Simple closed form Hankel transforms based on the central coefficients of certain Pascal-like triangles

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Abstract

We study the Hankel transforms of sequences related to the central coefficients of a family of Pascal-like triangles. The mechanism of Riordan arrays is used to elucidate the structure of these transforms.

1 Introduction

This note concerns the characterization of the Hankel transforms of the central coefficients T(2n, n, r) of a family of Pascal-like triangles that are parameterised by an integer r. Specifically, we define a family of number triangles with general term T(n, k, r) by

$$T(n,k,r) = \sum_{k=0}^{n-k} \binom{k}{j} \binom{n-k}{j} r^j.$$

For instance, r = 1 gives Pascal's triangle <u>A007318</u>, while r = 2 gives the triangle of Delannoy numbers [1], <u>A008288</u>.

Proposition 1. The Hankel transform of the sequence a(n,r) = T(2n,n,r) is given by

 $2^n r^{\binom{n+1}{2}}.$

Proof. We proceed as in [9] and [7] by means of the LDL^T decomposition of the Hankel matrix H(r) of T(2n, n, r). We take the example of r = 2. In this case,

$$H(2) = \begin{pmatrix} 1 & 3 & 13 & 63 & \dots \\ 3 & 13 & 63 & 321 & \dots \\ 13 & 63 & 321 & 1683 & \dots \\ 63 & 321 & 1683 & 8989 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then

$$\begin{aligned} H(2) &= L(2)D(2)L(2)^{T} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 3 & 1 & 0 & 0 & \dots \\ 13 & 6 & 1 & 0 & \dots \\ 63 & 33 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 4 & 0 & 0 & \dots \\ 0 & 0 & 8 & 0 & \dots \\ 0 & 0 & 0 & 16 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 3 & 13 & 63 & \dots \\ 0 & 1 & 6 & 33 & \dots \\ 0 & 0 & 1 & 9 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

Hence the Hankel transform of T(2n, n, 2) is equal to the sequence with general term

$$\prod_{k=0}^{n} (2.2^k - 0^k) = 2^n 2^{\binom{n+1}{2}}.$$

L(2) is in fact the Riordan array

$$\left(\frac{1}{\sqrt{1-6x+x^2}}, \frac{1-3x-\sqrt{1-6x+x^2}}{4x}\right)$$

or

$$\left(\frac{1-2x^2}{1+3x+2x^2}, \frac{x}{1+3x+2x^2}\right)^{-1}$$

In general, we can show that $H(r) = L(r)D(r)L(r)^T$ where L(r) is the Riordan array

$$\left(\frac{1}{\sqrt{1-2(r+1)x+(r-1)^2x^2}},\frac{1-(r+1)x-\sqrt{1-2(r+1)x+(r-1)^2x^2}}{2rx}\right)$$

and D(r) is the diagonal matrix with *n*-th term $2 \cdot r^n - 0^n$. Hence the Hankel transform of T(2n, n, r) is given by

$$\prod_{k=0}^{n} (2.r^{k} - 0^{k}) = 2^{n} r^{\binom{n+1}{2}}.$$

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We note that the Riordan array L(r)

$$\left(\frac{1}{\sqrt{1-2(r+1)x+(r-1)^2x^2}},\frac{1-(r+1)x-\sqrt{1-2(r+1)x+(r-1)^2x^2}}{2rx}\right)$$

is the inverse of the Riordan array

$$\left(\frac{1-rx^2}{1+(r+1)x+rx^2}, \frac{x}{1+(r+1)x+rx^2}\right).$$

Its general term is given by

$$\sum_{j=0}^{n} \binom{n}{j} \binom{n}{j-k} r^{j-k} = \sum_{j=0}^{n} \binom{n}{j} \binom{j}{n-k-j} r^{n-k-j} (r+1)^{2j-(n-k)}.$$

Its k-th column has exponential generating function given by

$$e^{(r+1)x}I_k(2\sqrt{r}x)/\sqrt{r}^k.$$

Corollary 2. The sequences with e.g.f. $I_0(2\sqrt{rx})$ have Hankel transforms given by $2^n r^{\binom{n+1}{2}}$. *Proof.* By [1] or otherwise, we know that the sequences T(2n, n, r) have e.g.f.

