The restricted Toda chain, exponential Riordan arrays, and Hankel transforms

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Abstract

We re-interpret results on the classification of Toda chain solutions given by Sheffer class orthogonal polynomials in terms of exponential Riordan arrays. We also examine associated Hankel transforms.

1 Introduction

The restricted Toda chain equation [18, 28] is simply described by

$$\dot{u}_n = u_n(b_n - b_{n-1}), \quad n = 1, 2, \dots \quad \dot{b}_n = u_{n+1} - u_n, \quad n = 0, 1, \dots \tag{1}$$

with $u_0 = 0$, where the dot indicates differentiation with respect to t. In this note, we shall show how solutions to this equation can be formulated in the context of exponential Riordan arrays. The Riordan arrays we shall consider may be considered as parameterised (or "time"-dependent) Riordan arrays. We have already considered such arrays in [2], wherein links between Riordan arrays and orthogonal polynomials are considered.

The restricted Toda chain equations are closely related to orthogonal polynomials, since the functions u_n and b_n can be considered as the coefficients in the usual three-term recurrence satisfied by orthogonal polynomials:

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \quad n = 1, 2, \dots \tag{2}$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x - b_0$.

2 Hermite polynomials and the Toda chain

We recall that the Hermite polynomials may be defined as

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}.$$

The generating function for $H_n(x)$ is given by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

The following result was proved in [2].

Proposition 1. *The proper exponential Riordan array*

$$\mathbf{L} = \left[e^{2rx-x^2}, x\right]$$

has as first column the Hermite polynomials $H_n(r)$. This array has a tri-diagonal production array.

Proof. The first column of $\mathbf{L}$ has generating function $e^{2rx-\frac{x^2}{2}}$, from which the first assertion follows. Standard Riordan array techniques show us that the production array of $\mathbf{L}$ is indeed tri-diagonal, beginning

$$\begin{pmatrix} 2r & 1 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 2r & 1 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 2r & 1 & 0 & 0 & \cdots \\ 0 & 0 & -6 & 2r & 1 & 0 & \cdots \\ 0 & 0 & 0 & -8 & 2r & 1 & \cdots \\ 0 & 0 & 0 & 0 & -10 & 2r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

□

We note that $\mathbf{L}$ starts

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2r & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2(2r^2-1) & 4r & 1 & 0 & 0 & 0 & \cdots \\ 4r(2r^2-3) & 6(2r^2-1) & 6r & 1 & 0 & 0 & \cdots \\ 4(4r^3-12r^2+3) & 16r(2r^2-3) & 12(2r^2-1) & 8r & 1 & 0 & \cdots \\ 8r(4r^4-20r^2+15) & 20(4r^4-12r^2+3) & 40r(2r^2-3) & 20(2r^2-1) & 10r & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus

$$\mathbf{L}^{-1} = \left[e^{-2rx+x^2}, x\right]$$

is the coefficient array of a set of orthogonal polynomials which have as moments the Hermite polynomials. These new orthogonal polynomials satisfy the three-term recurrence

$$\mathfrak{H}_{n+1}(x) = (x-2r)\mathfrak{H}_n(x) - 2n\mathfrak{H}_{n-1}(x),$$

with $\mathfrak{H}_0 = 1$, $\mathfrak{H}_1 = x - 2r$.

Proposition 2. *The exponential Riordan array*

$$\left[e^{-2(z-t)x+x^2}, x\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ *with*

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x),$$

where (u_n, b_n) *are a solution to the restricted Toda chain.*

Proof. We easily determine that the inverse matrix

$$\left[e^{2(z-t)x-x^2}, x\right]$$

has production matrix

$$\begin{pmatrix} 2(z-t) & 1 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 2(z-t) & 1 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 2(z-t) & 1 & 0 & 0 & \cdots \\ 0 & 0 & -6 & 2(z-t) & 1 & 0 & \cdots \\ 0 & 0 & 0 & -8 & 2(z-t) & 1 & \cdots \\ 0 & 0 & 0 & 0 & -10 & 2(z-t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This verifies that $P_n(x)$ is indeed a family of orthogonal polynomials, for which

$$u_n(t) = -2n, \quad b_n(t) = 2(z-t).$$

It is immediate that these satisfy Eq. (1).
We now note that the moments of this polynomial family (first column of the inverse matrix) m_n satisfy the following relation:

$$m_n = \frac{[x^n]}{n!} e^{2(z-t)x-x^2} = \frac{1}{e^{-t^2+2tz}} \frac{d^n}{dt^n} e^{-t^2+2tz}. \tag{3}$$

□

3 Charlier polynomials and the Toda chain

Proposition 3. *The exponential Riordan array*

$$\left[e^{xe^t}, \ln(1+x)\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ *with*

