A Catalan transform and related transformations on integer sequences

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Abstract

We introduce and study an invertible transformation on integer sequences related to the Catalan numbers. Transformation pairs are identified among classical sequences. A closely related transformation which we call the generalized Ballot transform is also studied, along with associated transformations. Results concerning the Fibonacci, Jacobsthal and Pell numbers are derived. Finally, we derive results about combined transformations.

1 Introduction

In this note, we report on a transformation of integer sequences that might reasonably be called the Catalan transformation. It is easy to describe both by formula (in relation to the general term of a sequence) and in terms of its action on the ordinary generating function of a sequence. It and its inverse can also be described succinctly in terms of the Riordan group.

Many classical 'core' sequences can be paired through this transformation. It is also linked to several other known transformations, most notably the binomial transformation.

2 Preliminaries

Unless otherwise stated, the integer sequences we shall study will be indexed by $\mathbf{N}$, the nonnegative integers. Thus the Catalan numbers, with general term $\mathrm{C}(n)$, are described by

$$\mathrm{C}(n) = \frac{\binom{2n}{n}}{n+1}$$

with ordinary generating function given by

$$\mathrm{c}(x) = \frac{1-\sqrt{1-4x}}{2x}$$

We use the notation 1^n to denote the all $1's$ sequence $1, 1, 1, \ldots$ with ordinary generating function $1/(1-x)$ and 0^n to denote the sequence $1, 0, 0, 0, \ldots$ with generating function 1. We have $0^n = \delta_{n,0} = \binom{0}{n}$ as an integer sequence. We note that this notation is consistent with the Binomial Theorem, as for instance in $1 = (1+0)^n = \sum_{k=0}^{n} \binom{n}{k} 0^k$. It has the added advantage of allowing us to regard $\ldots (-2)^n, (-1)^n, 0^n, 1^n, 2^n, \ldots$ as a sequence of successive binomial transforms (see next section). The ordinary generating function of the sequence 0^n is 1.

In order to characterize the effect of the so-called Catalan transformation, we shall look at its effect on some common sequence, including the Fibonacci and Jacobsthal numbers. The Fibonacci numbers [17] are amongst the most studied of mathematical objects. They are easy to define, and are known to have a rich set of properties. Closely associated to the Fibonacci numbers are the Jacobsthal numbers [18]. In a sense that will be made exact below, they represent the next element after the Fibonacci numbers in a one-parameter family of linear recurrences. These and many of the integer sequences that will be encountered in this note are to be found in The On-Line Encylopedia of Integer Sequences [9, 10]. Sequences in this database will be referred to by their Annnnnn number. For instance, the Catalan numbers are [A000108].

The Fibonacci numbers $\mathrm{F}(n)$ [A000045] are the solutions of the recurrence

$$a_n = a_{n-1} + a_{n-2},\ a_0 = 0,\ a_1 = 1 \tag{1}$$

with

$$\mathrm{F}(n) : 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots \tag{2}$$

The Jacobstahl numbers $\mathrm{J}(n)$ [A001045] are the solutions of the recurrence

$$a_n = a_{n-1} + 2a_{n-2},\ a_0 = 0,\ a_1 = 1 \tag{3}$$

with

$$\mathrm{J}(n) : 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, \ldots \tag{4}$$

$$\mathrm{J}(n) = \frac{2^n}{3} - \frac{(-1)^n}{3} \tag{5}$$

When we change the initial conditions to $a_0 = 1,\ a_1 = 0$, we get a sequence which we will denote by $\mathrm{J}_1(n)$ [A078008], given by

$$\mathrm{J}_1(n) : 1, 0, 2, 2, 6, 10, 22, 42, 86, 170, 342 \ldots \tag{6}$$

$$\mathrm{J}_1(n) = \frac{2^n}{3} + \frac{2(-1)^n}{3} \tag{7}$$

We see that

$$2^n = 2\mathrm{J}(n) + \mathrm{J}_1(n) \tag{8}$$

The Jacobsthal numbers are the case $k = 2$ for the one-parameter family of recurrences

$$a_n = a_{n-1} + ka_{n-2}, \ a_0 = 0, \ a_1 = 1 \tag{9}$$

where the Fibonacci numbers correspond to the case $k = 1$. The Pell numbers $\mathrm{Pell}(n)$ [A000129] are the solutions of the recurrence

$$a_n = 2a_{n-1} + a_{n-2}, \ a_0 = 0, \ a_1 = 1 \tag{10}$$

with

$$\mathrm{Pell}(n) : 0, 1, 2, 5, 12, 29, 70, 169, \ldots \tag{11}$$

The Pell numbers are the case $k = 2$ of the one-parameter family of recurrences

$$a_n = ka_{n-1} + a_{n-2}, \ a_0 = 0, \ a_1 = 1 \tag{12}$$

where again the Fibonacci numbers correspond to the case $k = 1$.

