# A note on a one-parameter family of Catalan-like numbers

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

**Abstract**

We study a family of sequences of Catalan-like numbers based on the series reversion process. Properties of these sequences are derived, including continued fraction expansions, associated orthogonal polynomials and associated Aigner matrices, which turn out to be Riordan arrays.

## 1 Introduction

The purpose of this note is to explore some properties of the family of sequences obtained by reverting the expression

$$\frac{x(1+rx)}{1+2rx+r(r+1)x^2}$$

where $r$ is an integer parameter. The analysis involves elements of the theory of Riordan arrays [13], orthogonal polynomials [4, 9], continued fractions [16], the Deleham construction (see A084938) and Hankel transforms [12]. The overall context of this note is that of "Catalan-like" numbers, a notion defined and developed by Martin Aigner [1, 2]. See also [7, 8, 17]. In the sequel, $[x^n]$ denotes the operator that extracts the coefficient of $x^n$ in a power series, Rev denotes the operation of reverting a sequence, $[P]$ is the Iverson operator [10] equal to 0 if $P$ is false, and 1 if $P$ is true, and

$$c(x)=\frac{1-\sqrt{1-4x}}{2x}=\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}}$$

is the g.f. of the Catalan numbers A000108. In addition, $(g,f)$ will denote a Riordan array whose $k$-th column has generating function $g(x)f(x)^k$. We recall that a number sequence $a_n$ is "Catalan-like" if none of the Hankel determinants $|a_{i+j}|_{i,j=0}^n$ is zero, while a lower-triangular matrix $(a_{n,k})$ is called an Aigner matrix if there exist two sequences $s_n$ and $t_n$ such that

$$a_{0,0}=1, \quad a_{0,k}=0 \quad (k>0)$$

and

$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1} \quad (\geq 1).$$

## 2 The sequences $a_n(r)$

We let

$$\begin{aligned}
a_n(r) &= [x^{n+1}]\mathrm{Rev}\frac{x(1+rx)}{1+2rx+r(r+1)x^2} \\
&= [x^n]\frac{\sqrt{1-4rx^2}+2rx-1}{2r(1-(r+1)x)} \\
&= [x^n]\frac{c(rx^2)}{1-rxc(rx^2)} \\
&= [x^n](c(rx^2), rxc(rx^2)) \cdot \frac{1}{1-x} \\
&= [x^n](c(rx^2), xc(rx^2)) \cdot \frac{1}{1-rx} \\
&= \sum_{k=0}^{n} \frac{k+1}{n+k+2}\binom{n}{\frac{n-k}{2}}(1+(-1)^{n-k})r^{\frac{n-k}{2}}r^k \\
&= \sum_{k=0}^{n} \frac{k+1}{n+k+2}\binom{n}{\frac{n-k}{2}}(1+(-1)^{n-k})r^{\frac{n+k}{2}} \\
&= \sum_{k=0}^{n} [n \leq 2k]\binom{n}{k}\frac{2k-n+1}{k+1}r^k.
\end{aligned}$$

A short table of these sequences is given below.

| $r$ | A-number | Reversion of |
|---|---|---|
| 1 | A001405 | $\frac{x(1+x)}{1+2x+2x^2}$ |
| 2 | A151281 | $\frac{x(1+2x)}{1+4x+6x^2}$ |
| 3 | A151162 | $\frac{x(1+3x)}{1+6x+12x^2}$ |
| 4 | A151254 | $\frac{x(1+4x)}{1+8x+20x^2}$ |
| 5 | A156195 | $\frac{x(1+5x)}{1+10x+30x^2}$ |
| 6 | A156361 | $\frac{x(1+6x)}{1+12x+42x^2}$ |

For example, $a_n(1)$ is the sequence of central binomial numbers $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, while $a_n(3)$ counts the number of walks within $\mathbf{N}^3$ (the first octant of $\mathbf{Z}^3$) starting at $(0,0,0)$ and consisting of $n$ steps taken from $\{(-1,0,0),(1,0,0),(1,0,1),(1,1,0)\}$ [3].