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x),$$

where (u_n, b_n) *are a solution to the restricted Toda chain.*

Proof. We determine that the inverse matrix

$$\left[e^{e^{t+x}-e^t}, e^x - 1\right]$$

has production matrix

$$\left(\begin{array}{ccccccc} e^t & 1 & 0 & 0 & 0 & 0 & \cdots \\ e^t & e^t+1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2e^t & e^t+2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3e^t & e^t+3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4e^t & e^t+4 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5e^t & e^t+5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This verifies that $P_n(x)$ is indeed a family of orthogonal polynomials, for which

$$u_n(t) = ne^t, \quad b_n(t) = n + e^t.$$

It is easy now to verify that with these values, (u_n, b_n) satisfy the Toda chain equations Eq. (1). □

The moments m_n of this family of orthogonal polynomials may be expressed as:

$$m_n = \frac{[x^n]}{n!} e^{e^{t+x}-e^t} = \frac{1}{e^{e^t-1}} \frac{d^n}{dt^n} e^{e^t-1}. \tag{4}$$

4 Laguerre polynomials and the Toda chain

Proposition 4. *The exponential Riordan array*

$$\left[\left(1 - \frac{x}{1+t}\right)^\alpha, \frac{x}{1 - \frac{x}{1+t}}\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ with

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x),$$

where (u_n, b_n) are a solution to the restricted Toda chain.

Proof. The inverse matrix

$$\left[\left(\frac{1+t+x}{1+t}\right)^\alpha, \frac{(1+t)x}{1+t+x}\right]$$

has production matrix

$$\left(\begin{array}{ccccccc} \frac{\alpha}{1+t} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{-\alpha}{(1+t)^2} & \frac{\alpha-2}{1+t} & 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{2(1-\alpha)}{(1+t)^2} & \frac{\alpha-4}{1+t} & 1 & 0 & 0 & \cdots \\ 0 & 0 & \frac{3(2-\alpha)}{(1+t)^2} & \frac{\alpha-6}{1+t} & 1 & 0 & \cdots \\ 0 & 0 & 0 & \frac{4(3-\alpha)}{(1+t)^2} & \frac{\alpha-8}{1+t} & 1 & \cdots \\ 0 & 0 & 0 & 0 & \frac{4(3-\alpha)}{(1+t)^2} & \frac{\alpha-10}{1+t} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This verifies that $P_n(x)$ is indeed a family of orthogonal polynomials, for which

$$u_n(t) = \frac{n(n-\alpha-1)}{1+t}, \quad b_n(t) = \frac{\alpha-2n}{1+t}.$$

It is easy now to verify that with these values, (u_n, b_n) satisfy the Toda chain equations Eq. (1). ◻

For this family of orthogonal polynomials, the moments m_n may be expressed as:

$$m_n = \frac{[x^n]}{n!}\left(1+\frac{x}{1+t}\right)^{\alpha} = \frac{1}{(1+t)^{\alpha}}\frac{d^n}{dt^n}(1+t)^{\alpha} = \frac{(\alpha)_n}{(1+t)^n}. \tag{5}$$

5 Meixner polynomials and the Toda chain

Proposition 5. *The exponential Riordan array*

$$\left[\frac{1}{\sqrt{1-2x\tanh(t)-x^2\text{sech}(t)^2}}, \ln\left(\sqrt{\frac{1+xe^{-t}\text{sech}(t)}{1-xe^{t}\text{sech}(t)}}\right)\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ with

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x),$$

where (u_n, b_n) are a solution to the restricted Toda chain.

Proof. The inverse matrix

$$\left[\frac{\text{sech}(x+t)}{\text{sech}(t)}, \sinh(x)\frac{\text{sech}(x+t)}{\text{sech}(t)}\right]$$

has production matrix

$$\begin{pmatrix} -\tanh(t) & 1 & 0 & 0 & 0 & 0 & \cdots \\ -\text{sech}^2(t) & -3\tanh(t) & 1 & 0 & 0 & 0 & \cdots \\ 0 & -4\,\text{sech}^2(t) & -5\tanh(t) & 1 & 0 & 0 & \cdots \\ 0 & 0 & -9\,\text{sech}^2(t) & -7\tanh(t) & 1 & 0 & \cdots \\ 0 & 0 & 0 & -16\,\text{sech}^2(t) & -9\tanh(t) & 1 & \cdots \\ 0 & 0 & 0 & 0 & -25\,\text{sech}^2(t) & -11\tanh(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This verifies that $P_n(x)$ is indeed a family of orthogonal polynomials, for which

$$u_n(t) = -n^2\text{sech}^2(t), \quad b_n(t) = -(2n+1)\tanh(t).$$

It is easy now to verify that with these values, (u_n, b_n) satisfy the Toda chain equations Eq. (1). ◻

We may describe the moments m_n of this family of polynomials by

$$m_n = \frac{[x^n]}{n!} \frac{\operatorname{sech}(x+t)}{\operatorname{sech}(t)} = \frac{1}{\operatorname{sech}(t)} \frac{d^n}{dt^n} \operatorname{sech}(t). \tag{6}$$

The Hankel transform of m_n is then given by

$$h_n = (-1)^{\binom{n+1}{2}} \operatorname{sech}(t)^{n(n+1)} \prod_{k=0}^{n} (k!)^2.$$

References

[1] P. Barry, P. Rajkovic & M. Petkovic, An application of Sobolev orthogonal polynomials to the computation of a special Hankel Determinant, in W. Gautschi, G. Rassias, M. Themistocles (Eds), *Approximation and Computation*, Springer, 2010.

