Riordan arrays, orthogonal polynomials as moments, and Hankel transforms

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Abstract

Taking the examples of Legendre and Hermite orthogonal polynomials, we show how to interpret the fact that these orthogonal polynomials are moments of other orthogonal polynomials in terms of their associated Riordan arrays. We use these means to calculate the Hankel transforms of the associated polynomial sequences.

1 Introduction

In this note, we shall re-interpret some of the results of [13] (see also [14]) in terms of Riordan arrays. In [13], the authors give functionals whose moments are the Hermite, Laguerre, and various Meixner families of polynomials. In this note, we shall confine ourselves to Legendre and Hermite polynomials. Indeed, the types of orthogonal polynomials representable with Riordan arrays is very limited (see below), but it is nevertheless instructive to show that a number of them can be exhibited as moments, again using (parameterized) Riordan arrays.

While partly expository in nature, the note assumes a certain familiarity with integer sequences, generating functions, orthogonal polynomials [4, 11, 25], Riordan arrays [20, 24], production matrices [9, 18], and the integer Hankel transform [1, 6, 16]. We provide background material in this note to provide a hopefully coherent narrative. Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [22, 23]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix **B** ("Pascal's triangle") is A007318.

The plan of the paper is as follows:

1. This Introduction
2. Preliminaries on integer sequences and Riordan arrays
3. Orthogonal polynomials and Riordan arrays
4. Exponential Riordan arrays and orthogonal polynomials
5. The Hankel transform of an integer sequence

6. Legendre polynomials

7. Legendre polynomials as moments

8. Hermite polynomials

9. Hermite polynomials as moments

2 Preliminaries on integer sequences and Riordan arrays

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \ldots$, A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$. We sometimes write $\bar{f} = \text{Rev} f$.

For a lower triangular matrix $(a_{n,k})_{n,k\geq 0}$ the row sums give the sequence with general term $\sum_{k=0}^{n} a_{n,k}$ while the diagonal sums form the sequence with general term

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-k,k}.$$

The *Riordan group* [20], [24], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \ldots$ and $f(x) = f_1 x + f_2 x^2 + \ldots$ where $f_1 \neq 0$ [24]. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is a_{n-k}. Such arrays are also called *Appell* arrays as they form the elements of the Appell subgroup.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences

$\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g,f):\mathcal{A}(x)\mapsto (g,f)\cdot\mathcal{A}(x)=g(x)\mathcal{A}(f(x)).$$

Example 1. The so-called *binomial matrix* $\mathbf{B}$ is the element $(\frac{1}{1-x},\frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx},\frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx},\frac{x}{1+mx})$.

Example 2. If a_n has generating function $g(x)$, then the generating function of the sequence

$$b_n=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} a_{n-2k}$$

is equal to

$$\frac{g(x)}{1-x^2}=\left(\frac{1}{1-x^2},x\right)\cdot g(x),$$

while the generating function of the sequence

$$d_n=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-k}{k}a_{n-2k}$$

is equal to

$$\frac{1}{1-x^2}g\left(\frac{x}{1-x^2}\right)=\left(\frac{1}{1-x^2},\frac{x}{1-x^2}\right)\cdot g(x).$$

The row sums of the matrix (g,f) have generating function

$$(g,f)\cdot\frac{1}{1-x}=\frac{g(x)}{1-f(x)}$$

while the diagonal sums of (g,f) (sums of left-to-right diagonals in the North East direction) have generating function $g(x)/(1-xf(x))$. These coincide with the row sums of the "generalized" Riordan array (g,xf):

$$(g,xf)\cdot\frac{1}{1-x}=\frac{g(x)}{1-xf(x)}.$$

For instance the Fibonacci numbers F_{n+1} are the diagonal sums of the binomial matrix $\mathbf{B}$ given by $\left(\frac{1}{1-x},\frac{x}{1-x}\right)$:

$$\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \dots\\
1 & 1 & 0 & 0 & 0 & 0 & \dots\\
1 & 2 & 1 & 0 & 0 & 0 & \dots\\
1 & 3 & 3 & 1 & 0 & 0 & \dots\\
1 & 4 & 6 & 4 & 1 & 0 & \dots\\
1 & 5 & 10 & 10 & 5 & 1 & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

while they are the row sums of the "generalized" or "stretched" (using the nomenclature of [5]) Riordan array $\left(\frac{1}{1-x}, \frac{x^2}{1-x}\right)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 4 & 3 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Each Riordan array $(g(x), f(x))$ has bi-variate generating function given by

$$\frac{g(x)}{1-yf(x)}.$$

For instance, the binomial matrix $\mathbf{B}$ has generating function

$$\frac{\frac{1}{1-x}}{1-y\frac{x}{1-x}} = \frac{1}{1-x(1+y)}.$$

For a sequence $a_0, a_1, a_2, \dots$ with g.f. $g(x)$, the "aeration" of the sequence is the sequence $a_0, 0, a_1, 0, a_2, \dots$ with interpolated zeros. Its g.f. is $g(x^2)$.

The aeration of a (lower-triangular) matrix $\mathbf{M}$ with general term $m_{i,j}$ is the matrix whose general term is given by

$$m^r_{\frac{i+j}{2}, \frac{i-j}{2}} \frac{1+(-1)^{i-j}}{2},$$

where $m^r_{i,j}$ is the i,j-th element of the reversal of $\mathbf{M}$:

$$m^r_{i,j} = m_{i,i-j}.$$

In the case of a Riordan array (or indeed any lower triangular array), the row sums of the aeration are equal to the diagonal sums of the reversal of the original matrix.

