penson, A. I. Solomon, A. Horzela and G. E. H. Duchamp, Some useful combinatorial formulae for bosonic operators , available electronically at http://arxiv.org/PS_cache/quant-ph/pdf/0405/0405103.pdf, 2007.

[33] P. Blasiak, G. Dattoli and A. Horzela, Motzkin numbers, Central Trinomial Coefficients and Hybrid Polynomials, *Journal of Integer Sequences* **11** (2008) Article 08.1.1.

[34] A. Bodin, Number of irreducible polynomials in several variables over finite fields, arXiv:0706.0157v2.

[35] B. Bouras, Generalized Jacobi orthogonal polynomials, *Integral Transforms and Special Functions*, 18(10) 2007, pp. 715–730.

[36] B. Bouras, Hankel determinants of Catalan numbers and Jacobi polynomials.

[37] L. Boyadjiev and R. Scherer, On the Chebyshev polynomials, *Kuwait J. Sci. Eng.*, 28(2) 2001, pp. 227–239.

[38] R. Brak, J. Essam, J. Osborn, A. L. Owczarek and A. Rechnitzer, Lattice Paths and the Constant Term, *Journal of Physics: Conference Series* **42** (2006) pp. 47–58.

[39] P. Branden, A. Claesson & E. Steingrimsson, Catalan continued fractions and increasing subsequences in permutations, available electronically at http://www.cs.chalmers.se/ claesson/papers/kb13.ps, 2009.

[40] P. Branden, The generating function of 2-stack sortable permutations by descents is real-rooted, arXiv:math/0303149v2.

[41] D. M. Bressoud, *Proofs and Confirmations*, Cambridge University Press, 1999.

[42] E. H. M. Brietzke, An identity of Andrews and a new method for the Riordan array proof of combinatorial identities, arXiv:0803.2803v1, 2008.

[43] V.H. Bui, M. H. Le, T. H. D. Phan and V. D. Phung, Generating trees for integer partitions and ECO method, available electronically at http://www.math.ac.vn/library/download/e-print/08/pdf/pthduong0801.pdf

[44] H. Burgiel, V. Reiner, Two Signed Associahedra, *New York Journal of Mathematics*, **4** (1998) pp. 83–95.

[45] A. Burstein, T. Mansour, Words restricted by patterns with at most 2 distinct letters, *Electron. J. Combin.*, **9 (2)**,#R3 (2002).

[46] G. S. Call, D. J. Velleman, Pascal's Matrices, *Amer. Math. Monthly*, **100** (1993), pp. 372–376.

[47] P. J. Cameron, J. H. van Lint, *Designs, Graphs, Codes and their Links*, Cambridge University Press, 2000.

[48] M. Chamberland, C. French, Generalized Catalan Numbers and Generalized Hankel Transformations, *Journal of Integer Sequences*, **10** (2007), article 07.1.1.

[49] F. Chapoton, Enumerative Properties of Generalized Associahedra, arXiv:math/0401237v1.

[50] W. Y. C. Chen, S. H. F. Yan and L. L. M. Yang, Identities from weighted 2-Motzkin paths, published electronically at http://arxiv.org/pdf/math.CO/0410200.pdf, 2006.

[51] W. Y.C. Chen, N. Y. Li, L.W.Shapiro, S.H.F. Yan, Matrix Identities on Weighted Partial Motzkin Paths, arXiv:math/0509255.

[52] A. Claesson, Counting segmented permutations using bicoloured Dyck paths, *El. Journal of Combin.*, **12** (2005) #R39.

[53] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[54] W. Chu, Inversion techniques and combinatorial identities-Strange evaluations of basic hypergeometric series, *Compositio Mathematica* **91** (1994) pp. 121–144.

[55] J. Cigler, Some relations between generalized Fibonacci and Catalan numbers, *Sitzungsberichte Abt. II* **211** (2002) pp. 143-154.

[56] C. Coker, Enumerating a class of lattice paths, *Discrete Mathematics*, 271 (2003) pp. 13-28.

[57] E. Coleman, The Baer-Specker Group, *Bulletin of the Irish Mathematical Society*, 40, 1998, pp. 9–23.

[58] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, Dordrecht, Holland, 1974.

[59] C. Corsani, D. Merlini, R. Sprugnoli, Left-inversion of combinatorial sums, *Discrete Math.* **180** (1998), pp. 107–122.

[60] P. Cromwell, *Knots and Links*, Cambridge University Press, Cambridge, UK, 2004.