3 The Binomial Transform

We shall introduce transformations that operate on integer sequences. An example of such a transformation that is widely used in the study of integer sequences is the so-called binomial transform [15], which associates to the sequence with general term a_n the sequence with general term b_n where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \tag{13}$$

If we consider the sequence to be the vector $(a_0, a_1, \ldots)$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector with the lower-triangle matrix $\mathbf{Bin}$ whose (i, j)-th element is equal to $\binom{i}{j}$:

$$\mathbf{Bin} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & 0 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that we index matrices starting at $(0, 0)$. This transformation is invertible, with

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k \tag{14}$$

We note that **Bin** corresponds to Pascal's triangle. Its row sums are 2^n, while its diagonal sums are the Fibonacci numbers $\text{Fib}(n+1)$.

If $\mathcal{A}(x)$ is the ordinary generating function of the sequence a_n, then the generating function of the transformed sequence b_n is $(1/(1-x))\mathcal{A}(x/(1-x))$.

4 The Riordan Group

The Riordan group is a set of infinite lower-triangular matrices, where each matrix is defined by a pair of generating functions $g(x) = 1+g_1x+g_2x^2+\ldots$ and $f(x) = f_1x + f_2x^2 + \ldots$ where $f_1 \neq 0$ [8]. The associated matrix is the matrix whose k-th column is generated by $g(x)f(x)^k$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) * (h, l) = (g(h \circ f), l \circ f)$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix (g, f), and $a = \{a_n\}$ is an integer sequence with ordinary generating function $\mathcal{A}$ (x), then the sequence $\mathbf{M}a$ has ordinary generating function $g(x)\mathcal{A}(f(x))$.

For example, the Binomial matrix **Bin** is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group, while its inverse is the element $(\frac{1}{1+x}, \frac{x}{1+x})$. It can be shown more generally [10] that the matrix with general term $\binom{n+ak}{m+bk}$ is the element $(x^m/(1-x)^{m+1}, x^{b-a}/(1-x)^b)$ of the Riordan group. This result will be used in a later section, along with characterizations of terms of the form $\binom{2n+ak}{n+bk}$ [11]. As an example, we cite the result that the lower triangular matrix with general term $\binom{2n}{n-k} = \binom{2n}{n+k}$ is given by the Riordan array $(\frac{1}{\sqrt{1-4x}}, \frac{1-2x-\sqrt{1-4x}}{2x}) = (\frac{1}{\sqrt{1-4x}}, xc(x)^2)$ where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers.

A lower-triangular matrix that is related to $(\frac{1}{\sqrt{1-4x}}, xc(x)^2)$ is the matrix $(\frac{1}{\sqrt{1-4x}}, x^2c(x)^2)$. This is no longer a *proper* Riordan array: it is a stretched Riordan array, as described in [1]. The row sums of this array are then the diagonal sums of $(\frac{1}{\sqrt{1-4x}}, xc(x)^2)$, and hence have expression $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2(n-k)}{n}$.

5 Introducing the Catalan transform

We initially define the Catalan transformation by its action on ordinary generating functions. For this, we let $\text{g}(x)$ be the generating function of a sequence. The *Catalan transform* of that sequence is defined to be the sequence whose generating function is $\mathrm{g}(x\mathrm{c}(x))$. That this transformation is invertible is demonstrated by

Proposition 1 *The inverse of the Catalan transformation is given by*

$$g(x) \rightarrow g(x(1-x))$$

Proof. We prove a more general result. Consider the Riordan matrix $(1, x(1-kx))$. Let $(g', \bar{f})$ denote its Riordan inverse. We then have

$$(g', \bar{f})(1, x(1-kx)) = (1, x)$$

Hence

$$\bar{f}(1-k\bar{f}) = x \Rightarrow k\bar{f}^2 - \bar{f} + x = 0 \Rightarrow \bar{f} = \frac{1-\sqrt{1-4kx}}{2k}$$

Since $g = 1$, $g' = 1$ also, and thus

$$(1, x(1-kx))^{-1} = (1, \frac{1-\sqrt{1-4kx}}{2k})$$

Setting $k = 1$, we obtain

$$(1, x(1-x))^{-1} = (1, x\mathrm{c}(x)) \Rightarrow (1, x\mathrm{c}(x))^{-1} = (1, x(1-x))$$

as required.