$$e^{(r+1)x}I_0(2\sqrt{rx}).$$

By the above proposition and the binomial invariance property of the Hankel transform [6], $\mathbf{B}^{-r-1}T(2n, n, r)$ has the desired Hankel transform. But $\mathbf{B}^{-r-1}T(2n, n, r)$ has e.g.f. given by

$$e^{-(r+1)x}e^{(r+1)x}I_0(2\sqrt{r}x) = I_0(2\sqrt{r}x)$$

2 Hankel transform of generalized Catalan numbers

Following [1], we denote by c(n; r) the sequence of numbers

$$c(n;r) = T(2n, n, r) - T(2n, n+1, r).$$

For instance, c(n; 1) = c(n), the sequence of Catalan numbers <u>A000108</u>. We have

Proposition 3. The Hankel transform of c(n;r) is $r^{\binom{n+1}{2}}$.

Proof. Again, we use the LDL^{T} decomposition of the associated Hankel matrices. For instance, when r = 3, we obtain

$$H(3) = \begin{pmatrix} 1 & 3 & 12 & 57 & \dots \\ 3 & 12 & 57 & 300 & \dots \\ 12 & 57 & 300 & 1686 & \dots \\ 57 & 300 & 1686 & 9912 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then

$$\begin{array}{rcl} H(3) &=& L(3)D(3)L(3)^T \\ &=& \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 3 & 1 & 0 & 0 & \dots \\ 12 & 7 & 1 & 0 & \dots \\ 57 & 43 & 11 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 0 & 0 & \dots \\ 0 & 0 & 9 & 0 & \dots \\ 0 & 0 & 0 & 27 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 3 & 12 & 57 & \dots \\ 0 & 1 & 7 & 43 & \dots \\ 0 & 0 & 1 & 11 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Hence the Hankel transform of c(n; 3) is

$$\prod_{k=0}^{n} 3^{k} = 3^{\binom{n+1}{2}}$$

In this case, L(3) is the Riordan array

$$(\frac{1}{1+3x}, \frac{x}{1+4x+3x^2})^{-1}.$$

In general, we can show that $H(r) = L(r)D(r)L(r)^T$ where

$$L(r) = \left(\frac{1}{1+rx}, \frac{x}{1+(r+1)x+rx^2}\right)^{-1}$$

and D(r) has *n*-th term r^n . Hence the Hankel transform of c(n; r) is given by

$$\prod_{k=0}^{n} r^k = r^{\binom{n+1}{2}}.$$

We finish this section with some notes concerning production matrices as found, for instance, in [4]. It is well known that the production matrix P(1) for the Catalan numbers C(n) = c(n, 1) is given by

$$P(1) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Following [4], we can associate a Riordan array $A_P(1)$ to P(1) as follows. The second column of P has generating function $\frac{1}{1-x}$. Solving the equation

$$u = \frac{1}{1 - xu}$$

we obtain $u(x) = \frac{1-\sqrt{1-4x}}{2x} = c(x)$. Since the first column is all 0's, this means that $A_P(1)$ is the Riordan array (1, xc(x)). This is the inverse of (1, x(1-x)). We have

$$A_P(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 2 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Multiplying on the right by B, the binomial matrix, we obtain

$$A_P(1)B = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 2 & 3 & 1 & 0 & \dots \\ 5 & 9 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = L(1)$$

which is the Riordan array

$$\left(\frac{1}{1-x}, xc(x)^2\right) = \left(\frac{1}{1+x}, \frac{1}{1+2x+x^2}\right)^{-1}.$$

Similarly the production matrix for the c(n; 2), or the large Schroeder numbers, is given by

$$P(2) = \begin{pmatrix} 0 & 2 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & \dots \\ 0 & 1 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Here, the generating function for the second column is $\frac{2-x}{1-x}$. Now solving

$$u = \frac{2 - xu}{1 - xu}$$

which gives $u = \frac{1+x-\sqrt{1-6x+x^2}}{2x}$. Hence in this case, $A_P(2)$ is the Riordan array $(1, \frac{1+x-\sqrt{1-6x+x^2}}{2})$. That is, $\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \end{pmatrix}$

$$A_P(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 2 & 4 & 0 & \dots \\ 0 & 6 & 8 & 8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (1, \frac{x(1-x)}{2-x})^{-1}.$$