[61] A. Cvetković, P. Rajković and M. Ivković, Catalan Numbers, the Hankel Transform and Fibonacci Numbers, *Journal of Integer Sequences*, **5**, May 2002, Article 02.1.3.

[62] S. J. Cyvin, I. Gutman, *Kekulé Structures in Benzenoid Hydrocarbons*, Lecture Notes in Chemistry 46, Springer-Verlag, 1988.

[63] G. Davidoff, P. Sarnak & A. Valette, *Elementary Number Theory, Group Theory and Ramanujan Graphs*, London Mathematical Society Student Texts **55**, Cambridge University Press, UK, 2003.

[64] P. J. Davis, *Circulant Matrices*, John Wiley, New York, 1979.

[65] A. Del Lungo, E. Duchi, A. Frosini and S. Rinaldi, Enumeration of convex polyominoes using the ECO method, *Discrete Mathematics and Theoretical Computer Science* **AB(DMCS)** (2003) pp 103–116.

[66] Ph. Delsarte, Nombres de Bell et polynômes de Charlier, *C. R. Acad. Sc. Paris* Serie A **287** pp. 271–273.

[67] E.Y.P. Deng, W-J. Yan, Some identities on the Catalan, Motzkin and Schröder numbers, *Discrete Applied Mathematics*, 156 (2008), pp. 2781–2789.

[68] M. De Sainte-Catherine, G. Viennot, Enumeration of certain Young tableaux with bounded heights, Combinatoire Enumerative (Montreal) (1985), *Lecture Notes in Mathematics*, 1234, 1986, pp. 58–67.

[69] E. Deutsch, Dyck path enumeration, *Discrete Mathematics*, **204** (1999) pp. 167–202.

[70] E. Deutsch, A bijective proof of the equation linking the Schröder numbers, large and small, *Discrete Mathematics* **241** (2001) pp. 235–240.

[71] E. Deutsch, L. Shapiro, A survey of the Fine numbers, *Discrete Math.* **241** (2001), pp. 241–265.

[72] E. Deutsch, A. Robertson and D. Saracino, Refined Restricted Involutions, preprint, arxiv:math.CO/0212267.

[73] E. Deutsch, L. Shapiro, Exponential Riordan Arrays, Lecture Notes, Nankai University, 2004, available electronically at http://www.combinatorics.net/ppt2004/Louis%20W.%20Shapiro/shapiro.htm

[74] E. Deutsch, L. Ferrari, and S. Rinaldi, Production Matrices, *Advances in Applied Mathematics* **34** (2005) pp. 101–122.

[75] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, http://arxiv.org/abs/math/0702638v1, February 22 2007.

[76] P. Duchon, On the enumeration and generation of generalized Dyck words, *Discrete Mathematics* **225** (2000) pp. 121–135.

[77] A. W. F. Edwards, *Pascal's Arithmetical Triangle*, Johns Hopkins University Press, 2002.

[78] Ö. Egecioglu, T. Redmond, C. Ryavec, Almost Product Evaluation of Hankel Determinants, *The Electronic Journal of Combinatorics*, **15**, (2008), #R6.

[79] Ö. Egecioglu, A. King, Random Walks and Catalan Factorization, *Congressus Numerantium*, **138** (1999), pp. 129–140.

[80] Ö. Egecioglu, T. Redmond, C. Ryavec, A Multilinear Operator for Almost Product Evaluation of Hankel Determinants, arXiv:0804.0440v1.

[81] G. P. Egorychev, *Integral Representation and the Computation of Combinatorial Sums*, Transl. of Math. Monographs, Vol. 59, Amer. Soc. Soc., 1984, 2nd edn. 1989.

[82] G. P. Egorychev, E. V. Zima, On integral representation and algebraic approaches to the evaluation of combinatorial sums, Technical Report, University of Waterloo, 2002, available electronically at http://www.cs.uwaterloo.ca/research/tr/2002/02/CS-2002-02.pdf.

[83] G. P. Egorychev, E. V. Zima, Decomposition and Group Theoretic Characterization of Pairs of Inverse Relations of the Riordan Type, *Acta Applicandae Math.* **85** (2005), 93–109.

[84] S. Elizalde, T. Mansour, Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials, Formal Power Series and Algebraic Combinatorics, Vancouver 2004.

[85] R. Ehrenborg, The Hankel Determinant of Exponential Polynomials, *Am. Math. Monthly*, 107 (2000), pp. 557–560.