## 3 Continued fractions

We have

$$\begin{aligned}
\frac{c(rx^2)}{1-rxc(rx^2)} &= \frac{1}{\frac{1}{c(rx^2)}-rx} \\
&= \frac{1}{1-rx+(\frac{1}{c(rx^2)}-1)} \\
&= \frac{1}{1-rx+\frac{1-c(rx^2)}{c(rx^2)}} \\
&= \frac{1}{1-rx-rx^2c(rx^2)} \\
&= \cfrac{1}{1-rx-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cdots}}}}.
\end{aligned}$$

An immediate consequence of this is that $a_n(r)$ has Hankel transform $r^{\binom{n+1}{2}}$ [11, 16]. This proves that these numbers are "Catalan-like" ($r \neq 0$). We note that the above implies that the bi-variate generating function of the Riordan array

$$(c(rx^2), xc(rx^2))$$

is given by

$$\cfrac{1}{1-xy-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cdots}}}}$$

while that of the generalized Riordan array

$$(c(rx^2), rxc(rx^2))$$

is given by

$$\cfrac{1}{1-rxy-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cdots}}}}.$$

There is another link to continued fractions, via the *Deleham construction*. For the purposes of this note, we define this as follows. Given two sequences $r_n$ and $s_n$,we use the notation

$$r \quad \Delta \quad s = [r_0, r_1, r_2, \ldots] \quad \Delta \quad [s_0, s_1, s_2, \ldots]$$

to denote the number triangle whose bi-variate generating function is given by

$$\cfrac{1}{1-\cfrac{(r_0x+s_0xy)}{1-\cfrac{(r_1x+s_1xy)}{1-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}}.$$

In this instance, we follow Deleham in A120730 by taking $r_n$ to be the sequence that begins

$$0,1,-1,0,0,1,-1,0,0,1,-1,0,\ldots$$

and $s_n$ to be the sequence that begins

$$1,0,0,-1,1,0,0,-1,1,0,0,-1,1,0,\ldots$$

(extending periodically). We thus arrive at the number triangle with bi-variate generating function

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{xy}{1-\cfrac{xy}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{xy}{1-\cdots}}}}}}}}}.$$

This is the triangle with general term

$$[n \le 2k]\binom{n}{k}\frac{2k-n+1}{k+1}.$$

A consequence of the fact that

$$a_n(r)=\sum_{k=0}^{n}[n \le 2k]\binom{n}{k}\frac{2k-n+1}{k+1}r^k$$

is that the generating function of $a_n(r)$ may also be expressed as

$$\cfrac{1}{1-\cfrac{rx}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{rx}{1-\cfrac{rx}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{rx}{1-\cdots}}}}}}}}}.$$

For example, the generating function of the central binomial numbers $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ can be expressed as

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1-\cdots}}}}}}}}} = \cfrac{1}{1-x-\cfrac{x^2}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-\cdots}}}}.$$

# 4 $LDL^t$ decomposition and orthogonal polynomials

We let $\mathbf{H} = \mathbf{H}(r)$ denote the Hankel matrix of the sequence $a_n(r)$, with general element $a_{i+j}(r)$. The theory of "Catalan-like" numbers ensures us that

$$\mathbf{H} = \mathbf{LDL}^t$$

where $\mathbf{D}$ is a diagonal matrix, and $\mathbf{L} = \mathbf{L}(r)$ is a lower-triangular matrix with 1's on the diagonal. Moreover, $\mathbf{L}^{-1}$ is the coefficient array of a family of orthogonal polynomials. Given that the Hankel transform of $a_n(r)$ is $r^{\binom{n+1}{2}}$, it is clear from the theory that in fact