Example 3. The Riordan array $(c(x^2), xc(x^2))$ is the aeration of $(c(x), xc(x))$ A033184. Here

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the g.f. of the Catalan numbers. Indeed, the reversal of $(c(x), xc(x))$ is the matrix with general element

$$[k \leq n+1] \binom{n+k}{k} \frac{n-k+1}{n+1},$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 0 & 0 & 0 & \dots \\ 1 & 3 & 5 & 5 & 0 & 0 & \dots \\ 1 & 4 & 9 & 14 & 14 & 0 & \dots \\ 1 & 5 & 14 & 28 & 42 & 42 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A009766. Then $(c(x^2), xc(x^2))$ has general element

$$\binom{n+1}{\frac{n-k}{2}}\frac{k+1}{n+1}\frac{(1+(-1)^{n-k}}{2},$$

and begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & 4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A053121. Note that

$$(c(x^2), xc(x^2)) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}.$$

We note that the diagonal sums of the reverse of $(c(x), xc(x))$ coincide with the row sums of $(c(x^2), xc(x^2))$, and are equal to the central binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ A001405.

An important feature of Riordan arrays is that they have a number of sequence characterizations [3, 12]. The simplest of these is as follows.

Proposition 4. *[12] Let $D = [d_{n,k}]$ be an infinite triangular matrix. The D is a Riordan array if and only if there exit two sequences $A = [a_0, a_1, a_2, \dots]$ and $Z = [z_0, z_1, z_2, \dots]$ with $a_0 \neq 0$, $z_0 \neq 0$ such that*

- $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}, \quad (k, n = 0, 1, \dots)$
- $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, \dots).$

The coefficients $a_0, a_1, a_2, \dots$ and $z_0, z_1, z_2, \dots$ are called the A-sequence and the Z-sequence of the Riordan array $D = (g(x), f(x))$, respectively.

3 Orthogonal polynomials and Riordan arrays

By an *orthogonal polynomial sequence* $(p_n(x))_{n\ge 0}$ we shall understand [4, 11] an infinite sequence of polynomials $p_n(x)$, $n \ge 0$, of degree n, with real coefficients (often integer coefficients) that are mutually orthogonal on an interval $[x_0, x_1]$ (where $x_0 = -\infty$ is allowed, as well as $x_1 = \infty$), with respect to a weight function $w : [x_0, x_1] \to \mathbb{R}$:

$$\int_{x_0}^{x_1} p_n(x)p_m(x)w(x)dx = \delta_{nm}\sqrt{h_n h_m},$$

where

$$\int_{x_0}^{x_1} p_n^2(x)w(x)dx = h_n.$$

We assume that w is strictly positive on the interval (x_0, x_1). Every such sequence obeys a so-called "three-term recurrence" :

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

for coefficients a_n, b_n and c_n that depend on n but not x. We note that if

$$p_j(x) = k_j x^j + k'_j x^{j-1} + \dots \qquad j = 0, 1, \dots$$

then

$$a_n = \frac{k_{n+1}}{k_n}, \qquad b_n = a_n\left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n}\right), \qquad c_n = a_n\left(\frac{k_{n-1}h_n}{k_n h_{n-1}}\right),$$

where

$$h_i = \int_{x_0}^{x_1} p_i(x)^2 w(x)\,dx.$$

Since the degree of $p_n(x)$ is n, the coefficient array of the polynomials is a lower triangular (infinite) matrix. In the case of monic orthogonal polynomials the diagonal elements of this array will all be 1. In this case, we can write the three-term recurrence as

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0.$$

The *moments* associated to the orthogonal polynomial sequence are the numbers

$$\mu_n = \int_{x_0}^{x_1} x^n w(x)dx.$$

We can find $p_n(x)$, α_n and β_n from a knowledge of these moments. To do this, we let Δ_n be the Hankel determinant $|\mu_{i+j}|_{i,j\ge 0}^n$ and $\Delta_{n,x}$ be the same determinant, but with the last row equal to $1, x, x^2, \dots$. Then

$$p_n(x) = \frac{\Delta_{n,x}}{\Delta_{n-1}}.$$

More generally, we let $H\begin{pmatrix} u_1 & \dots & u_k \\ v_1 & \dots & v_k \end{pmatrix}$ be the determinant of Hankel type with (i,j)-th term $\mu_{u_i+v_j}$. Let

$$\Delta_n = H\begin{pmatrix} 0 & 1 & \dots & n \\ 0 & 1 & \dots & n \end{pmatrix}, \qquad \Delta' = H\begin{pmatrix} 0 & 1 & \dots & n-1 & n \\ 0 & 1 & \dots & n-1 & n+1 \end{pmatrix}.$$

Then we have

$$\alpha_n = \frac{\Delta'_n}{\Delta_n} - \frac{\Delta'_{n-1}}{\Delta_{n-1}}, \qquad \beta_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}.$$

Given a family of monic orthogonal polynomials

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0,$$

we can write

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k.$$

Then we have

$$\sum_{k=0}^{n+1} a_{n+1,k} x^k = (x - \alpha_n) \sum_{k=0}^{n} a_{n,k} x^k - \beta_n \sum_{k=0}^{n-1} a_{n-1,k} x^k$$

from which we deduce

$$a_{n+1,0} = -\alpha_n a_{n,0} - \beta_n a_{n-1,0} \tag{1}$$

and

$$a_{n+1,k} = a_{n,k-1} - \alpha_n a_{n,k} - \beta_n a_{n-1,k} \tag{2}$$

The question immediately arises as to the conditions under which a Riordan array (g, f) can be the coefficient array of a family of orthogonal polynomials. A partial answer is given by the following proposition.