[86] L. Ferrari, R. Pinzani, Catalan-like numbers and succession rules, arXiv:math.CO/0507210v1.

[87] G. Ferri, M. Faccio and A. D'Amico, A new numerical triangle showing links with Fibonacci numbers, *Fibonacci Quart.* **29** (1991), pp. 316–320.

[88] G. Ferri, M. Faccio and A. D'Amico, Fibonacci numbers and ladder network impedance, *Fibonacci Quart.* **30** (1992), pp. 62–67.

[89] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Mathematics*, 306 (2006) pp. 992–1021

[90] P. Feinsilver, J. Kocik, Krawtchouk polynomials and Krawtchouk matrices, in R. Baeza-Yates, J. Glaz, H. Gzyl, J. Husler, and J. L. Palacios, eds., *Recent Advances in Applied Probability*, Springer, 2005, pp. 115–141.

[91] P. Feinsilver, J. Kocik, Krawtchouk matrices from classical and quantum random walks, http://arxiv.org/abs/quant-ph/0702173v1, February 16 2007.

[92] L. Ferrari, R. Pinzani, Catalan-like numbers and succession numbers, available electronically at http://www.arXiv.org/math.CO/0507210, 2006.

[93] T. Fink, Y. Mao, *The* 85 *ways to tie a tie*, Fourth Estate, London, 2001.

[94] S. Fomin, N. Reading, Root Systems and Generalized Associahedra, arXiv:math/0505518.

[95] P. J. Forrester, A. Gamburd, Counting formulas associated with some random matrix averages, arXiv:math/0503002v1.

[96] C. French, Transformations Preserving the Hankel Transform, *Journal of Integer Sequences*, Vol 10 (2007), Article 07.7.3.

[97] D. D. Frey and J. A. Sellers, Jacobsthal Numbers and Alternating Sign Matrices, *J. Integer Seqs.*, Vol. 3 (2000), Article 00.2.3.

[98] G. Gaspar and M. Rahman, *Basic Hypergeometric Series*, 2nd ed, Cambridge University Press, Cambridge, UK, 2004.

[99] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press, Oxford, 2003.

[100] I. M. Gessel, X. G. Viennot, Determinants, Paths, and Plane Partitions, available electronically at http://people.brandeis.edu/∼gessel/homepage/papers/pp.pdf, 2006.

[101] I. M. Gessel, Guoce Xin, A Combinatorial Interpretation of the Numbers 6(2n)!/n!(n+2)!, *Journal of Integer Sequences*, Vol. 8 (2005), Article 05.2.3.

[102] I. M. Gessel, Guoce Xin, The generating function of ternary trees and continued fractions, arXiv:math/05052v1.

[103] S. Getu, L.W. Shapiro, W-J. Woan & L. C. Woodson, How to guess a generating function, *SIAM J. Disc. Math.*, 5 (4) (1992) pp. 497–499.

[104] C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.

[105] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.

[106] I. Graham, D. E. Knuth & O. Patashnik, *Concrete Mathematics*, Addison–Wesley, Reading, MA, 1995.

[107] R. P. Grimaldi, *Discrete and Combinatorial Mathematics*, 4th ed, Addison–Wesley, Reading, MA, 2000.

[108] E. Grosswald, *Bessel Polynomials*, Springer-Verlag, 1978

[109] V. J. W Guo, J. Zeng, The Number of Convex Polyominoes and the Generating Function of Jacobi Polynomials, *Discrete Applied Mathematics* **154** 3 (2006) pp 587–593.

[110] Tian-Xiao He, L. C. Hsu, P. J.-S. Shiue & D. C. Torney, A symbolic operator approach to several summation formulas for power series, *J. Comput. Appl. Math.*, **177** (2005), pp. 17–33.

[111] Tian-Xiao He, L. C. Hsu & P. J.-S. Shiue, Multivariate Expansion Associated with Sheffer-type Polynomials and Operators,*Bulletin Institute of Mathematics*, Academia Sinica, New Series, **1** (2006), No. 4.

[112] Tian-Xiao He, L. C. Hsu & P. J.-S. Shiue, The Sheffer Group and the Riordan Group, *Discrete Applied Math* **155** (2007), pp. 1895–1909.

[113] Tian-Xiao He, A Symbolic Operator Approach to Power Series Transformation-Expansion Formulas, *Journal of Integer Sequences*, **11** (2008) Article 08.2.7.