The row sums of this matrix are $1, 2, 6, 22, 90, \ldots$ as expected. Multiplying $A_P(2)$ on the right by the binomial matrix B, we obtain

$$A_P(2)B = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 2 & 2 & 0 & 0 & \dots \\ 6 & 10 & 4 & 0 & \dots \\ 22 & 46 & 32 & 8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the array

$$(\frac{1-x-\sqrt{1-6x+x^2}}{2x},\frac{1-3x-\sqrt{1-6x+x^2}}{2x}).$$

Finally

$$A_{P}B\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{4} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = A_{P}B(1, \frac{x}{2}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 6 & 5 & 1 & 0 & \dots \\ 22 & 23 & 8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = L(2)$$

which is

$$(\frac{1-x-\sqrt{1-6x+x^2}}{2x}, \frac{1-3x-\sqrt{1-6x+x^2}}{4x})$$

or

$$L(2) = \left(\frac{1}{1+2x}, \frac{x}{1+3x+2x^2}\right)^{-1}.$$

We can generalize these results to give the following proposition.

Proposition 4. The production matrix for the generalized Catalan sequence c(n; r) is given by

$$P(r) = \begin{pmatrix} 0 & r & 0 & 0 & \dots \\ 0 & 1 & r & 0 & \dots \\ 0 & 1 & 1 & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The associated matrix $A_P(r)$ is given by

$$A_P(r) = (1, \frac{x(1-x)}{r - (r-1)x})^{-1} = (1, \frac{1 + (r-1)x - \sqrt{1 - 2(r+1)x + (r-1)^2 x^2}}{2}).$$

The matrix L(r) in the decomposition $L(r)D(r)L(r)^T$ of the Hankel matrix H(r) for c(n; r), which is equal to $A_P(r)B(1, x/r)$, is given by

$$L(r) = \left(\frac{1 - (r-1)x - \sqrt{1 - 2(r+1)x + (r-1)^2 x^2}}{2x}, \frac{1 - (r+1)x - \sqrt{1 - 2(r+1)x + (r-1)^2 x^2}}{2rx}\right)$$

We have

$$L(r) = \left(\frac{1}{1+rx}, \frac{x}{1+(r+1)x+rx^2}\right)^{-1}.$$

We note that the elements of $L(r)^{-1}$ are in fact the coefficients of the orthogonal polynomials associated to H(r).

Proposition 5. The elements of the rows of the Riordan array $(\frac{1}{1+rx}, \frac{x}{1+(r+1)x+rx^2})$ are the coefficients of the orthogonal polynomials associated to the Hankel matrix determined by the generalized Catalan numbers c(n; r).

3 Hankel transform of the sum of consecutive generalized Catalan numbers

We now look at the Hankel transform of the sum of two consecutive generalized Catalan numbers. That is, we study the Hankel transform of c(n; r) + c(n + 1; r). For the case r = 1(the ordinary Catalan numbers) this was dealt with in [3], while the general case was studied in [8]. We use the methods developed above to gain greater insight. We start with the case r = 1. For this, the Hankel matrix for c(n) + c(n+1) is given by

$$H = \begin{pmatrix} 2 & 3 & 7 & 19 & \dots \\ 3 & 7 & 19 & 56 & \dots \\ 7 & 19 & 56 & 174 & \dots \\ 19 & 56 & 174 & 561 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proceeding to the LDL^T decomposition, we get

$$H = LDL^{T}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{3}{2} & 1 & 0 & 0 & \dots \\ \frac{7}{2} & \frac{17}{5} & 1 & 0 & \dots \\ \frac{19}{2} & 11 & \frac{70}{13} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 & \dots \\ 0 & \frac{5}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{13}{5} & 0 & \dots \\ 0 & 0 & 0 & \frac{34}{13} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{2} & \frac{7}{2} & \frac{19}{2} & \dots \\ 0 & 1 & \frac{17}{5} & 11 & \dots \\ 0 & 0 & 1 & \frac{70}{13} & \dots \\ 0 & 0 & 0 & \frac{34}{13} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This indicates that the Hankel transform of c(n) + c(n+1) is given by

$$\prod_{k=0}^{n} \frac{F(2k+3)}{F(2k+1)} = F(2n+3).$$

This is in agreement with [3]. We note that in this case, L^{-1} takes the form

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -\frac{3}{2} & 1 & 0 & 0 & \dots \\ \frac{8}{5} & -\frac{17}{5} & 1 & 0 & \dots \\ -\frac{21}{13} & \frac{95}{13} & -\frac{70}{13} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{5} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{13} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -3 & 2 & 0 & 0 & \dots \\ 8 & -17 & 5 & 0 & \dots \\ -21 & 95 & -70 & 13 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where we see the sequences F(2n+1) and $(-1)^n F(2n+2)$ in evidence.