In this context we note the following identities

$$x(1-x)\mathrm{c}(x(1-x)) = x(1-x)\tfrac{1-\sqrt{1-4x(1-x)}}{2x} = \tfrac{1}{2}(1-\sqrt{1-4x+4x^2})$$

$$= \tfrac{1}{2}(1-\sqrt{(1-2x)^2}) = \tfrac{1}{2}(1-(1-2x)) = x \Rightarrow \mathrm{c}(x(1-x)) = \tfrac{1}{1-x}$$

Similarly, we have

$$x\mathrm{c}(x)(1-x\mathrm{c}(x)) = x\mathrm{c}(x) - x^2\mathrm{c}(x)^2 = x\tfrac{1-\sqrt{1-4x}}{2x} - x^2\tfrac{(1-\sqrt{1-4x})^2}{4x^2}$$

$$= \tfrac{1}{2}(1-\sqrt{1-4x}) - \tfrac{1}{4}(1-2\sqrt{1-4x}+(1-4x))$$

$$= \tfrac{1}{2}(1-\sqrt{1-4x}) + \tfrac{1}{2}(\sqrt{1-4x}-1+2x)$$

$$= \tfrac{1}{2}(1-\sqrt{1-4x}+\sqrt{1-4x}-1+2x)$$

$$= x \Rightarrow \mathrm{c}(x) = \tfrac{1}{1-xc(x)}$$

6 Matrix representation

In terms of the Riordan group, this means that the Catalan transform and its inverse are given by the elements $(1, x\mathrm{c}(x))$ and $(1, x(1-x))$. The lower-triangular matrix representing the Catalan transformation thus has the form

$$\mathbf{Cat} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 5 & 5 & 3 & 1 & 0 & \dots \\ 0 & 14 & 14 & 9 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Where convenient, we shall denote this transformation by **Cat**. The inverse Catalan transformation $\mathbf{Cat}^{-1}$ has the form

$$\mathbf{Cat}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 3 & -4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The general term of the matrix $(1, x(1-x))$ is given by $\binom{k}{n-k}(-1)^{n-k}$. This can be shown by observing that the $k-$th column of $(1, x(1-x))$ has generating function $(x(1-x))^k$. But $[x^n](x(1-x))^k = \binom{k}{n-k}(-1)^{n-k}$. We note that the generating function of $(1, x(1-x))$ is $\frac{1}{1-xy(1-x)}$.

We now characterise the general term of the matrix for the Catalan transform.

Proposition 2 *The general term* $\mathrm{T}(n,k)$ *of the Riordan matrix* $(1, \mathrm{c}(x))$ *is given by*

$$\mathrm{T}(n,k) = \sum_{j=0}^{k} \binom{k}{j} \binom{j/2}{n} (-1)^{n+j} 2^{2n-k}$$

Proof. We seek $[x^n](x\mathrm{c}(x))^k$. To this end, we develop the term $(x\mathrm{c}(x))^k$ as follows:

$$x^k \mathrm{c}(x)^k = x^k \left(\tfrac{1-\sqrt{1-4x}}{2x}\right)^k = \tfrac{1}{2^k}(1-\sqrt{1-4x})^k = \tfrac{1}{2^k}\textstyle\sum_{j=0}^k \binom{k}{j}(-\sqrt{1-4x})^k$$
$$= \tfrac{1}{2^k}\textstyle\sum_j \binom{k}{j}(-1)^j(1-4x)^{j/2} = \tfrac{1}{2^k}\sum_j \binom{k}{j}(-1)^j \sum_i \binom{j/2}{i}(-4x)^i$$
$$= \tfrac{1}{2^k}\textstyle\sum_j \binom{k}{j}(-1)^j \sum_i \binom{j/2}{i}(-4)^i x^i = \sum_j \binom{k}{j} \sum_i \binom{j/2}{i}(-1)^{i+j} 2^{2i-k} x^i$$

Thus $[x^n](x\mathrm{c}(x))^k = \sum_{j=0}^k \binom{k}{j}\binom{j/2}{n}(-1)^{n+j}2^{2n-k}$.

This result can be obtained in a similar fashion by noting that the generating function of the Riordan matrix $(1, x\mathrm{c}(x))$ is $G(x,y) = \frac{1}{1-xyc(x)}$. Expanding this bivariate expression and taking $[x^n y^k]G(x,y)$ leads to the same result.

The above proposition shows that the Catalan transform of a sequence a_n has general term b_n given by

$$b_n = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{j/2}{n}(-1)^{n+j}2^{2n-k}a_k$$

The following proposition gives alternative versions for this expression.