$$\mathbf{D} = \text{Diag}\{1, r, r^2, r^3, \ldots\}.$$

We obtain

$$\mathbf{L} = \left( \frac{c(rx^2)}{1 - rxc(rx^2)}, xc(rx^2) \right),$$

and

$$\mathbf{L}^{-1} = \left( \frac{1-rx}{1+rx^2}, \frac{x}{1+rx^2} \right).$$

This latter matrix has general term

$$(-r)^{\lfloor \frac{n-k+1}{2} \rfloor} \binom{n - \lfloor \frac{n-k+1}{2} \rfloor}{\lfloor \frac{n-k}{2} \rfloor}.$$

The matrix $\mathbf{L}^{-1}$ is the coefficient array of a set of generalized Chebyshev polynomials of the third kind. To see this, we first let

$$U_n(x;r) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-r)^k (2x)^{n-2k}.$$

The family of polynomials

$$U_n(x/2;r)$$

has coefficient array

$$\left( \frac{1}{1+rx^2}, \frac{x}{1+rx^2} \right).$$

The *generalized Chebyshev polynomials of the third kind* can then be defined to be

$$V_n(r,x) = U_n(x;r) - rU_{n-1}(x;r).$$

Then $\mathbf{L}^{-1}$ is the coefficient array of the orthogonal polynomials $V_n(x/2;r)$. We can easily verify that these polynomials satisfy the recurrence

$$V_{n+2}(x/2;r) = xV_{n+1}(x/2;r) - rV_n(x/2;r).$$

**Example 1.** We take the case $r$=3. Then we get

$$\mathbf{L}(3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 12 & 3 & 1 & 0 & 0 & 0 & \dots \\ 45 & 15 & 3 & 1 & 0 & 0 & \dots \\ 180 & 54 & 18 & 3 & 1 & 0 & \dots \\ 702 & 234 & 63 & 21 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\mathbf{L}(3)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -3 & 1 & 0 & 0 & 0 & 0 & \dots \\ -3 & -3 & 1 & 0 & 0 & 0 & \dots \\ 9 & -6 & -3 & 1 & 0 & 0 & \dots \\ 9 & 18 & -9 & -3 & 1 & 0 & \dots \\ -27 & 27 & 27 & -12 & -3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\mathbf{L}(3)$ is an Aigner matrix for the sequences $3, 0, 0, 0, \ldots$ and $3, 3, 3, \ldots$. Thus for instance

$$702 = 3 \cdot 180 + 3 \cdot 54, \qquad 234 = 180 + 3 \cdot 18.$$

The generalized Chebyshev polynomials of the third kind associated to $a_n(3)$ begin

$$1, x-3, x^2-3x-3, x^3-3x^2-6x-9, \ldots$$

# 5 Moment representation

By means of the Stieltjes transform, we can establish that

$$a_n(r) = \frac{1}{2\pi} \int_{-2\sqrt{r}}^{2\sqrt{r}} x^n \frac{\sqrt{4r-x^2}}{r(r+1-x)} dx + \frac{r-1}{r}(r+1)^n.$$

# 6 Production matrices

It is instructive to examine the production matrices [5, 6] of some of the Riordan arrays involved in this note. The Riordan array

$$(c(rx^2), xc(rx^2))$$

has production matrix

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ r & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & r & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & r & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & r & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & r & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

while the generalized Riordan array

$$(c(rx^2), rxc(rx^2))$$

has production matrix

$$\left(\begin{array}{ccccccc} 0 & r & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & r & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & r & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & r & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & r & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

The Riordan array

$$\mathbf{L} = \left( \frac{c(rx^2)}{1 - rxc(rx^2)}, xc(rx^2) \right)$$

has production matrix

$$\begin{pmatrix} r & 1 & 0 & 0 & 0 & 0 & \dots \\ r & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & r & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & r & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & r & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & r & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which confirms that its first column, which is $a_n(r)$, has generating function given by the continued fraction

$$\cfrac{1}{1-rx-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cdots}}}.$$

$\mathbf{L}$ is thus the Aigner matrix for the sequences $s_n$ given by $r, 0, 0, 0, \dots$ and $t_n$ given by $r, r, r, \dots$.