Proposition 5. *Every Riordan array of the form*

$$\left(\frac{1}{1 + rx + sx^2}, \frac{x}{1 + rx + sx^2}\right)$$

is the coefficient array of a family of monic orthogonal polynomials.

Proof. By [15], the array $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ has a C-sequence $C(x) = \sum_{n \geq 0} c_n x^n$ given by

$$\frac{x}{1 + rx + sx^2} = \frac{x}{1 - xC(x)},$$

and thus

$$C(x) = -r - sx.$$

Thus the Riordan array $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ is determined by the fact that

$$a_{n+1,k} = a_{n,k-1} + \sum_{i \geq 0} c_i d_{n-i,k} \quad \text{for } n, k = 0, 1, 2, \ldots$$

where $a_{n,-1} = 0$. In the case of $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ we have

$$a_{n+1,k} = a_{n,k-1} - r a_{n,k} - s a_{n-1,k}.$$

Working backwards, this now ensures that

$$p_{n+1}(x) = (x - r)p_n(x) - s p_{n-1}(x),$$

where $p_n(x) = \sum_{k=0}^{n} a_{n,k} x^n$. $\square$

We note that in this case the three-term recurrence coefficients α_n and β_n are constants. We have in fact the following proposition (see the next section for information on the Chebyshev polynomials).

Proposition 6. *The Riordan array* $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ *is the coefficient array of the modified Chebyshev polynomials of the second kind given by*

$$P_n(x) = (\sqrt{s})^n U_n\left(\frac{x-r}{2\sqrt{s}}\right), \quad n = 0, 1, 2, \ldots$$

Proof. The production array of $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ is given by

$$\begin{pmatrix} r & 1 & 0 & 0 & 0 & 0 & \ldots \\ s & r & 1 & 0 & 0 & 0 & \ldots \\ 0 & s & r & 1 & 0 & 0 & \ldots \\ 0 & 0 & s & r & 1 & 0 & \ldots \\ 0 & 0 & 0 & r & s & 1 & \ldots \\ 0 & 0 & 0 & 0 & r & s & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The result now follows by [10], for instance. □

The complete answer can be found by considering the associated production matrix of a Riordan arrray, in the following sense.

The concept of a *production matrix* [8, 9] is a general one, but for this work we find it convenient to review it in the context of Riordan arrays. Thus let P be an infinite matrix (most often it will have integer entries). Letting $\mathbf{r}_0$ be the row vector

$$\mathbf{r}_0 = (1, 0, 0, 0, \ldots),$$

we define $\mathbf{r}_i = \mathbf{r}_{i-1}P$, $i \geq 1$. Stacking these rows leads to another infinite matrix which we denote by A_P. Then P is said to be the *production matrix* for A_P.
If we let

$$u^T = (1, 0, 0, 0, \ldots, 0, \ldots)$$

then we have

$$A_P = \begin{pmatrix} u^T \\ u^T P \\ u^T P^2 \\ \vdots \end{pmatrix}$$

and

$$DA_P = A_P P$$

where $D = (\delta_{i,j+1})_{i,j\geq 0}$ (where δ is the usual Kronecker symbol).

In [18, 21] P is called the Stieltjes matrix associated to A_P.

The sequence formed by the row sums of A_P often has combinatorial significance and is called the sequence associated to P. Its general term a_n is given by $a_n = u^T P^n e$ where

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

In the context of Riordan arrays, the production matrix associated to a proper Riordan array takes on a special form :

Proposition 7. *[9] Let P be an infinite production matrix and let A_P be the matrix induced by P. Then A_P is an (ordinary) Riordan matrix if and only if P is of the form*

$$P = \begin{pmatrix} \xi_0 & \alpha_0 & 0 & 0 & 0 & 0 & \dots \\ \xi_1 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \dots \\ \xi_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \dots \\ \xi_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots \\ \xi_4 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \dots \\ \xi_5 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Moreover, columns 0 and 1 of the matrix P are the Z- and A-sequences, respectively, of the Riordan array A_P.

Example 8. We consider the Riordan array $\mathbf{L}$ where

$$L^{-1} = \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right).$$

The production matrix (Stieltjes matrix) of

$$L = \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}$$

is given by

$$P = S_L = \begin{pmatrix} a+\lambda & 1 & 0 & 0 & 0 & 0 & \dots \\ b+\mu & a & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that since

$$\begin{aligned} L^{-1} &= \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) \\ &= (1-\lambda x-\mu x^2, x)\cdot\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right), \end{aligned}$$

we have

$$L=\left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2},\frac{x}{1+ax+bx^2}\right)^{-1}=\left(\frac{1}{1+ax+bx^2},\frac{x}{1+ax+bx^2}\right)^{-1}\cdot\left(\frac{1}{1-\lambda x-\mu x^2},x\right).$$

If we now let

$$L_1=\left(\frac{1}{1+ax},\frac{x}{1+ax}\right)\cdot L,$$

then (see [17]) we obtain that the Stieltjes matrix for L_1 is given by

$$S_{L_1}=\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & \ldots \\ b+\mu & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have in fact the following general result [17]:

Proposition 9. *If $L=(g(x),f(x))$ is a Riordan array and $P=S_L$ is tridiagonal, then necessarily*

$$P=S_L=\begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ b_1 & a & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$f(x)=Rev\frac{x}{1+ax+bx^2}\qquad and \qquad g(x)=\frac{1}{1-a_1x-b_1xf},$$

and vice-versa.

