A note on a one-parameter family of Catalan-like numbers

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Abstract
We study a family of sequences of Catalan-like numbers based on the series reversion process. Properties of these sequences are derived, including continued fraction expansions, associated orthogonal polynomials and associated Aigner matrices, which turn out to be Riordan arrays.

1 Introduction
The purpose of this note is to explore some properties of the family of sequences obtained by reverting the expression

\[ \frac{x(1 + rx)}{1 + 2rx + r(r + 1)x^2} \]

where \( r \) is an integer parameter. The analysis involves elements of the theory of Riordan arrays [13], orthogonal polynomials [4, 9], continued fractions [16], the Deleham construction (see A084938) and Hankel transforms [12]. The overall context of this note is that of “Catalan-like” numbers, a notion defined and developed by Martin Aigner [1, 2]. See also [7, 8, 17]. In the sequel, \([x^n]\) denotes the operator that extracts the coefficient of \( x^n \) in a power series, \( \text{Rev} \) denotes the operation of reverting a sequence, \([P]\) is the Iverson operator \([10] \) equal to 0 if \( P \) is false, and 1 if \( P \) is true, and

\[ c(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \frac{1}{x} \frac{1}{1 - \frac{x}{1 - \frac{x}{1 - \ldots}}} \]

is the g.f. of the Catalan numbers A000108. In addition, \((g, f)\) will denote a Riordan array whose \( k \)-th column has generating function \( g(x)f(x)^k \). We recall that a number sequence \( a_n \) is “Catalan-like” if none of the Hankel determinants \(|a_{i+j}|_{i,j=0}^n \) is zero, while a lower-triangular matrix \((a_{n,k})\) is called an Aigner matrix if there exist two sequences \( s_n \) and \( t_n \) such that

\[ a_{0,0} = 1, \quad a_{0,k} = 0 \quad (k > 0) \]

1
and

\[ a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1} \quad (\geq 1). \]

## 2 The sequences \( a_n(r) \)

We let

\[
\begin{align*}
    a_n(r) &= [x^{n+1}] \text{Rev} \left( \frac{x(1+rx)}{1+2rx+r(r+1)x^2} \right) \\
    &= [x^n] \frac{\sqrt{1-4rx^2} + 2rx - 1}{2r(1-(r+1)x)} \\
    &= [x^n] \frac{c(rx^2)}{1-rxc(rx^2)} \\
    &= [x^n] (c(rx^2), rxc(rx^2)) \cdot \frac{1}{1-x} \\
    &= [x^n] (c(rx^2), xc(rx^2)) \cdot \frac{1}{1-rx} \\
    &= \sum_{k=0}^{n} \frac{k+1}{n+k+2} \left( \frac{n-k}{2} \right) (1 + (-1)^{n-k}) r^{n-k} r^k \\
    &= \sum_{k=0}^{n} \frac{k+1}{n+k+2} \left( \frac{n-k}{2} \right) (1 + (-1)^{n-k}) r^{n+k} \\
    &= \sum_{k=0}^{n} [n \leq 2k] \binom{n}{k} \frac{2k-n+1}{k+1} r^k.
\end{align*}
\]

A short table of these sequences is given below.

<table>
<thead>
<tr>
<th>( r )</th>
<th>A-number</th>
<th>Reversion of</th>
<th>( x^{n+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A001405</td>
<td>( x(1+x) )</td>
<td>( 1+2x+2x^2 )</td>
</tr>
<tr>
<td>2</td>
<td>A151281</td>
<td>( x(1+2x) )</td>
<td>( 1+4x+6x^2 )</td>
</tr>
<tr>
<td>3</td>
<td>A151162</td>
<td>( x(1+3x) )</td>
<td>( 1+6x+12x^2 )</td>
</tr>
<tr>
<td>4</td>
<td>A151254</td>
<td>( x(1+4x) )</td>
<td>( 1+8x+20x^2 )</td>
</tr>
<tr>
<td>5</td>
<td>A156195</td>
<td>( x(1+5x) )</td>
<td>( 1+10x+30x^2 )</td>
</tr>
<tr>
<td>6</td>
<td>A156361</td>
<td>( x(1+6x) )</td>
<td>( 1+12x+42x^2 )</td>
</tr>
</tbody>
</table>

For example, \( a_n(1) \) is the sequence of central binomial numbers \( \binom{n}{\lfloor \frac{n}{2} \rfloor} \), while \( a_n(3) \) counts the number of walks within \( \mathbb{N}^3 \) (the first octant of \( \mathbb{Z}^3 \)) starting at \((0,0,0)\) and consisting of \( n \) steps taken from \( \{(−1,0,0),(1,0,0),(0,1,0),(1,1,0)\} \) [3].
3 Continued fractions