Proposition 3 *Given a sequence a_n, its Catalan transform b_n is given by*

$$b_n = \sum_{k=0}^{n}\frac{k}{2n-k}\binom{2n-k}{n-k}a_k = \sum_{k=0}^{n}\frac{k}{n}\binom{2n-k-1}{n-k}a_k$$

or

$$b_n = \sum_{j=0}^{n}\sum_{k=0}^{n}\frac{2k+1}{n+k+1}(-1)^{k-j}\binom{2n}{n-k}\binom{k}{j}a_j$$

The inverse transformation is given by

$$a_n = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}(-1)^k b_{n-k} = \sum_{k=0}^{n}\binom{k}{n-k}(-1)^{n-k}b_k$$

Proof. We have

$\mathrm{T}(n,k) = 2^{2n-k}(-1)^n\sum_{j=0}^k\binom{k}{j}\binom{j/2}{n}(-1)^j =$

$2^{2n-k}(-1)^n((-1)^n 2^{k-2n}\left(\binom{2n-k-1}{n-1} - \binom{2n-k-1}{n}\right)) =$

$\binom{2n-k-1}{n-1} - \binom{2n-k-1}{n} = \frac{k}{n}\binom{2n-k-1}{n-1} = \frac{k}{n}\binom{2n-k-1}{n-k}$

This proves the first two assertions of the proposition. The last assertion follows from the expression for the general term of the matrix $(1, x(1-x))$ obtained above. The equivalence of this and the accompanying expression is easily obtained. The remaining assertion will be a consequence of results in Section 9.

7 Catalan pairs

Using the last proposition, and [9], it is possible to draw up the following representative list of Catalan pairs, that is, sequences and their Catalan transforms.

Table 1. Catalan pairs

a_n	b_n	a_n	b_n
0^n	0^n	A000007	A000007
1^n	$\mathrm{C}(n)$	1^n	A000108
2^n	$\binom{2n}{n}$	A000079	A000984
2^n-1	$\binom{2n}{n-1}$	A000225	A001791
$2^n-\sum_{j=0}^{k-1}\binom{n}{j}$	$\binom{2n}{n-k}$	various	various
n	$3n\mathrm{C}(n)/(n+2)$	A001477	A000245
$n+1$	$\mathrm{C}(n+1)$	A000027	A000108$(n+1)$
$\binom{n}{2}$	$5\binom{2n}{n-2}/(n+3)$	A000217$(n-1)$	A000344
$\binom{n+1}{2}$	$4\binom{2n+1}{n-1}/(n+3)$	A000217	A002057
$\binom{n+2}{2}$	$3\binom{2n+2}{n}/(n+3)$	A000217$(n+1)$	A000245
$0^n-(-1)^n$	Fine's sequence	-	A000957
$(1+(-1)^n)/2$	Fine's sequence	A000035$(n+1)$	A000957$(n+1)$
$2-0^n$	$(2-0^n)\mathrm{C}(n)$	A040000	A068875
$\mathrm{F}(n)$	G.f.$\frac{xc(x)}{x+\sqrt{1-4x}}$	A000045	-
$\mathrm{F}(n+1)$	G.f.$\frac{1}{x+\sqrt{1-4x}}$	A000045$(n+1)$	A081696
$\mathrm{J}(n)$	$\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-2}{n-1}$	A001045	A014300
$\mathrm{J}(n+1)$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{2n-2j-1}{n-1}+0^n$	A001045$(n+1)$	A026641$(n)+0^n$
$\mathrm{J}_1(n)$	$\sum_{k=0}^{n}\binom{n+k-1}{k}(-1)^{n-k}$	A078008	A072547
$2\sin(\frac{\pi n}{3}+\frac{\pi}{3})/\sqrt{3}$	1^n	A010892	1^n
$(-1)^n\mathrm{F}(n+1)$	$(-1)^n$	$(-1)^n$A000045$(n+1)$	$(-1)^n$
$2^{\frac{n}{2}}(\cos(\frac{\pi n}{4})+\sin(\frac{\pi n}{4}))$	2^n	A009545	A000079

We note that the result above concerning Fine's sequence [A000957] is implicit in the work [2]. We deduce immediately that the generating function for Fine's sequence can be written as

$$\frac{x}{1-(x\mathrm{c}(x))^2}=\frac{2x}{1+2x+\sqrt{1-4x}}=\frac{x}{1+x-x\mathrm{c}(x)}$$

See also [7].