[114] Tian-Xiao He, L. C. Hsu & P. J.-S. Shiue, A symbolic operator approach to several summation formulas for power series II, *Discrete Math.* **308** (2008) pp. 3427–3440.

[115] G. Helms, Binomial-("Pascal")-Matrix: a beautiful identity, available electronically at 141.51.96.22/files/PascalDreieckMatrixLog.pdf, 2006.

[116] P. Henrici, *Applied and Computational Complex Analysis*, vol. 2, John Wiley & Sons, New York.

[117] P. Hilton and J. Pedersen, Catalan numbers, their generalizations, and their uses, *Math. Intelligencer* **13** (2) (1991), pp. 64–75.

[118] L. C. Hsu, Theory and Applications of Generalized Stirling Number Pairs, *Journal of Mathematical Research and Exposition*, **9** No. 2 (1989) pp. 211–220.

[119] M. E. H. Ismail, D. Stanton, Classical orthogonal polynomials as moments, *Canad. Journal Math.*, **49** 3 (1997), pp. 520–542.

[120] M. E. H. Ismail, D. Stanton, More orthogonal polynomials as moments, in :Mathematical Essays in Honor of Gian-Carlo Rota, B. E. Sagan, R. P. Stanley, eds.,*Progress in Math.* vol. 161, Birkhauser, Boston, 1998, pp 377-396.

[121] A. Junod, Hankel Determinants and Orthogonal Polynomials, *Expositiones Mathematicae* **21** (2003) pp. 63–74.

[122] S. Kamioka, A combinatorial derivation with Schröder paths of a determinant representation of Laurent biorthogonal polynomials, *The electronic journal of combinatorics* **15** (2008), #R76.

[123] S. Karlin, G. Szegö, On certain determinants whose elements are orthogonal polynomials, *J. d'Analyse Math.* **8** (1960-1961) pp. 1–157.

[124] H. Kesten, Symmetric random walks on groups, *Trans. Amer. Math. Soc.*, **92** (1959), pp. 336–354.

[125] D. Kim, D. Stanton & J. Zeng, The combinatorics of the Al-Salam-Chihara q-Charlier polynomials, *Seminaire Lotharingien de Combinatoire* **54** (2006), Article B54i

[126] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, TU Delft, The Netherlands, 1998; available electronically at http://aw.twi.tudelft.nl/ koekoek/research.html, (p. 43), 2008.

[127] D. E. Knuth, *The Art of Computer Programming*, Addison-Wesley, Reading, MA, Vol. 1, pp. 84,492.

[128] C. Krattenthaler, *The Major Counting of Nonintersecting Lattice Paths and Generating Functions for Tableaux*, American Mathematical Society, 1995.

[129] C. Krattenthaler, Permutations with Restricted Patterns and Dyck Paths, *Advances in Applied Mathematics*, **27** (2001) pp. 510–530.

[130] C. Krattenthaler, Determinants of (generalised) Catalan numbers, Preprint.

[131] C. Krattenthaler, Advanced Determinant Calculus, available electronically at `http://arxiv.org/PS_cache/math/pdf/9902/9902004.pdf`, 2006

[132] C. Krattenthaler, Advanced determinant calculus: A complement, *Linear Algebra and its Applications* 411 (2005) 68166.

[133] C. Krattenthaler, S. G. Mohanty, Lattice Path Combinatorics - Applications to Probability and Statistics, *Encyclopedia of Statistical Sciences*, S. Kotz, N.L. Johnson, C.B. Read, N. Balakrhishnan, B. Vidakovic, eds., Wiley, New York, 2003.

[134] C. Krattenthaler, Personal communication, 2007.

[135] E. Krot, An Introduction to Finite Fibonomial Calculus, published electronically at `http://arxiv.org/pdf/math.CO/0503210.pdf`, 2006.

[136] S. Khrushchev, *Orthogonal Polynomials and Continued Fractions*, Encyclopedia of Mathematics, Cambridge University Press, Cambridge, 2008.

[137] W. Lang, On Polynomials Related to Powers of the Generating Function of Catalan's Numbers, *Fibonacci Quarterly* **38** (2000) pp. 408–419.

[138] A. Lascoux, *Symmetric Functions and Combinatorial Operators on Polynomials* AMS, 2001.1

[139] J. W. Layman, The Hankel Transform and Some of Its Properties, *Journal of Integer Sequences*, **4**, Article 01.1.5, 2001.

[140] M. E. Larsen, Wronskian Harmony, *Mathematics Magazine*, **63** 1 (1990), pp. 33–37.