We have the important corollary

Corollary 10. *If $L=(g(x),f(x))$ is a Riordan array and $P=S_L$ is tridiagonal, with*

$$P=S_L=\begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ b_1 & a & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3}$$

then L^{-1} is the coefficient array of the family of orthogonal polynomials $p_n(x)$ where $p_0(x) = 1$, $p_1(x) = x - a_1$, and

$$p_{n+1}(x) = (x-a)p_n(x) - b_n p_{n-1}(x), \qquad n \geq 2,$$

where b_n is the sequence $0, b_1, b, b, b, \ldots$.

We note that the elements of the rows of L^{-1} can be identified with the coefficients of the characteristic polynomials of the successive principal sub-matrices of P.

Example 11. We consider the Riordan array

$$\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right).$$

Then the production matrix (Stieltjes matrix) of the inverse Riordan array $\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}$ left-multiplied by the k-th binomial array

$$\left(\frac{1}{1-kx}, \frac{x}{1-kx}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)^k$$

is given by

$$P = \begin{pmatrix} a+k & 1 & 0 & 0 & 0 & 0 & \ldots \\ b & a+k & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a+k & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a+k & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a+k & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a+k & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and vice-versa. This follows since

$$\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) \cdot \left(\frac{1}{1+kx}, \frac{x}{1+kx}\right) = \left(\frac{1}{1+(a+k)x+bx^2}, \frac{x}{1+(a+k)x+bx^2}\right).$$

In fact we have the more general result:

$$\left(\frac{1+\lambda x+\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) \cdot \left(\frac{1}{1+kx}, \frac{x}{1+kx}\right) = \left(\frac{1+\lambda x+\mu x^2}{1+(a+k)x+bx^2}, \frac{x}{1+(a+k)x+bx^2}\right).$$

The inverse of this last matrix therefore has production array

$$\begin{pmatrix} a+k-\lambda & 1 & 0 & 0 & 0 & 0 & \ldots \\ b-\mu & a+k & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a+k & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a+k & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a+k & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a+k & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Finally, we note that if $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal of the form given in Eq. (3), then the first column of L gives the moment sequence for the weight function associated to the orthogonal polynomials whose coefficient array is L^{-1}.

4 Exponential Riordan arrays

The *exponential Riordan group* [2], [9], [7], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1 x + g_2 x^2 + \dots$ and $f(x) = f_1 x + f_2 x^2 + \dots$ where $f_1 \neq 0$. The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. It is *monic* if $g_0 = 1$. The group law is then given by

$$[g, f] * [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f. We use the notation $e\mathcal{R}$ to denote this group.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n \geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}(x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g, f]$ are given by $g(x)e^{f(x)}$ since the sequence $1, 1, 1, \dots$ has exponential generating function e^x.

As an element of the group of exponential Riordan arrays, we have $\mathbf{B} = [e^x, x]$. By the above, the exponential generating function of its row sums is given by $e^x e^x = e^{2x}$, as expected (e^{2x} is the e.g.f. of 2^n).

Example 12. We consider the exponential Riordan array $[\frac{1}{1-x}, x]$, A094587. This array has elements

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

and general term $[k \leq n]\frac{n!}{k!}$ with inverse

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which is the array $[1-x, x]$. In particular, we note that the row sums of the inverse, which begin $1, 0, -1, -2, -3, \ldots$ (that is, $1-n$), have e.g.f. $(1-x)\exp(x)$. This sequence is thus the binomial transform of the sequence with e.g.f. $(1-x)$ (which is the sequence starting $1, -1, 0, 0, 0, \ldots$).

Example 13. We consider the exponential Riordan array $[1, \frac{x}{1-x}]$. The general term of this matrix may be calculated as follows:

$$\begin{aligned}
T_{n,k} &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\
&= \frac{n!}{k!}[x^{n-k}](1-x)^{-k} \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j \\
&= \frac{n!}{k!}\binom{k+n-k-1}{n-k} \\
&= \frac{n!}{k!}\binom{n-1}{n-k}.
\end{aligned}$$

Thus its row sums, which have e.g.f. $\exp\left(\frac{x}{1-x}\right)$, have general term $\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{n-k}$. This is A000262, the 'number of "sets of lists": the number of partitions of $\{1, .., n\}$ into any number of lists, where a list means an ordered subset'. Its general term is equal to $(n-1)!L_{n-1}(1,-1)$.

The production matrix of the inverse of this matrix is given by

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 2 & 1 & 0 & 0 & 0 & \ldots \\
0 & 2 & 4 & 1 & 0 & 0 & \ldots \\
0 & 0 & 6 & 6 & 1 & 0 & \ldots \\
0 & 0 & 0 & 12 & 8 & 1 & \ldots \\
0 & 0 & 0 & 0 & 20 & 10 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

The inverse of $\left[1, \frac{x}{1-x}\right]$ is the exponential Riordan array $\left[1, \frac{x}{1+x}\right]$, A111596. The row sums of this sequence have e.g.f. $\exp\left(\frac{x}{1+x}\right)$, and start $1, 1, -1, 1, 1, -19, 151, \ldots$. This is A111884.