8 Transforms of a Jacobsthal family

From the above, we see that the Jacobsthal numbers $\mathrm{J}(n)$ transform to give the sequence with general term

$$\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-2}{n-1}$$

and generating function

$$\frac{x\mathrm{c}(x)}{(1+x\mathrm{c}(x))(1-2x\mathrm{c}(x))}$$

This prompts us to characterize the family of sequences with general term

$$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n-2j-2}{n-1} k^j$$

where the transform of the Jacobsthal numbers corresponds to the case $k = 1$. Re-casting the ordinary generating function of the Jacobsthal numbers as

$$\frac{x}{(1+x)(1-2x)} = \frac{x(1-x)}{(1-x^2)(1-2x)}$$

we see that the Jacobsthal numbers are the case $k = 1$ of the one-parameter family of sequences with generating functions

$$\frac{x(1-x)}{(1-kx^2)(1-2x)}$$

For instance, the sequence for $k = 0$ has g.f. $\frac{x(1-x)}{1-2x}$ which is $0, 1, 1, 2, 4, 8, 16, \ldots$. For $k = 2$, we obtain $0, 1, 1, 4, 6, 16, \ldots$ or [A007179]. The general term for these sequences is given by

$$\frac{(\sqrt{k})^{n-1}(1-\sqrt{k})}{2(2-\sqrt{k})} + \frac{(-\sqrt{k})^{n-1}(1+\sqrt{k})}{2(2+\sqrt{k})} + \frac{2^n}{4-k}$$

for $k \neq 4$.

They are solutions of the family of recurrences

$$a_n = 2a_{n-1} + ka_{n-2} - 2ka_{n-3}$$

where $a_0 = 0$, $a_1 = 1$ and $a_2 = 1$.

Proposition 4 *The Catalan transform of the generalized Jacobsthal sequence with ordinary generating function*

$$\frac{x(1-x)}{(1-kx^2)(1-2x)}$$

has ordinary generating function given by

$$\frac{x}{\sqrt{1-4x}(1-k(xc(x))^2)}$$

and general term

$$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n-2j-2}{n-1} k^j$$

Proof. By definition, the Catalan transform of $\frac{x(1-x)}{(1-kx^2)(1-2x)}$ is $\frac{xc(x)(1-xc(x))}{(1-kx^2c(x)^2)(1-2xc(x))}$. But $\mathrm{c}(x)(1-x\mathrm{c}(x))=1$ and $1-2x\mathrm{c}(x)=\sqrt{1-4x}$. Hence we obtain the first assertion.

We now recognise that

$$\frac{1}{\sqrt{1-4x}(1-k(x\mathrm{c}(x))^2)}=\left(\frac{1}{\sqrt{1-4x}},x^2\mathrm{c}(x)^2\right)\frac{1}{1-kx}$$

But this is $\sum_{j=0}^{\lfloor n/2\rfloor}\binom{2n-2j}{n}k^j$. The second assertion follows from this.

For example, the transform of the sequence $0,1,1,2,4,8,\ldots$ can be recognized as $\binom{2n-2}{n-1}=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-2}{n-1}0^j$.

As noted in [A014300], the Catalan transform of the Jacobsthal numbers corresponds to the convolution of the central binomial numbers (with generating function $\frac{1}{1-4x}$) and Fine's sequence [A000957] (with generating function $\frac{x}{1-(xc(x))^2}$). The above proposition shows that the Catalan transform of the generalized Jacobsthal numbers corresponds to a convolution of the central binomial numbers and the 'generalized' Fine numbers with generating function $\frac{x}{1-k(xc(x))^2}$.

Using the inverse Catalan transform, we can express the general term of this Jacobsthal family as

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}(-1)^i\sum_{j=0}^{\lfloor (n-i-1)/2\rfloor}\binom{2n-2i-2j-2}{n-i-1}k^j$$

This provides us with a closed form for the case $k=4$ in particular.

We now wish to find an expression for the transform of $\mathrm{J}_1(n)$. To this end, we note that

$$\mathrm{J}(n+1)\rightarrow\sum_{j=0}^{\lfloor n/2\rfloor}\binom{2n-2j-1}{n-1}+0^n=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-1}{n-1}+\frac{1+(-1)^n}{2}$$

Then

$$\mathrm{J}_1(n)=\mathrm{J}(n+1)-\mathrm{J}(n)\rightarrow\sum_{j=0}^{\lfloor\frac{n-1)}{2}\rfloor}\binom{2n-2j-1}{n-1}+\frac{1+(-1)^n}{2}-\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{2n-2j-2}{n-1}$$

$$=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-2}{n-2}+\frac{1+(-1)^n}{2}$$

The first term of the last expression deserves comment. Working with generating functions, it is easy to show that under the Catalan transform, we have

$$(-1)^n\mathrm{F}(n+1)/2+\cos(\frac{\pi n}{3})/2+\sqrt{3}\sin(\frac{\pi n}{3})/6 \to \frac{1+(-1)^n}{2}$$

Hence $\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2n-2j-2}{n-2}$ is the Catalan transform of

$$\mathrm{J}_1(n)-(-1)^n\mathrm{F}(n+1)/2-\cos(\frac{\pi n}{3})/2-\sqrt{3}\sin(\frac{\pi n}{3})/6$$