We have

\[
\frac{c(rx^2)}{1 - rx c(rx^2)} = \frac{1}{\frac{1}{c(rx^2)} - rx} = \frac{1}{1 - rx + \frac{1 - c(rx^2)}{c(rx^2)}} = \frac{1}{1 - rx - rx^2 c(rx^2)} = \frac{1}{1 - rx - \frac{r x^2}{1 - \frac{r x^2}{1 - \ldots}}}
\]

An immediate consequence of this is that \(a_n(r)\) has Hankel transform \(r^{\binom{n+1}{2}}\) [11, 16]. This proves that these numbers are “Catalan-like” \((r \neq 0)\). We note that the above implies that the bi-variate generating function of the Riordan array

\((c(rx^2), xc(rx^2))\)

is given by

\[
\frac{1}{1 - xy - \frac{rx^2}{1 - \frac{r x^2}{1 - \ldots}}}
\]

while that of the generalized Riordan array

\((c(rx^2), rxc(rx^2))\)

is given by

\[
\frac{1}{1 - rxy - \frac{r x^2}{1 - \frac{r x^2}{1 - \ldots}}}
\]

There is another link to continued fractions, via the Deleham construction. For the purposes of this note, we define this as follows. Given two sequences \(r_n\) and \(s_n\), we use the notation

\[r \triangle s = [r_0, r_1, r_2, \ldots] \triangle [s_0, s_1, s_2, \ldots]\]
to denote the number triangle whose bi-variate generating function is given by

\[
\frac{1}{1 - (r_0 x + s_0 xy)} \cdot \frac{1}{1 - (r_1 x + s_1 xy)} \cdot \frac{1}{1 - (r_2 x + s_2 xy)} \cdots
\]

In this instance, we follow Deleham in A120730 by taking \( r_n \) to be the sequence that begins

\[
0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0, \ldots
\]

and \( s_n \) to be the sequence that begins

\[
1, 0, 0, -1, 1, 0, 0, -1, 1, 0, 0, -1, 1, 0, \ldots
\]

( extending periodically). We thus arrive at the number triangle with bi-variate generating function

\[
\frac{1}{1 - xy} \cdot \frac{1}{1 - x} \cdot \frac{1}{1 + x} \cdot \frac{1}{1 + xy} \cdot \frac{1}{1 - x} \cdot \frac{1}{1 + xy} \cdots
\]

This is the triangle with general term

\[
[n \leq 2k] \binom{n}{k} \frac{2k - n + 1}{k + 1}.
\]

A consequence of the fact that

\[
a_n(r) = \sum_{k=0}^{n} [n \leq 2k] \binom{n}{k} \frac{2k - n + 1}{k + 1} r^k
\]
is that the generating function of $a_n(r)$ may also be expressed as

$$\frac{1}{1 - \frac{rx}{x} + \frac{rx}{1 + \frac{x}{1 - \ldots}}}.\]

For example, the generating function of the central binomial numbers $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ can be expressed as

$$\frac{1}{1 - \frac{x}{x} + \frac{x}{1 + \frac{x}{1 - \ldots}}} = \frac{1}{1 - x - \frac{x^2}{1 - \frac{x^2}{1 - \ldots}}}.\]

4 \ LDL_t \ decomposition \ and \ orthogonal \ polynomials

We let $H = H(r)$ denote the Hankel matrix of the sequence $a_n(r)$, with general element $a_{i+j}(r)$. The theory of “Catalan-like” numbers ensures us that

$$H = LDL^t$$

where $D$ is a diagonal matrix, and $L = L(r)$ is a lower-triangular matrix with 1’s on the diagonal. Moreover, $L^{-1}$ is the coefficient array of a family of orthogonal polynomials. Given that the Hankel transform of $a_n(r)$ is $r^\binom{n+1}{2}$, it is clear from the theory that in fact

$$D = \text{Diag}\{1, r, r^2, r^3, \ldots\}.$$ 

We obtain

$$L = \left(\frac{c(rx^2)}{1 - rxc(rx^2), xc(rx^2)}\right),$$
and

\[ \mathbf{L}^{-1} = \left( \frac{1- rx}{1+rx^2}, \frac{x}{1+rx^2} \right). \]

This latter matrix has general term

\[ (-r)^{\left\lfloor \frac{n-k+1}{2} \right\rfloor} \left( n - \left\lfloor \frac{n-k+1}{2} \right\rfloor \right). \]

The matrix \( \mathbf{L}^{-1} \) is the coefficient array of a set of generalized Chebyshev polynomials of the third kind. To see this, we first let

\[ U_n(x; r) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k} (-r)^k (2x)^{n-2k}. \]

The family of polynomials

\[ U_n(x/2; r) \]

has coefficient array

\[ \left( \frac{1}{1+rx^2}, \frac{x}{1+rx^2} \right). \]

The generalized Chebyshev polynomials of the third kind can then be defined to be

\[ V_n(r,x) = U_n(x;r) - rU_{n-1}(x;r). \]