## 7 $a_n(r)$ as row sums

Following [8], it is possible to exhibit the sequences $a_n(r)$ as the row sums of a given number triangle. In effect, the matrix $\mathbf{LB}^{-1}$, where $\mathbf{B}$ is the Binomial matrix with general term $\binom{n}{k}$ (Pascal's triangle, A007318), has row sums equal to $a_n(r)$. This matrix is the inverse of

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right) \cdot \left(\frac{1-rx}{1+rx^2}, \frac{x}{1+rx^2}\right) = \left(\frac{1-(r+1)x}{1-2x+(r+1)x^2}, \frac{x(1-x)}{1-2x+(r+1)x^2}\right).$$

We may verify this algebraically: the row sums of

$$\mathbf{LB}^{-1} = \left(\frac{c(rx^2)}{1-rxc(rx^2)} \frac{1}{1+xc(rx^2)}, \frac{xc(rx^2)}{1+xc(rx^2)}\right)$$

do have generating function $\frac{c(rx^2)}{1-rxc(rx^2)}$.

# References


[1] M. Aigner, Catalan-like numbers and determinants, *J. Combinatorial Theory*, Series A **87** (1999) pp. 33–51.

[2] M. Aigner, Catalan and other numbers: a recurrent theme, in: Grapo, Senato, eds., *Algebraic Combinatorics and Computer Science*, Springer Italia, Milan, 2001, pp. 347–390.

[3] A. Bostan, M. Kauers, Automatic Classification of Restricted Lattice Walks, arXiv:0811.2899v1, 2008.

[4] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[5] E. Deutsch, L. Ferrari, and S. Rinaldi, Production Matrices, *Advances in Applied Mathematics* **34** (2005) pp. 101–122.

[6] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, http://arxiv.org/abs/math/0702638v1, February 22 2007.

[7] L. Ferrari, E. Pergola, R. Pinzani & S. Rinaldi, Some applications arising from the interactions between the theory of Catalan-like numbers and the ECO method, available electronically as www.dsi.unifi.it/ ferrari/applecoadm.pdf, 2009.

[8] L. Ferrari, R. Pinzani, Catalan-like numbers and succession rules, *Pure Mathematics and Applications*,**16** (2005) pp. 229–250.

[9] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press, Oxford, 2003.

[10] R.L. Graham, D.E. Knuth & O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2e, Addison-Wesley, 1994.

[11] C. Krattenthaler, Advanced Determinant Calculus, available electronically at http://arxiv.org/PS_cache/math/pdf/9902/9902004.pdf, 2006

[12] J. W. Layman, The Hankel Transform and Some of Its Properties, *Journal of Integer Sequences*, **4**, Article 01.1.5, 2001.

[13] L. W. Shapiro, S. Getu, W-J. Woan and L.C. Woodson, The Riordan Group, *Discr. Appl. Math.* **34** (1991) pp. 229–239.

[14] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://www.research.att.com/∼njas/sequences/, 2008.

[15] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994) pp. 267–290.

[16] H. S. Wall, *Analytic Theory of Continued Fractions*, AMS Chelsea Publishing, 2000.

[17] Zhizheng Zhang, A characterization of Aigner-Catalan-like numbers, *Indian Journal of Pure and Applied Mathematics*, **32** (11) (2001) pp. 1615-1619.

---

2000 *Mathematics Subject Classification*: Primary 11B83; Secondary 11Y55, 42C05.
*Keywords:* Integer sequence, Catalan-like, Riordan arrays, Hankel transform, Chebyshev polynomials.

---

Concerns sequences A000108,A001405, A007318, A084938, A120730, A151281, A151162, A151254, A156195, A156361.