9 The Generalized Ballot Transform

In this section, we introduce and study a transformation that we will link to the generalized ballot numbers studied in [6]. For this, we define a new transformation **Bal** as the composition of the Catalan transform and the Binomial transform:

$$\mathbf{Bal}=\mathbf{Cat}\circ\mathbf{Bin}$$

The Riordan matrix formulation of this transformation is thus given by

$$\mathbf{Bal}=\mathbf{Cat}\circ\mathbf{Bin}=(1,\mathrm{c}(x))(\tfrac{1}{1-x},\tfrac{x}{1-x})=(\tfrac{1}{1-xc(x)},\tfrac{xc(x)}{1-xc(x)})=$$
$$(\mathrm{c}(x),\mathrm{c}(x)-1)=(\mathrm{c}(x),x\mathrm{c}(x)^2)$$

In similar fashion, we can find the Riordan description of the inverse of this transformation by

$$\mathbf{Bal}^{-1}=\mathbf{Bin}^{-1}\circ\mathbf{Cat}^{-1}=(\tfrac{1}{1-x},\tfrac{x}{1-x})^{-1}(1,\mathrm{c}(x))^{-1}=(\tfrac{1}{1+x},\tfrac{x}{1+x})(1,x(1-x))$$

$$=(\tfrac{1}{1+x}.1,\tfrac{x}{1+x}(1-\tfrac{x}{1+x}))=(\tfrac{1}{1+x},\tfrac{x}{(1+x)^2})$$

The general term of $\mathbf{Bal}^{-1}$ is easily derived from the last expression: it is $[x^n](x^k(1+x)^{-2k+1})=\binom{n+k}{n-k}(-1)^{n-k}=\binom{n+k}{2k}(-1)^{n-k}$. Alternatively we can find the general term in the matrix product of $\mathbf{Bin}^{-1}$, or $\binom{n}{n}(-1)^{n-k}$, with $\mathbf{Cat}^{-1}$, or $\binom{k}{n-k}(-1)^{n-k}$ to get the equivalent expression $\sum_{j=0}^{k+n}\binom{n}{j}\binom{k}{j-k}(-1)^{n-k}$.

We now examine the general term of the transformation $\mathbf{Bal}=(\mathrm{c}(x),\mathrm{c}(x)-1)=(\mathrm{c}(x),x\mathrm{c}(x)^2)$. An initial result is given by

Proposition 5 *The general term* $\mathrm{T}(n,k)$ *of the Riordan matrix* $(\mathrm{c}(x),\mathrm{c}(x)-1)$ *is given by*

$$\mathrm{T}(n,k)=\sum_{j=0}^{k}\sum_{i=0}^{j+1}\binom{k}{j}\binom{j+1}{i}\binom{i/2}{n+j+1}(-1)^{k+n+i+1}2^{2n+j+1}$$

and

$$\mathrm{T}(n,k)=2.4^n\sum_{j=0}^{2k+1}\binom{2k+1}{j}\binom{j/2}{n+k+1}(-1)^{n+k+j+1}$$

Proof. The first assertion follows by observing that the $k-$th column of $(\mathrm{c}(x), \mathrm{c}(x)-1)$ has generating function $\mathrm{c}(x)(\mathrm{c}(x)-1)^k$. We are thus looking for $[x^n]\mathrm{c}(x)(\mathrm{c}(x)-1)^k$. Expanding and substituting for $\mathrm{c}(x)$ yields the result. The second assertion follows by taking $[x^n]\mathrm{c}(x)(x\mathrm{c}(x)^2)^k$.

We now show that this transformation has in fact a much easier formulation, corresponding to the generalized Ballot numbers of [6]. We recall that the generalized Ballot numbers or generalized Catalan numbers [6] are defined by

$$\mathrm{B}(n,k) = \binom{2n}{n+k}\frac{2k+1}{n+k+1}$$

$\mathrm{B}(n,k)$ can be written as

$$\mathrm{B}(n,k) = \binom{2n}{n+k}\frac{2k+1}{n+k+1} = \binom{2n}{n-k} - \binom{2n}{n-k-1}$$

where the matrix with general term $\binom{2n}{n-k}$ is the element

$$\left(\frac{1}{\sqrt{1-4x}}, \frac{1-2x-\sqrt{1-4x}}{2x}\right) = \left(\frac{1}{\sqrt{1-4x}}, x\frac{\mathrm{c}(x)-1}{x}\right) = \left(\frac{1}{\sqrt{1-4x}}, x\mathrm{c}(x)^2\right)$$

of the Riordan group [11]. The inverse of this matrix is $(\frac{1-x}{1+x}, \frac{x}{(1+x)^2})$ with general term $(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{2k}$.