Now looking at the case r = 2, we get

$$H = \begin{pmatrix} 3 & 8 & 28 & 112 & \dots \\ 8 & 28 & 112 & 484 & \dots \\ 28 & 112 & 484 & 2200 & \dots \\ 112 & 484 & 2200 & 10364 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proceeding to the LDL^T decomposition, we obtain

$$\begin{array}{rcl} H &=& LDL^{T} \\ &=& \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{8}{3} & 1 & 0 & 0 & \dots \\ \frac{28}{3} & \frac{28}{5} & 1 & 0 & \dots \\ \frac{112}{3} & \frac{139}{5} & \frac{146}{17} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 & \dots \\ 0 & \frac{272}{3} & 0 & 0 & \dots \\ 0 & 0 & \frac{7424}{272} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & \frac{8}{2} & \frac{28}{3} & \frac{112}{3} & \dots \\ 0 & 1 & \frac{28}{5} & \frac{139}{5} & \dots \\ 0 & 0 & 1 & \frac{146}{17} & \dots \\ 0 & 0 & 0 & \frac{7424}{272} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus the Hankel transform of c(n; 2) + c(n+1; 2) is 3, 20, 272, 7424.... This is in agreement with [8]. We note that different factorizations of L^{-1} can lead to different formulas for $h_n(2)$, the Hankel transform of c(n; 2) + c(n+1; 2). For instance, we can show that

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -\frac{8}{3} & 1 & 0 & 0 & \dots \\ \frac{28}{5} & -\frac{28}{15} & -\frac{28}{17} & 1 & \dots \\ -\frac{192}{17} & \frac{345}{17} & -\frac{146}{17} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{3} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{5} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{17} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -8 & 3 & 0 & 0 & \dots \\ 28 & -28 & 5 & 0 & \dots \\ -192 & 345 & -146 & 17 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We note that the diagonal elements of the last matrix correspond to the sequence a(n) of terms $1, 3, 5, 17, 29, 99, \ldots$ with generating function

$$\frac{1+3x-x^2-x^3}{1-6x^2+x^4}$$

This is <u>A079496</u>. It is the interleaving of bisections of the Pell numbers <u>A000129</u> and their associated numbers <u>A001333</u>. We have

$$a(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} {\binom{n+1}{2k}} 2^{n+1-k-\lfloor \frac{n+2}{2} \rfloor}$$

= $-(\sqrt{2}-1)^n ((\frac{\sqrt{2}}{8}-\frac{1}{4})(-1)^n - \frac{\sqrt{2}}{8} - \frac{1}{4}) - (\sqrt{2}+1)^n ((\frac{\sqrt{2}}{8}-\frac{1}{4})(-1)^n - \frac{\sqrt{2}}{8} - \frac{1}{4})$

Multiplying a(n) by $4^{\lfloor \frac{(n+1)^2}{4} \rfloor}$, we obtain 1, 3, 20, 272, 7424, Hence

$$\begin{aligned} 1, 3, 20, 272, \dots &= 4^{\lfloor \frac{(n+1)^2}{4} \rfloor} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1}{2k} 2^{n+1-k-\lfloor \frac{n+2}{2} \rfloor} \\ &= 4^{\lfloor \frac{(n+1)^2}{4} \rfloor} 2^{n+1-\lfloor \frac{n+2}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1}{2k} 2^{-k} \\ &= 2^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1}{2k} 2^{-k} \end{aligned}$$

That is, the Hankel transform $h_n(2)$ of c(n; 2) + c(n+1; 2) is given by

$$h_n(2) = 2^{\binom{n+2}{2}} \sum_{k=0}^{\lfloor \frac{n+2}{2} \rfloor} {\binom{n+2}{2k}} 2^{-k}.$$

For our purposes, the following factorization of L^{-1} is more convenient.