Then \( \mathbf{L}^{-1} \) is the coefficient array of the orthogonal polynomials \( V_n(x/2; r) \). We can easily verify that these polynomials satisfy the recurrence

\[ V_{n+2}(x/2; r) = xV_{n+1}(x/2; r) - rV_n(x/2; r). \]

**Example 1.** We take the case \( r=3 \). Then we get

\[ \mathbf{L}(3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 3 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 12 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 45 & 15 & 3 & 1 & 0 & 0 & \ldots \\ 180 & 54 & 18 & 3 & 1 & 0 & \ldots \\ 702 & 234 & 63 & 21 & 3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

and

\[ \mathbf{L}(3)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -3 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -3 & -3 & 1 & 0 & 0 & 0 & \ldots \\ 9 & -6 & -3 & 1 & 0 & 0 & \ldots \\ 9 & 18 & -9 & -3 & 1 & 0 & \ldots \\ -27 & 27 & 27 & -12 & -3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]
\( \mathbf{L}(3) \) is an Aigner matrix for the sequences 3, 0, 0, 0, \ldots and 3, 3, 3, \ldots. Thus for instance
\[
702 = 3 \cdot 180 + 3 \cdot 54, \quad 234 = 180 + 3 \cdot 18.
\]
The generalized Chebyshev polynomials of the third kind associated to \( a_n(3) \) begin
\[
1, x - 3, x^2 - 3x - 3, x^3 - 3x^2 - 6x - 9, \ldots
\]

5 \hspace{1em} \textbf{Moment representation}

By means of the Stieltjes transform, we can establish that
\[
a_n(r) = \frac{1}{2\pi} \int_{-2\sqrt{r}}^{2\sqrt{r}} x^n \sqrt{4r - x^2} dx + \frac{r - 1}{r} (r + 1)^n.
\]

6 \hspace{1em} \textbf{Production matrices}

It is instructive to examine the production matrices [5, 6] of some of the Riordan arrays involved in this note. The Riordan array
\[
(c(rx^2), xc(rx^2))
\]
has production matrix
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
r & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & r & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & r & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & r & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & r & 0 & \cdots \\
: & : & : & : & : & : & \ddots
\end{bmatrix}
\]
while the generalized Riordan array
\[
(c(rx^2), rxc(rx^2))
\]
has production matrix
\[
\begin{bmatrix}
0 & r & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & r & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & r & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & r & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & r & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
: & : & : & : & : & : & \ddots
\end{bmatrix}
\]
The Riordan array
\[
\mathbf{L} = \left( \frac{c(rx^2)}{1 - rxc(rx^2)}, xc(rx^2) \right)
\]
has production matrix

\[
\begin{pmatrix}
  r & 1 & 0 & 0 & 0 & 0 & \ldots \\
  r & 0 & 1 & 0 & 0 & 0 & \ldots \\
  0 & r & 0 & 1 & 0 & 0 & \ldots \\
  0 & 0 & r & 0 & 1 & 0 & \ldots \\
  0 & 0 & 0 & r & 0 & 1 & \ldots \\
  0 & 0 & 0 & 0 & r & 0 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

which confirms that its first column, which is \(a_n(r)\), has generating function given by the continued fraction

\[
\frac{1}{1 - rx - \frac{rx^2}{1 - \frac{rx^2}{1 - \ldots}}},
\]

\(L\) is thus the Aigner matrix for the sequences \(s_n\) given by \(r, 0, 0, \ldots\) and \(t_n\) given by \(r, r, r, \ldots\).

7 \(a_n(r)\) as row sums

Following [8], it is possible to exhibit the sequences \(a_n(r)\) as the row sums of a given number triangle. In effect, the matrix \(LB^{-1}\), where \(B\) is the Binomial matrix with general term \(\binom{n}{k}\) (Pascal’s triangle, \(\text{A007318}\)), has row sums equal to \(a_n(r)\). This matrix is the inverse of

\[
\left( \frac{1}{1 - x}, \frac{x}{1 - x} \right) \cdot \left( \frac{1 - rx}{1 + rx^2}, \frac{x}{1 + rx^2} \right) = \left( \frac{1 - (r + 1)x}{1 - 2x + (r + 1)x^2}, \frac{x(1 - x)}{1 - 2x + (r + 1)x^2} \right).
\]

We may verify this algebraically: the row sums of

\[
LB^{-1} = \left( \frac{c(rx^2)}{1 - rxc(rx^2)}, \frac{1}{1 + xc(rx^2)}, \frac{xc(rx^2)}{1 + xc(rx^2)} \right)
\]

do have generating function \(\frac{c(rx^2)}{1 - rxc(rx^2)}\).

References


---

2000 Mathematics Subject Classification: Primary 11B83; Secondary 11Y55, 42C05.
Keywords: Integer sequence, Catalan-like, Riordan arrays, Hankel transform, Chebyshev polynomials.