[141] N. N. Lebedev, *Special Functions & Their Applications*, trans. R. Silverman, Dover Publications, Inc. New York, 1972.

[142] B. Leclerc, On certain formulas of Karlin and Szegö, Preprint, availabel electronically at `citeseer.ist.psu.edu/554905.html`, 2008.

[143] F. Lehner, Cumulants, Lattice Paths, and Orthogonal Polynomials, *Discrete Mathematics*, **270** no. 1-3 (2003), pp. 177–191.

[144] S. Ling, C. Xing, *Coding Theory*, Cambridge University Press, Cambridge, UK, 2004.

[145] W. F. Lunnon, The Pascal matrix, Fib. Quart. 15 (1977), pp. 201–204.

[146] A. Luzón, M.A. Morón, Ultrametrics, Banach's fixed point theorem and the Riordan group, *Discrete Applied Mathematics*, 156 (2008), pp. 2620–2635.

[147] F. J. MacWilliams, N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North Holland, Amsterdam, NL, 2003.

[148] T. Mansour, A. Vainshtein, Restricted Permutations, Continued Fractions, and Chebyshev Polynomials, *El. Journal Combin.*, **7** 1 (2000) #R17.

[149] T. Mansour, Y. Sun, Identities involving Narayana polynomials and Catalan numbers, arXiv:0805.1274v1.

[150] F. Marcellan, A. Ronveaux, On a class of polynomials orthogonal with respect to a discrete Sobolev inner product, *Indag. Mathem.*, N.S. 1 (4), pp. 451–464.

[151] M. E. Mays, J. Wojciechowski, A determinant property of Catalan numbers, *Discrete Mathematics* **211** (2000) pp. 125–133.

[152] D. Merlini, Generating functions for the area below some lattice paths, *Discrete Mathematics and Theoretical Computer Science* **AC** (2003) pp. 217–228.

[153] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, On some alternative characterizations of Riordan arrays, *Canadian J. Mathematics*, **49** (2) (1997), pp. 301–320.

[154] D. Merlini, R. Sprugnoli and M. C. Verri, The area determined by underdiagonal lattice paths, *Lecture Notes in Computer Science*, **1059**, Proceedings of the 21st International Colloquium on Trees in Algebra and Programming, (1996), pp. 59–71.

[155] D. Merlini, R. Sprugnoli and M. C. Verri, Algebraic and combinatorial properties of simple, coloured walks, in: *Proceedings of the 19th International Colloquium on Trees in Algebra and Programming*, Lecture Notes In Computer Science, **787** (1994) pp. 218–233.

[156] D. Merlini, R. Sprugnoli and M. C. Verri, Combinatorial inversions and implicit Riordan arrays, *El. Notes in Discrete Mathematics* **26** (2006), pp. 103–110.

[157] D. Merlini, R. Sprugnoli and M. C. Verri, Lagrange Inversion: When and How, *Acta Applicandae Mathematicae*, **94** (2006), pp. 233–249.

[158] D. Merlini, R. Sprugnoli and M. C. Verri, The Method of Coefficients, *Amer. Math. Monthly*, **114** (2007), pp. 40–57.

[159] D. Merlini, R. Sprugnoli, M. C. Verri, Combinatorial sums and implicit Riordan arrays, *Discrete Mathematics* (2008) doi:10.1016/j.disc.2007.12.039.

[160] S. C. Milne, New infinite families of exact sums of squares formulas, Jacobi elliptic functions, and Ramanujan's tau function, *Proc. Natl. Acad. Sci.*, vol. 93 (1996), pp. 15004–15008.

[161] S. C. Milne, Infinite families of exact sums of squares formulas, Jacobi elliptic functions, continued fractions, and Schur functions, *Ramanujan Journal*, **6** (2002), no. 1, pp. 7–149.

[162] Tan Mingshu, Wang Tianming, Lah Matrix and its Algebraic Properties, *Ars Combinatoria* **70** (2004), pp. 97–108.

[163] Y. Nakamura, A. Zhedanov, Toda Chain, Sheffer Class of Orthogonal Polynomials and Combinatorial Numbers, *Proceedings of Inst. Math. NAS Ukraine*, **50** (2004), pp. 450–457.

[164] T. V. Narayana, *Lattice path combinations with statistical applications*, University of Toronto Press, Toronto, 1979.

[165] S. A. Ng, Some identities and formulas involving generalized Catalan number, 2004. Preprint available at http://saturn.cs.unp.ac.za/∼siuahn/default.htm.