Example 14. The exponential Riordan array $\mathbf{A} = \left[\frac{1}{1-x}, \frac{x}{1-x}\right]$, or

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
2 & 4 & 1 & 0 & 0 & 0 & \ldots \\
6 & 18 & 9 & 1 & 0 & 0 & \ldots \\
24 & 96 & 72 & 16 & 1 & 0 & \ldots \\
120 & 600 & 600 & 200 & 25 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

has general term

$$T_{n,k} = \frac{n!}{k!}\binom{n}{k}.$$

It is closely related to the Laguerre polynomials. Its inverse is $\left[\frac{1}{1+x}, \frac{x}{1+x}\right]$ with general term $(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$. This is A021009, the triangle of coefficients of the Laguerre polynomials $L_n(x)$.

The production matrix of the inverse of this matrix is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 0 & 4 & 5 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9 & 7 & 1 & 0 & \dots \\ 0 & 0 & 0 & 16 & 9 & 1 & \dots \\ 0 & 0 & 0 & 0 & 25 & 11 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that

$$\mathbf{A} = \exp(\mathbf{S}),$$

where

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 9 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 16 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 25 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 15. The exponential Riordan array $\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]$, or

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 8 & 6 & 1 & 0 & 0 & \dots \\ 1 & 24 & 29 & 10 & 1 & 0 & \dots \\ 1 & 89 & 145 & 75 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the coefficient array for the polynomials

$${}_2F_0(-n, x; -1)$$

which are an unsigned version of the Charlier polynomials (of order 0) [11, 19, 25]. This is A094816. It is equal to

$$[e^x, x]\left[1, \ln\left(\frac{1}{1-x}\right)\right],$$

or the product of the binomial array $\mathbf{B}$ and the array of (unsigned) Stirling numbers of the first kind. The production matrix of the inverse of this matrix is given by

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & -3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3 & -4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4 & -5 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5 & -6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which indicates the orthogonal nature of these polynomials. We can prove this as follows. In [9], we find the following result concerning matrices that are production matrices for exponential Riordan arrays.

Proposition 16. *Let $A = (a_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \dots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \dots \tag{4}$$

be two formal power series that that

$$r(f(x)) = f'(x) \tag{5}$$

$$c(f(x)) = \frac{g'(x)}{g(x)}. \tag{6}$$

Then

$$(i) \qquad a_{n+1,0} = \sum_i i! c_i a_{n,i} \tag{7}$$

$$(ii) \qquad a_{n+1,k} = r_0 a_{n,k-1} + \frac{1}{k!} \sum_{i\geq k} i!(c_{i-k} + k r_{i-k+1}) a_{n,i} \tag{8}$$

or, defining $c_{-1} = 0$,

$$a_{n+1,k} = \frac{1}{k!} \sum_{i\geq k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}. \tag{9}$$

Conversely, starting from the sequences defined by (4), the infinite array $(a_{n,k})_{n,k\geq 0}$ defined by (9) is an exponential Riordan array.

A consequence of this proposition is that $P = (p_{i,j})_{i,j\geq 0}$ where

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{r-j+1}) \qquad (c_{-1} = 0).$$

Furthermore, the bivariate exponential function

$$\phi_P(t,z) = \sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}$$

of the matrix P is given by

$$\phi_P(t,z) = e^{tz}(c(z) + tr(z)).$$

Now for our example, we have

$$\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]^{-1} = \left[e^{-(1-e^{-x})}, 1-e^{-x}\right].$$

Hence $g(x) = e^{-(1-e^{-x})}$ and $f(x) = 1 - e^{-x}$. We are thus led to the equations

$$\begin{aligned} r(1-e^{-x}) &= e^{-x}, \\ c(1-e^{-x}) &= -e^{-x}, \end{aligned}$$

with solutions $r(x) = 1 - x$, $c(x) = x - 1$. Thus the bi-variate generating function for the production matrix of the inverse array is

$$e^{tz}(z - 1 + t(1-z)),$$

which is what is required.

When we come to study Hermite polynomials, we shall be working with elements of the exponential Appell subgroup. By the *exponential Appell subgroup* of $e\mathcal{R}$ we understand the set of arrays of the form $[f(x), x]$.

Let $\mathbf{A} \in e\mathcal{R}$ correspond to the sequence $(a_n)_{n\geq 0}$, with e.g.f. $f(x)$. Let $\mathbf{B} \in e\mathcal{R}$ correspond to the sequence (b_n), with e.g.f. $g(x)$. Then we have

1. The row sums of $\mathbf{A}$ are the partial sums of (a_n).
2. The inverse of $\mathbf{A}$ is the sequence array for the sequence with e.g.f. $\frac{1}{f(x)}$.
3. The product $\mathbf{AB}$ is the sequence array for the exponential convolution $a * b(n) = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ with e.g.f. $f(x)g(x)$.