[2] P. Barry, Riordan arrays, orthogonal polynomials as moments, and Hankel transforms, preprint, Waterford Institute of Technology 2010.

[3] P. Barry, On a family of generalized Pascal triangles defined by exponential Riordan arrays, *J. Integer Sequences*, **10** (2007), Article 07.3.5.

[4] G-S. Cheon, H. Kim & L. W. Shapiro, Riordan group involution, *Linear Algebra and its Applications*, **428** (2008) pp. 941–952.

[5] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[6] C. Corsani, D. Merlini, R. Sprugnoli, Left-inversion of combinatorial sums, *Discrete Math.* **180** (1998), pp. 107–122.

[7] A. Cvetković, P. Rajković and M. Ivković, Catalan Numbers, the Hankel Transform and Fibonacci Numbers, *Journal of Integer Sequences*, **5**, (2002), Article 02.1.3.

[8] E. Deutsch, L. Shapiro, Exponential Riordan Arrays, Lecture Notes, Nankai University, 2004, available electronically at `http://www.combinatorics.net/ppt2004/Louis%20W.%20Shapiro/shapiro.htm`

[9] E. Deutsch, L. Ferrari, and S. Rinaldi, Production Matrices, *Advances in Applied Mathematics* **34** (2005) pp. 101–122.

[10] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, `http://arxiv.org/abs/math/0702638v1`, February 22 2007.

[11] M. Elouafi, A. D. A. Hadj, On the powers and the inverse of a tridiagonal matrix, *Applied Math. and Computation*, **211** (2009) pp. 137–141.

[12] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press - Oxford, 2003.

[13] Tian-Xiao He, R. Sprugnoli, Sequence characterization of Riordan arrays, *Discrete Mathematics* **2009** (2009), pp. 3962–3974.

[14] M.E.H. Ismail, D. Stanton, Classical orthogonal polynomials as moments, *Can. J. Math.* **49** (1997) pp. 520–542.

[15] M.E.H. Ismail, D. Stanton, More orthogonal polynomials as moments, pp. 377–396, in R. Stanley, B. Sagan (Eds), *Festschrift in Honor of Gian-Carlo Rota (Progress in Mathematics)* , Birkhauser, (1998).

[16] S-T. Jin, A characterization of the Riordan Bell subgroup by C-sequences, *Korean J. Math.* **17** (2009), pp. 147–154.

[17] J. W. Layman, The Hankel Transform and Some of Its Properties, *Journal of Integer Sequences*, **4**, (2001) Article 01.1.5.

[18] Y. Nakamura, A. Zhedanov, Toda chain, Sheffer class of orthogonal polynomials and combinatorial numbers, *Proceedings of Institute of Mathematics of NAS of Ukraine*, **50** (2004), pp. 450–457.

[19] P. Peart, L. Woodson, Triple Factorisation of some Riordan Matrices, *Fibonacci Quarterly*, 31 (1993) pp. 121–128.

[20] P. Peart, W-J. Woan, Generating functions via Hankel and Stieltjes matrices, *Journal of Integer Sequences*, **3** (2000) Article 00.2.1.

[21] S. Roman, *The Umbral Calculus*, Dover Publications, 2005.

[22] L. W. Shapiro, S. Getu, W-J. Woan and L.C. Woodson, The Riordan Group, *Discr. Appl. Math.* **34** (1991) pp. 229–239.

[23] L. W. Shapiro, Bijections and the Riordan group, *Theoretical Comp. Sci.* **307** (2003) pp. 403–413.

[24] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at `http://www.research.att.com/∼njas/sequences/`, 2009.

[25] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, *Notices of the AMS*, **50** (2003), pp. 912–915.

[26] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994) pp. 267–290.

[27] G. Szegö, *Orthogonal Polynomials*, 4th ed. Providence, RI, Amer. Math. Soc., (1975) pp. 35–37.

[28] L. Vinet, A. Zhedanov, Elliptic solutions of the restricted Toda chain, Lamé polynomials and generalization of the elliptic Stieltjes polynomials, *J. Phys. A: Math. Theor.*, **42** (2009) 454024 (16pp).

[29] W-J Woan, Hankel matrices and lattice paths, *Journal of Integer Sequences*, **4** (2001), Article 01.1.2

2010 *Mathematics Subject Classification*: Primary 42C05; Secondary 11B83, 11C20, 15B05, 15B36,33C45.
Keywords: Legendre polynomials, Hermite polynomials, integer sequence, orthogonal polynomials, moments, Riordan array, Hankel determinant, Hankel transform.

Concerns sequences A000007, A000045, A000108, A000262, A001405, A007318, A009766, A021009, A033184, A053121, A094587, A094816, A111596, A111884, A119467, A119879.