[166] S. A. Ng, A Generalization of the Catalan numbers, preprint, 2004.

[167] E. M. Nikishin, V. N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs vol. 92, American Mathematical Society, 1991.

[168] A. Nkwanta, A Riordan matrix approach to unifying a selected class of combinatorial arrays, *Congressus Numerantium*, **160** (2003), pp. 33–45.

[169] A. Nkwanta, Lattice paths and RNA secondary structures, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, **34** (1997) pp. 137–147.

[170] A. Nkwanta, N. Knox, A note on Riordan matrices, *Contemporary Mathematics* 252 (1999) pp. 99–107.

[171] T. D. Noe, On the divisibility of generalized central trinomial coefficients, Preprint, 2005, available electronically at http://www.sspectra.com/math/Trinomial.pdf, 2006.

[172] R. Ortner, W. Woess, Non-Backtracking Random Walks and Cogrowth of Graphs, *Canad. J. Math.* **59** 4 (2007), pp. 828–844.

[173] P. Peart and W.-J. Woan, Dyck paths with no peaks at height k, *J. Integer Sequences*, **4** (2001), Article 01.1.3.

[174] P. Peart, W-J. Woan, Generating Functions via Hankel and Stieltjes Matrices, *Journal of Integer Sequences*, Vol. 3 (2000), Article 00.2.1.

[175] P. Peart, L. Woodson, Triple Factorisation of some Riordan Matrices, *Fibonacci Quarterly*, 31 (1993) pp. 121–128.

[176] K. A. Penson, J.-M. Sixdeniers, Integral Representations of Catalan and Related Numbers, *Journal of Integer Sequences*, 4 (2001), Art. 01.2.5.

[177] E. Pergola, R. Pinzani, A Combinatorial Interpretation of the Area of Schröder Paths, *Electronic J. of Combin.*, vol. **6** (1999), #R40.

[178] E. Pergola, Two bijections for the area of Dyck paths, *Discrete Mathematics*, 241 (2001) pp. 435–447.

[179] W. W. Peterson, E. J. Weldon, *Error-Correcting Codes*, 2nd ed. MIT Press, Cambridge, US, 1984.

[180] A. Postnikov, V. Reiner & L. Williams, Faces of Generalized Permutohedra, *Documenta Math.* **13** (2008) pp. 207–273.

[181] Ch. Radoux, Calcul effectif de certains déterminants de Hankel, *Bull. Soc. Math. Belg.*, XXXI, Fascicule 1, serie B (1979), pp. 49–55.

[182] Ch. Radoux, Déterminant de Hankel construit sur les polynômes de Hermite, *Annales de la Societe Scientifiques de Bruxelles*, 104 (2) (1991), pp. 59–61.

[183] Ch. Radoux, Déterminant de Hankel construit sur des polynômes liés aux nombres de dérangements, *Europ. J. Combinatorics* (1991) 12, pp. 327–329.

[184] Ch. Radoux, Nombres de Catalan généralisés, *Bull. Belg. Math. Soc. Simon Stevin* 4 (2) (1997) pp. 289–292.

[185] Ch. Radoux, Addition formulas for polynomials built on classical combinatorial sequences, *J. of Computational and Applied Mathematics*, 115 (2000) pp. 471–477.

[186] Ch. Radoux, The Hankel Determinant of Exponential Polynomials: A very short proof and a new result concerning Euler numbers, *Amer. Math. Monthly* 109 (2002) pp. 277–278.

[187] Ch. Radoux, Déterminants de Hankel et théorème de Sylvester, available electronically at `www.mat.univie.ac.at/ slc/opapers/s28radoux.pdf`, 2006.

[188] P.M. Rajković, M.D. Petković, P. Barry The Hankel Transform of the Sum of Consecutive Generalized Catalan Numbers, *Integral Transforms and Special Functions*, Vol. 18, Issue 4, 2007, 285–296.

[189] A. Reifegerste, On the Diagam of 132-Avoiding Permuations, arXiv:math/0208006v3.

[190] T. M. Richardson, The Filbert matrix, published electronically at http://arxiv.org/pdf/math.RA/9905079.pdf, 2006.

[191] J. Riordan, *Combinatorial Identities*, John Wiley & Sons Inc., 1968.

[192] J. Riordan, *An Introduction to Combinatorial Analysis*, Dover Publications Inc, 2002.

[193] D. G. Rogers, Pascal triangles, Catalan numbers and renewal arrays, *Discrete Mathematics* **22** (1978), pp. 301–310.