Example 17. We consider the matrix $[\cosh(x), x]$, A119467, with elements

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & 6 & 0 & 1 & 0 & \ldots \\ 0 & 5 & 0 & 10 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of this matrix have e.g.f. $\cosh(x)\exp(x)$, which is the e.g.f. of the sequence $1, 1, 2, 4, 8, 16, \ldots$. The inverse matrix is $[\mathrm{sech}(x), x]$, A119879, with entries

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & -3 & 0 & 1 & 0 & 0 & \ldots \\ 5 & 0 & -6 & 0 & 1 & 0 & \ldots \\ 0 & 25 & 0 & -10 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The row sums of this matrix have e.g.f. $\operatorname{sech}(x)\exp(x)$.

5 The Hankel transform of an integer sequence

The *Hankel transform* of a given sequence $A = \{a_0, a_1, a_2, ...\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ where $h_n = |a_{i+j}|_{i,j=0}^{n}$, i.e

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \rightarrow \quad h = \{h_n\}_{n\in\mathbb{N}_0} : \quad h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix}. \tag{10}$$

The Hankel transform of a sequence a_n and its binomial transform are equal.

In the case that a_n has g.f. $g(x)$ expressible in the form

$$g(x) = \cfrac{a_0}{1-\alpha_0 x - \cfrac{\beta_1 x^2}{1-\alpha_1 x - \cfrac{\beta_2 x^2}{1-\alpha_2 x - \cfrac{\beta_3 x^2}{1-\alpha_3 x - \cdots}}}}$$

then we have

$$h_n = a_0^n \beta_1^{n-1}\beta_2^{n-2}\cdots\beta_{n-1}^2\beta_n = a_0^n \prod_{k=1}^{n} \beta_k^{n-k}. \tag{11}$$

Note that this independent from α_n.

We note that α_n and β_n are in general not integers. Now let $H\begin{pmatrix} u_1 & \dots & u_k \\ v_1 & \dots & v_k \end{pmatrix}$ be the determinant of Hankel type with (i,j)-th term $\mu_{u_i+v_j}$. Let

$$\Delta_n = H\begin{pmatrix} 0 & 1 & \dots & n \\ 0 & 1 & \dots & n \end{pmatrix}, \qquad \Delta' = H\begin{pmatrix} 0 & 1 & \dots & n-1 & n \\ 0 & 1 & \dots & n-1 & n+1 \end{pmatrix}.$$

Then we have

$$\alpha_n = \frac{\Delta'_n}{\Delta_n} - \frac{\Delta'_{n-1}}{\Delta_{n-1}}, \qquad \beta_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}. \tag{12}$$

6 Legendre polynomials

We recall that the Legendre polynomials $P_n(x)$ can be defined by

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \left(\frac{1+x}{2}\right)^{n-k} \left(\frac{1-x}{2}\right)^k.$$

Their generating function is given by

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty} P_n(x)t^n.$$

We note that the production matrix of the inverse of the coefficient array of these polynomials is given by

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & \ldots \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 & \ldots \\ 0 & 0 & \frac{3}{7} & 0 & \frac{4}{7} & 0 & \ldots \\ 0 & 0 & 0 & \frac{4}{9} & 0 & \frac{5}{9} & \ldots \\ 0 & 0 & 0 & 0 & \frac{5}{11} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

which corresponds to the fact that the $P_n(x)$ satisfy the following three-term recurrence

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x).$$

The shifted Legendre polynomials $\tilde{P}_n(x)$ are defined by

$$\tilde{P}_n(x)=P_n(2x-1).$$

They satisfy

$$\tilde{P}_n(x)=(-1)^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-x)^k=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{2k}\binom{2k}{k}x^k.$$

Their coefficient array begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -6 & 6 & 0 & 0 & 0 & \ldots \\ -1 & 12 & -30 & 20 & 0 & 0 & \ldots \\ 1 & -20 & 90 & -140 & 70 & 0 & \ldots \\ -1 & 30 & -210 & 560 & -630 & 252 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

and so the first few terms begin

$$1, 2x-1, 6x^2-6x+1, 20x^3-30x^2+12x-1, \ldots$$

We clearly have

$$\frac{1}{\sqrt{1-2(2x-1)t+t^2}}=\sum_{n=0}^{\infty}\tilde{P}_n(x)t^n.$$

7 Legendre polynomials as moments

Our goal in this section is to represent the Legendre polynomials as the first column of a Riordan array whose production matrix is tri-diagonal. We first of all consider the so-called shifted Legendre polynomials. We have

Proposition 18. *The inverse* $\mathbf{L}$ *of the Riordan array*

$$\left(\frac{(1+r(1-r)x^2}{1+(2r-1)x+r(r-1)x^2}, \frac{x}{1+(2r-1)x+r(r-1)x^2}\right)$$

has as its first column the shifted Legendre polynomials $\tilde{P}_n(r)$*. The production matrix of* $\mathbf{L}$ *is tri-diagonal.*

Proof. Indeed, standard Riordan array techniques show that we have

$$\begin{aligned}\mathbf{L} &= \left(\frac{1+r(1-r)x^2}{1+(2r-1)x+r(r-1)x^2}, \frac{x}{1+(2r-1)x+r(r-1)x^2}\right)^{-1} \\ &= \left(\frac{1}{\sqrt{1-2(2r-1)x+x^2}}, \frac{1-(2r-1)x-\sqrt{1-2(2r-1)x+x^2}}{2r(r-1)x}\right).\end{aligned}$$

This establishes the first part. Again, standard Riordan array techniques now show that the production matrix of $\mathbf{L}$ is given by

$$\begin{pmatrix} 2r-1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2r(r-1) & 2r-1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & r(r-1) & 2r-1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & r(r-1) & 2r-1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & r(r-1) & 2r-1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & r(r-1) & 2r-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

□

Thus the shifted Legendre polynomials are moments of the family of orthogonal polynomials whose coefficient array is given by

$$\mathbf{L}^{-1} = \left(\frac{1+r(1-r)x^2}{1+(2r-1)x+r(r-1)x^2}, \frac{x}{1+(2r-1)x+r(r-1)x^2}\right).$$

In this case, the Riordan array is a proper Riordan array.