Proposition 6 *The general term of the Riordan matrix* $(\mathrm{c}(x), \mathrm{c}(x)-1)$ *is given by*

$$\mathrm{T}(n,k) = \mathrm{B}(n,k) = \binom{2n}{n+k}\frac{2k+1}{n+k+1}$$

Proof. We provide two proofs - one indirect, the other direct. The first, indirect proof is instructive as it uses properties of Riordan arrays.

We have $\mathrm{B}(n,k) = \binom{2n}{n-k} - \binom{2n}{n-k-1}$. Hence the generating function of the $k-$th column of the matrix with general term $\mathrm{B}(n,k)$ is given by

$$\frac{1}{\sqrt{1-4x}}(c(x)-1)^k - \frac{1}{\sqrt{1-4x}}(c(x)-1)^{k+1}$$

But

$$\tfrac{1}{\sqrt{1-4x}}(c(x)-1)^k - \tfrac{1}{\sqrt{1-4x}}(c(x)-1)^{k+1} = (c(x)-1)^k(\tfrac{1}{\sqrt{1-4x}} - \tfrac{c(x)-1}{\sqrt{1-4x}})$$

$$= (c(x)-1)^k(\tfrac{1}{\sqrt{1-4x}}(1-(c(x)-1))) = (c(x)-1)^k(\tfrac{-c(x)+2}{\sqrt{1-4x}}) = (c(x)-1)^k c(x)$$

But this is the generating function of the $k-$th column of $(\mathrm{c}(x), \mathrm{c}(x)-1)$.

The second, direct proof follows from the last proposition. From this, we have

$$\begin{aligned}&\mathrm{T}(n,k) = 2.4^n(-1)^{n+k+1}\textstyle\sum_{j=0}^{2k+1}\binom{2k+1}{j}\binom{j/2}{n+k+1}(-1)^j\\ &= 2.4^n(-1)^{n+k+1}\left\{(-1)^{n+k+1}2^{2k+1-2(n+k+1)}\left(\binom{2n+2k+2-2k-1-1}{n+k}-\binom{2n+2k+2-2k-1-1}{n+k+1}\right)\right\}\\ &= \left\{\binom{2n}{n+k}-\binom{2n}{n+k+1}\right\} = \left\{\binom{2n}{n-k}-\binom{2n}{n-k-1}\right\} = \binom{2n}{n+k}\frac{2k+1}{n+k+1}\end{aligned}$$

The numbers $\mathrm{B}(n,k)$ have many combinatorial uses. For instance,

$$\mathrm{B}(n,k) = \mathrm{D}(n+k+1, 2k+1)$$

where $\mathrm{D}(n,k)$ is the number of Dyck paths of semi-length n having height of the first peak equal to k [7]. $\mathrm{B}(n,k)$ also counts the number of paths from $(0,-2k)$ to $(n-k, n-k)$ with permissible steps $(0,1)$ and $(1,0)$ that don't cross the diagonal $y = x$ [6].

We recall that the classical ballot numbers are given by $\frac{k}{2n+k}\binom{2n+k}{n} = \frac{k}{2n+k}\binom{2n+k}{n+k}$ [5].

We now define the *generalized Ballot transform* to be the transformation corresponding to the Riordan array $\mathbf{Bal} = \mathbf{Cat} \circ \mathbf{Bin} = (\mathrm{c}(x), \mathrm{c}(x) - 1) = (\mathrm{c}(x), x\mathrm{c}(x)^2)$. By the above, the generalized Ballot transform of the sequence a_n is the sequence b_n where

$$b_n = \sum_{k=0}^{n}\binom{2n}{n+k}\frac{2k+1}{n+k+1}a_k$$

In terms of generating functions, the generalized Ballot transform maps the sequence with ordinary generating function $g(x)$ to the sequence with generating function $\mathrm{c}(x)g(\mathrm{c}(x)-1) = \mathrm{c}(x)\mathrm{g}(x\mathrm{c}(x)^2)$ where $\mathrm{c}(x)$ is the generating function of the Catalan numbers. We then have

Proposition 7 *Given a sequence a_n, its inverse generalized Ballot transform is given by*

$$b_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{2k}a_k$$

If a_n has generating function $g(x)$ then b_n has generating function $\frac{1}{1+x}g(\frac{x}{(1+x)^2})$

Proof We have seen that $\mathbf{Bal}^{-1} = \mathbf{Bin}^{-1} \circ \mathbf{Cat}^{-1} = (\frac{1}{1+x}, \frac{x}{(1+x)^2})$. The second statement follows from this. We have also seen that the general term of $\mathbf{Bal}^{-1}$ is $(-1)^{n-k}\binom{n+k}{2k}$. Hence the transform of the sequence a_n is as asserted.