[194] D. Romik, Some formulas for the central trinomial and Motzkin numbers, *Journal of Integer Sequences* **6** (2003) Article 03.2.4.

[195] S. Roman, *The Umbral Calculus*, Dover Publications, 2005.

[196] W.A. Al-Salam, L. Carlitz, Some determinants of Bernoulli, Euler and Related Numbers, *Portugaliae Mathematica* textbf18 pp. 91–99.

[197] A. Sapounakis, P. Tsikouras, On k-colored Motzkin words, *J. of Integer Sequences*, vol. **7**, http://www.cs.uwaterloo.ca/journals/JIS/VOL7/Tsikouras/tsikouras43.ps.

[198] A. Sapounakis, P. Tsikouras, Counting peaks and valleys in k-colored Motzkin paths, *El. Journal of Combinatorics* **12** (2005) #R16.

[199] S. Schroer, Baer's Result: The infinite product of the integers has no basis, *Amer. Math. Monthly*, **115** (2008) pp. 660–663.

[200] I. Schur, On Faber polynomials, *Amer. J. Math.*, **14** (1976) pp. 33–41.

[201] L. W. Shapiro, A Catalan Triangle, *Discrete Math.* **14** (1976) pp. 83–90.

[202] L. W. Shapiro, S. Getu, W-J. Woan and L.C. Woodson, The Riordan Group, *Discr. Appl. Math.* **34** (1991) pp. 229–239.

[203] L. W. Shapiro, Bijections and the Riordan group, *Theoretical Comp. Sci.* **307** (2003) pp. 403–413.

[204] L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, UK, 1996.

[205] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://www.research.att.com/∼njas/sequences/, 2008.

[206] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, *Notices of the AMS*, **50** (2003), pp. 912–915.

[207] M. Z. Spivey, L. L. Steil, The k-Binomial Transforms and the Hankel Transform, *Journal of Integer Sequences*, Vol. 9 (2006), Article 06.1.1.

[208] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.*,**132** (1994) pp. 267–290.

[209] R. Sprugnoli, *Riordan Array Proofs of Identities in Gould's Book*. Published electronically at http://www.dsi.unifi.it/∼resp/GouldBK.pdf, 2008.

[210] R. P. Stanley, *Enumerative Combinatorics*, vol. I, Cambridge University Press, New York, 7th printing, 2006.

[211] R. P. Stanley, *Enumerative Combinatorics*, vol. II, Cambridge University Press, New York, 2001.

[212] R. A. Sulanke, Counting lattice paths by Narayana polynomials, *Electronic J. Combin.* **7** (2000), #R40.

[213] R. A. Sulanke, The Narayana distribution, *J. Statist. Plann. and Infer.* **101** (2002) pp. 311–346.

[214] R. A. Sulanke, Generalizing Narayana and Schröder numbers to higher dimensions, *Electron. J. Combin.* **11** (1) (2004), #R54.

[215] Y. Sun, The statistic 'number of *udu's*' in Dyck paths, *Discrete Math.* **287** (2004), pp. 177–186.

[216] Y. Sun, Numerical triangles and several classical sequences, 2004. Preprint available at `http://cfc.nankai.edu.cn/publications/04-accepted/Sun-04A2/Triangle.pdf`.

[217] M. N. S. Swamy, Further Properties of Morgan-Voyce polynomials, *Fibonacci Quarterly* **6** (1968) pp. 167–175.

[218] G. Szegö, *Orthogonal Polynomials*, 4th ed. Providence, RI, Amer. Math. Soc., (1975) pp. 35-37.

[219] U. Tamm, Some aspects of Hankel matrices in coding theory and combinatorics, *Electronic J. Combin.* **8** (2001) #A1.

[220] J. Touchard, Sur certaines equations fonctionelles, in: *Proceedings of the International Congress on Mathematics*, University of Toronto Press, Toronto, 1928, pp. 465–472.

[221] W. Trzaska, Modified numerical triangle and the Fibonacci sequence, *Fibonacci Quart.* **29** (1991), pp. 316–320.

[222] J. H. van Lint, R. M. Wilson, *A Course in Combinatorics*, 2nd ed. Cambridge University Press, Cambridge, 2001.

[223] G. Viennot, A Combinatorial Theory for General Orthogonal Polynomials with Extensions and Applications, in: Polynomes Orthogonaux et Applications, Lecture Notes in Mathematics, 1171 (1985) pp. 139–157.