Proposition 19. *The Hankel transform of the sequence* $\tilde{P}_n(r)$ *is given by* $2^n(r(r-1))^{\binom{n+1}{2}}$.

Proof. From the above, the g.f. of $\tilde{P}_n(r)$ is given by

$$\cfrac{1}{1-(2r-1)x-\cfrac{2r(r-1)x^2}{1-(2r-1)x-\cfrac{r(r-1)x^2}{1-(2r-1)x-\cfrac{r(r-1)x^2}{1-\cdots}}}}.$$

The result now follows from this. □

We note that

$$\tilde{P}_n(r)=\frac{1}{\pi}\int_{-2\sqrt{r(r-1)}+2r-1}^{2\sqrt{r(r-1)}+2r-1}\frac{x^n}{\sqrt{-x^2+2(2r-1)x-1}}\,dx$$

gives an explicit moment representation for $\tilde{P}_n(r)$.

Turning now to the Legendre polynomials $P_n(x)$, we have the following result.

Proposition 20. *The inverse* $\mathbf{L}$ *of the Riordan array*

$$\left(\frac{1+\frac{1-r^2}{4}x^2}{1+rx+\frac{r^2-1}{2}x^2},\frac{x}{1+rx+\frac{r^2-1}{2}x^2}\right)$$

has as its first column the Legendre polynomials $P_n(r)$*. The production matrix of* $\mathbf{L}$ *is tri-diagonal.*

Proof. We have

$$\begin{aligned}\mathbf{L} &= \left(\frac{1+\frac{1-r^2}{4}x^2}{1+rx+\frac{r^2-1}{2}x^2},\frac{x}{1+rx+\frac{r^2-1}{2}x^2}\right)^{-1}\\ &= \left(\frac{1}{\sqrt{1-2rx+x^2}},\frac{2(1-rx-\sqrt{1-2rx+x^2})}{x(r^2-1)}\right).\end{aligned}$$

This proves the first assertion. Using standard techniques of Riordan arrays, we obtain the following tri-diagonal matrix as the production matrix of $\mathbf{L}$:

$$\begin{pmatrix} r & 1 & 0 & 0 & 0 & 0 & \ldots\\ \frac{r^2-1}{2} & r & 1 & 0 & 0 & 0 & \ldots\\ 0 & \frac{r^2-1}{4} & r & 1 & 0 & 0 & \ldots\\ 0 & 0 & \frac{r^2-1}{4} & r & 1 & 0 & \ldots\\ 0 & 0 & 0 & \frac{r^2-1}{4} & r & 1 & \ldots\\ 0 & 0 & 0 & 0 & \frac{r^2-1}{4} & r & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

□

Thus

$$\mathbf{L}^{-1} = \left(\frac{1 + \frac{1-r^2}{4}x^2}{1 + rx + \frac{r^2-1}{2}x^2}, \frac{x}{1 + rx + \frac{r^2-1}{2}x^2} \right)$$

is the coefficient array of a set of orthogonal polynomials for which the Legendre polynomials are moments.

Proposition 21. *The Hankel transform of $P_n(r)$ is given by*

$$\frac{(r^2-1)^{\binom{n+1}{2}}}{2^{n^2}}.$$

Proof. From the above, we obtain that the g.f. of $P_n(r)$ can be expressed as

$$\cfrac{1}{1 - rx - \cfrac{\frac{r^2-1}{2}x^2}{1 - rx - \cfrac{\frac{r^2-1}{4}x^2}{1 - rx - \cfrac{\frac{r^2-1}{4}x^2}{1 - \cdots}}}}.$$

The result now follows. □

We end this section by noting that

$$P_n(r) = \frac{1}{\pi} \int_{r-\sqrt{r^2-1}}^{r+\sqrt{r^2-1}} \frac{x^n}{\sqrt{-x^2 + 2rx - 1}}\, dx$$

gives an explicit moment representation for $P_n(r)$.

8 Hermite polynomials

The Hermite polynomials may be defined as

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}.$$

The generating function for $H_n(x)$ is given by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

The unitary Hermite polynomials (also called normalized Hermite polynomials) are given by

$$He_n(x) = 2^{-\frac{n}{2}} H_n(\sqrt{2}x) = \sum_{k=0}^{n} \frac{n!}{(-2)^{\frac{n-k}{2}} k! \left(\frac{n-k}{2}\right)!} \frac{1 + (-1)^{n-k}}{2} x^k.$$

Their generating function is given by

$$e^{xt-\frac{t^2}{2}} = \sum_{n=0}^{\infty} He_n(x)\frac{t^n}{n!}.$$

We note that the coefficient array of He_n is a proper exponential Riordan array, equal to

$$\left[e^{-\frac{x^2}{2}}, x\right].$$

This array A066325 begins

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & -3 & 0 & 1 & 0 & 0 & \dots \\ 3 & 0 & -6 & 0 & 1 & 0 & \dots \\ 0 & 15 & 0 & -10 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

It is the aeration of the alternating sign version of the Bessel coefficient array A001497. The inverse of this matrix has production matrix

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 4 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 5 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

which corresponds to the fact that we have the following three-term recurrence for He_n:

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x).$$

9 Hermite polynomials as moments

Proposition 22. *The proper exponential Riordan array*

$$\mathbf{L} = \left[e^{rx-x^2}, x\right]$$

has as first column the unitary Hermite polynomials $He_n(r)$. This array has a tri-diagonal production array.