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -\frac{8}{3} & 1 & 0 & 0 & \dots \\ \frac{56}{10} & -\frac{56}{10} & 1 & 0 & \dots \\ -\frac{384}{34} & \frac{690}{34} & -\frac{292}{34} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{3} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{34} & \dots \\ 0 & 0 & 0 & \frac{1}{34} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -8 & 3 & 0 & 0 & \dots \\ 56 & -56 & 10 & 0 & \dots \\ -384 & 690 & -292 & 34 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We now note that the sequence $\frac{h_n(2)}{2\binom{n+1}{2}}$ is the sequence $b_2(n+1)$, where $b_2(n)$ is the sequence $1, 3, 10, 34, 116, \ldots$ with generating function $\frac{1-x}{1-4x+2x^2}$ and general term

$$b_2(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-2)^k 4^{n-2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} (-2)^k 4^{n-2k-1}.$$

Hence

$$h_n(2) = 2^{\binom{n+1}{2}} b_2(n+1).$$

Noting that $b_2(n)$ is the binomial transform of the Pell <u>A000129</u>(n + 1) numbers whose generating function is $\frac{1}{1-2x-x^2}$, we have the following alternative expressions for $b_2(n)$:

$$b_2(n) = \sum_{k=0}^n \binom{n}{k} \sum_{j=0}^k \binom{j}{k-j} 2^{2j-k}$$
$$= \sum_{k=0}^n \binom{n}{k} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} 2^{k-2j}.$$

For r = 3, we have

$$H = \begin{pmatrix} 4 & 15 & 69 & 357 & \dots \\ 15 & 69 & 357 & 1986 & \dots \\ 69 & 357 & 1986 & 11598 & \dots \\ 357 & 1986 & 11598 & 70125 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We find that

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{4} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{17} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{73} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -15 & 4 & 0 & 0 & \dots \\ 198 & -131 & 17 & 0 & \dots \\ -2565 & 2875 & -854 & 73 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the sequence $b_3(n)$ or $1, 4, 17, 73, 314, \ldots, \underline{A018902}$ has generating function $\frac{1-x}{1-5x+3x^2}$ and

$$b_{3}(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} {\binom{n-k}{k}} (-3)^{k} 5^{n-2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} {\binom{n-k-1}{k}} (-3)^{k} 5^{n-2k-1}$$
$$= \sum_{k=0}^{n} {\binom{n}{k}} \sum_{j=0}^{k} {\binom{j}{k-j}} 3^{2j-k}$$
$$= \sum_{k=0}^{n} {\binom{n}{k}} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} {\binom{k-j}{j}} 3^{k-2j}.$$

Then $3^{\binom{n}{2}}b_3(n)$ is the sequence 1, 4, 51, 1971, 228906,.... In other words, we have

$$h_n(3) = 3^{\binom{n+1}{2}} b_3(n+1).$$

We now note that F(2n+1) has generating function $\frac{1-x}{1-3x+x^2}$ with

$$F(2n+1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} {\binom{n-k}{k}} (-1)^k 3^{n-2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} {\binom{n-k-1}{k}} (-1)^k 3^{n-2k-1}.$$

We can generalize this result as follows.

Proposition 6. Let $h_n(r)$ be the Hankel transform of the sum of the consecutive generalized Catalan numbers c(n; r) + c(n + 1; r). Then

$$h_n(r) = r^{\binom{n+1}{2}} \left(\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k} (-r)^k (r+2)^{n-2k+1} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-r)^k (r+2)^{n-2k}\right).$$

In other words, $h_n(r)$ is the product of $r^{\binom{n+1}{2}}$ and the (n+1)-st term of the sequence with generating function $\frac{1-x}{1-(r+2)x+rx^2}$. Equivalently,

$$h_{n}(r) = r^{\binom{n+1}{2}} \left(\sum_{k=0}^{n+1} \binom{k}{n-k+1} (r+2)^{2k-n-1} (-r)^{n-k+1} - \sum_{k=0}^{n} \binom{k}{n-k} (r+2)^{2k-n} (-r)^{n-k}\right)$$

$$= r^{\binom{n+1}{2}} \sum_{k=0}^{n+1} \binom{n+1}{k} \sum_{j=0}^{k} \binom{j}{k-j} r^{2j-k}$$

$$= r^{\binom{n+1}{2}} \sum_{k=0}^{n+1} \binom{n+1}{k} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} r^{k-2j}.$$

The two last expressions are a result of the fact that $\frac{1-x}{1-(r+2)x+rx^2}$ is the binomial transform of $\frac{1}{1-rx-x^2}$.