[224] N. Ja. Vilenkin, A. U. Klimyk, *Representation of Lie Groups and Special Functions*, Vol 1, Kluwer Academic Publishers, Vol 1, 1991.

[225] N. Ja. Vilenkin, A. U. Klimyk, *Representation of Lie Groups and Special Functions*, Vol 2, Kluwer Academic Publishers, Vol 1, 1992.

[226] W. Wang, T. Wang, Generalized Riordan arrays, *Discrete Mathematics*, 2008, doi:10.10.16/j.disc.2007.12.037.

[227] H. S. Wall, *Analytic Theory of Continued Fractions*, AMS Chelsea Publishing, 2000.

[228] S. Warner, *Modern Algebra*, Dover Publications Inc, New York, 1990.

[229] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University Press, Cambridge, UK, 1966.

[230] E. W. Weisstein, http://mathworld.wolfram.com/BinomialTransform.html/, 2008.

[231] E. W. Weisstein, http://mathworld.wolfram.com/BesselPolynomial.html/, 2008.

[232] E. W. Weisstein, http://mathworld.wolfram.com/CatalanNumber.html/, 2008.

[233] E. W. Weisstein, http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html/,2008.

[234] E. W. Weisstein, http://mathworld.wolfram.com/ChebyshevPolynomialoftheSecondKind.html/,2008.

[235] E. W. Weisstein, http://mathworld.wolfram.com/DelannoyNumber.html/, 2008.

[236] E. W. Weisstein, http://mathworld.wolfram.com/FibonacciCoefficient.html/, 2008.

[237] E. W. Weisstein, http://mathworld.wolfram.com/FibonacciNumber.html/, 2008.

[238] E. W. Weisstein, http://mathworld.wolfram.com/HermitePolynomial.html/, 2008.

[239] E. W. Weisstein, http://mathworld.wolfram.com/JacobsthalNumber.html/, 2008.

[240] E. W. Weisstein, http://mathworld.wolfram.com/KrawtchoukPolynomial.html/, 2008.

[241] E. W. Weisstein, http://mathworld.wolfram.com/LaguerrePolynomial.html/, 2008.

[242] E. W. Wesstein, http://mathworld.wolfram.com/LucasSequence.html/, 2008.

[243] E. W. Weisstein, http://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html/, 2008.

[244] E. W. Weisstein, http://mathworld.wolfram.com/Morgan-VoycePolynomials.html/, 2008.

[245] E. W. Weisstein, http://mathworld.wolfram.com/PellNumber.html/, 2008.

[246] E. W. Weisstein, http://mathworld.wolfram.com/SierpinksiSieve.html/, 2008.

[247] E. W. Weisstein, http://mathworld.wolfram.com/ZarankiewiczsConjecture.html/, 2009.

[248] D. B. West, *Introduction to Graph Theory*, Prentice Hall, New Jersey, 2001.

[249] R. C. White, L. C. Barrett, *Advanced Engineering Mathematics*, McGraw Hill, New York, 1982.

[250] H. S. Wilf, *Generatingfunctionology*, Academic Press, 1994.

[251] J. Wimp, Hankel determinants of some polynomials arising in combinatorial analysis, *Numerical Algorithms* **24** (2000) pp. 179–193.

[252] W-J. Woan, Hankel Matrices and Lattice Paths, *Journal of Integer Sequences*, Vol. **4** (2001), Article 01.1.2.

[253] H. F. Yan, L. L. M. Yang, Hilly poor noncrossing partitions and $(2,3)$-Motzkin paths, Preprint.

[254] S.-L. Yang, H. You, On a connection between the Pascal, Stirling and Vandermonde matrices, *Discrete Applied Mathematics* **155** (2007) pp. 2025–2030.

[255] Zhang Zhizheng, The Linear Algebra of the Generalized Pascal Matrix, *Linear Algebra and its Applications*, **250** (1997) pp. 51–60.

[256] Zhizheng Zhang, A characterization of Aigner-Catalan-like numbers, *Indian Journal of Pure and Applied Mathematics*, **32** (11) (2001) pp. 1615-1619.

[257] Zhizheng Zhang, M. Liu, An Extension of the Generalized Pascal Matrix and its Algebraic Properties, *Linear Algebra and its Applications*, **271** (1998) pp. 169–177.

[258] X. Zhao, T. Wang, The algebraic properties of the generalized Pascal functional matrices associated with the exponential families, *Linear Algebra and its Applications*, **318** (2000) pp. 45–52.