Proof. The first column of $\mathbf{L}$ has generating function e^{rx-x^2}, from which the first assertion follows. Standard Riordan array techniques show us that the production array of $\mathbf{L}$ is indeed tri-diagonal, beginning

$$\left(\begin{array}{ccccccc} r & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & r & 1 & 0 & 0 & 0 & \dots \\ 0 & -2 & r & 1 & 0 & 0 & \dots \\ 0 & 0 & -3 & r & 1 & 0 & \dots \\ 0 & 0 & 0 & -4 & r & 1 & \dots \\ 0 & 0 & 0 & 0 & -5 & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

□

We note that $\mathbf{L}$ starts

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ r & 1 & 0 & 0 & 0 & 0 & \dots \\ r^2-1 & 2r & 1 & 0 & 0 & 0 & \dots \\ r(r^2-3) & 3(r^2-1) & 3r & 1 & 0 & 0 & \dots \\ r^4-6r^2+3 & 4r(r^2-3) & 6(r^2-1) & 4r & 1 & 0 & \dots \\ r(r^4-10r^2+15) & 5(r^4-6r^2+3) & 10r(r^2-3) & 10(r^2-1) & 5r & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Thus

$$\mathbf{L}^{-1} = \left[e^{-rx+x^2}, x\right]$$

is the coefficient array of a set of orthogonal polynomials which have as moments the unitary Hermite polynomials. These new orthogonal polynomials satisfy the three-term recurrence

$$\mathfrak{H}_{n+1}(x) = (x-r)\mathfrak{H}_n(x) + n\mathfrak{H}_{n-1}(x),$$

with $\mathfrak{H}_0 = 1$, $\mathfrak{H}_1 = x + r$.

Proposition 23. *The Hankel transform of the sequence $He_n(r)$ is given by $(-1)^{\binom{n+1}{2}} \prod_{k=0}^n k!$*

Proof. By the above, the g.f. of $H_n(r)$ is given by

$$\cfrac{1}{1-r+\cfrac{2x^2}{1-r+\cfrac{3x^2}{1-r+\cfrac{4x^2}{1-\dots}}}}.$$

The result follows from this. □

Turning now to the Hermite polynomials $H_n(x)$, we have the following result.

Proposition 24. *The proper exponential Riordan array*

$$\mathbf{L} = \left[e^{2rx-x^2}, x\right]$$

has as first column the Hermite polynomials $H_n(r)$. This array has a tri-diagonal production array.

Proof. The first column of $\mathbf{L}$ has generating function $e^{2rx-\frac{x^2}{2}}$, from which the first assertion follows. Standard Riordan array techniques show us that the production array of $\mathbf{L}$ is indeed tri-diagonal, beginning

$$\left(\begin{array}{ccccccc} 2r & 1 & 0 & 0 & 0 & 0 & \dots \\ -2 & 2r & 1 & 0 & 0 & 0 & \dots \\ 0 & -4 & 2r & 1 & 0 & 0 & \dots \\ 0 & 0 & -6 & 2r & 1 & 0 & \dots \\ 0 & 0 & 0 & -8 & 2r & 1 & \dots \\ 0 & 0 & 0 & 0 & -10 & 2r & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

□

We note that $\mathbf{L}$ starts

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2r & 1 & 0 & 0 & 0 & 0 & \dots \\ 2(2r^2-1) & 4r & 1 & 0 & 0 & 0 & \dots \\ 4r(2r^2-3) & 6(2r^2-1) & 6r & 1 & 0 & 0 & \dots \\ 4(4r^3-12r^2+3) & 16r(2r^2-3) & 12(2r^2-1) & 8r & 1 & 0 & \dots \\ 8r(4r^4-20r^2+15) & 20(4r^4-12r^2+3) & 40r(2r^2-3) & 20(2r^2-1) & 10r & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Thus

$$\mathbf{L}^{-1} = \left[e^{-2rx+x^2}, x\right]$$

is the coefficient array of a set of orthogonal polynomials which have as moments the Hermite polynomials. These new orthogonal polynomials satisfy the three-term recurrence

$$\mathfrak{H}_{n+1}(x) = (x-2r)\mathfrak{H}_n(x) - 2n\mathfrak{H}_{n-1}(x),$$

with $\mathfrak{H}_0 = 1$, $\mathfrak{H}_1 = x - 2r$.

Proposition 25. *The Hankel transform of the sequence $H_n(r)$ is given by $(-1)^{\binom{n+1}{2}} \prod_{k=0}^n 2^k k!$*

Proof. By the above, the g.f. of $H_n(r)$ is given by

$$\cfrac{1}{1-2r+\cfrac{2x^2}{1-4r+\cfrac{4x^2}{1-6r+\cfrac{6x^2}{1-\dots}}}}.$$

The result follows from this. □

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