4 Berlekamp Massey triangles associated to generalized Catalan numbers

A natural question that arises when dealing with Hankel matrices is one that is inspired by consideration of the Hankel matrix interpretation of the Berlekamp Massey algorithm [2]. In our context, this question is that of characterizing the solutions of equations such as the following (taking c(n, 3) as an example)

$$\begin{pmatrix} 1 & 3 & 12 & 57 \\ 3 & 12 & 57 & 300 \\ 12 & 57 & 300 & 1686 \\ 57 & 300 & 1686 & 9912 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} 300 \\ 1686 \\ 9912 \\ 60213 \end{pmatrix}$$

or

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} -81 \\ 142 \\ -75 \\ 15 \end{pmatrix}$$

We can define the *B-M triangle* of a sequence a_1, a_2, a_3, \ldots to be the lower triangular matrix whose *n*-th row is the solution of the Berlekamp Massey equations determined by the *n*-th order Hankel matrix of the sequence.

Example 7. The B-M triangle of the Catalan numbers. We must solve

$$(1)(x) = (1)$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ 42 \end{pmatrix}$$

and so on. We obtain the triangle

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 3 & 0 & 0 & \dots \\ 1 & -6 & 5 & 0 & \dots \\ -1 & 10 & -15 & 7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term $(-1)^{n-k} \begin{pmatrix} n+k+1\\ 2k \end{pmatrix} - \begin{pmatrix} 0\\ n-k+1 \end{pmatrix}$ and generating function

$$\frac{(1+x) + xy}{(1-xy)(1+2x+x^2-xy)}$$

Regarding the entries as polynomial coefficients, we see that these polynomials are related to the Morgan-Voyce polynomials, themselves a transformation of the Jacobi polynomials.

In fact, the generating function of the B-M triangle for c(n; r) has generating function

$$\frac{r(1+x) + xy}{(1-xy)(1+(r+1)x + rx^2 - xy)}$$

These matrices are closely related to the matrices L already studied. The standard Berlekamp Massey theory studies the polynomials

$$x^d - \sum_{i=0}^{d-1} g_i x^i.$$

For the above example, this is

$$x^4 - 15x^3 + 75x^2 - 142x + 81.$$

Note that the companion matrix of this polynomial is given by

$$\begin{pmatrix} 1 & 3 & 12 & 57 \\ 3 & 12 & 57 & 300 \\ 12 & 57 & 300 & 1686 \\ 57 & 300 & 1686 & 9912 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 12 & 57 & 300 \\ 12 & 57 & 300 & 1686 \\ 57 & 300 & 1686 & 9912 \\ 300 & 1686 & 9912 & 60213 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -81 \\ 1 & 0 & 0 & 142 \\ 0 & 1 & 0 & -75 \\ 0 & 0 & 1 & 15 \end{pmatrix}$$

In other words, the characteristic polynomial of the last matrix is

$$81 - 142x + 75x^2 - 15x^3 + x^4.$$

Thus we get

$$(1)^{-1}(3) = (3) \text{ gives } -3+x$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 12 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 57 \end{pmatrix} = \begin{pmatrix} -9 \\ 7 \end{pmatrix} \text{ gives } 9-7x+x^2$$

$$\begin{pmatrix} 1 & 3 & 12 \\ 3 & 12 & 57 \\ 12 & 57 & 300 \end{pmatrix}^{-1} \begin{pmatrix} 57 \\ 300 \\ 1686 \end{pmatrix} = \begin{pmatrix} 27 \\ -34 \\ 11 \end{pmatrix} \text{ gives } -27+34x-11x^2+x^3.$$

and so on. Forming the matrix of coefficients, we obtain

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \dots \\
-3 & 1 & 0 & 0 & 0 & \dots \\
9 & -7 & 1 & 0 & 0 & \dots \\
-27 & 34 & -11 & 1 & 0 & \dots \\
81 & -142 & 75 & -15 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

which is the Riordan matrix

$$(\frac{1}{1+3x}, \frac{x}{1+4x+3x^2}).$$

The inverse of this matrix is the L matrix in the LDL^T decomposition of the Hankel matrix for c(n; 3). The corresponding B-M matrix as defined above is given by the negative of the sub-diagonal matrix.

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