We finish this section by noting that

$$\mathbf{Cat} = \mathbf{Bal} \circ \mathbf{Bin}^{-1}$$

Since the general term of $\mathbf{Bin}^{-1}$ is $(-1)^{n-k}\binom{n}{k}$ we immediately obtain the expression $\sum_{j=0}^{n}\sum_{k=0}^{n}\frac{2k+1}{n+k+1}(-1)^{k-j}\binom{2n}{n-k}\binom{k}{j}$ for the general term of $\mathbf{Cat}$.

10 Matrix representations of the generalized Ballot transform and its inverse

In terms of the Riordan group, the above implies that the generalized Ballot transform and its inverse are given by the elements $(\mathrm{c}(x), x(\mathrm{c}(x)-1)/x)$ and $(\frac{1}{1+x}, \frac{x}{(1+x)^2})$. The lower-triangular matrix representing the Ballot transformation thus has the form

$$\mathbf{Bal} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 5 & 9 & 5 & 1 & 0 & 0 & \ldots \\ 14 & 28 & 20 & 7 & 1 & 0 & \ldots \\ 42 & 90 & 75 & 35 & 9 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while the inverse Ballot transformation is represented by the matrix

$$\mathbf{Bal}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -3 & 1 & 0 & 0 & 0 & \ldots \\ -1 & 6 & -5 & 1 & 0 & 0 & \ldots \\ 1 & -10 & 15 & -7 & 1 & 0 & \ldots \\ -1 & 15 & -35 & 28 & -9 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The first matrix is [A039599], while the absolute value of the second matrix is [A085478].

11 Generalized Ballot transform pairs

The following table identifies some Ballot transform pairings [9].

Table 2. Generalized Ballot transform pairs

a_n	b_n	a_n	b_n
$(-1)^n$	0^n	$(-1)^n$	A000007
0^n	$\mathrm{C}(n)$	A000007	A000108
1^n	$\binom{2n}{n}$	1^n	A000984
2^n	$\frac{1}{1-3xc(x)}$	A000079	A007854
n	$\frac{xc(x)}{1-4c(x)}$	A001477	A000346$(n-1)+0^n/2$
$n+1$	$\sum_{k=0}^{n}\binom{2n}{k}$	A000027	A032443
$2n+1$	4^n	A005408	A000302
$(1+(-1)^n)/2$	$\binom{2n+1}{n}$	A059841	A088218
$2-0^n$	$\binom{2n+1}{n+1}$	A040000	A001700
$(2^n+0^n)/2$	-	A011782	A090317
$\cos(\frac{2\pi n}{3})+\sin(\frac{2\pi n}{3})/\sqrt{3}$	1^n	A057078	1^n
$\cos(\frac{\pi n}{2})+\sin(\frac{\pi n}{2})$	2^n	-	A000079
$\cos(\frac{\pi n}{3})+\sqrt{3}\sin(\frac{\pi n}{3})$	3^n	A057079	A000244
$\mathrm{F}(n)$	$-$	A000045	A026674
$\mathrm{F}(n+1)$	$-$	A000045$(n+1)$	A026726

12 The Signed Generalized Ballot transform

For completeness, we consider a transformation that may be described as the signed generalized Ballot transform. As a member of the Riordan group, this is the element $(\mathrm{c}(-x), 1-\mathrm{c}(-x)) = (\mathrm{c}(-x), x\mathrm{c}(-x)^2)$. For a given sequence a_n, it yields the sequence with general term

$$b_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n+k}\frac{2k+1}{n+k+1}a_k$$

Looking at generating functions, we get the mapping

$$\mathcal{A}(x) \to \mathrm{c}(-x)\mathcal{A}(1-\mathrm{c}(-x)) = \mathrm{c}(-x)\mathcal{A}(x\mathrm{c}(-x)^2)$$

The inverse of this map is given by

$$b_n = \sum_{k=0}\binom{n+k}{2k}a_k$$

or, in terms of generating functions

$$\mathcal{A}(x) \to \frac{1}{1-x}\mathcal{A}\left(\frac{x}{(1-x)^2}\right)$$

Example mappings under this transform are $0^n \to (-1)^n\mathrm{C}(n)$, $1^n \to 0^n$, $\mathrm{F}(2n+1) \to 1^n$.

We can characterize the effect of its inverse on the power sequences $n \to k^n$ as follows : the image of k^n under the inverse signed generalized Ballot transform is the solution to the recurrence

$$a_n = (k+2)a_{n-1} - a_{n-2}$$

with $a_0 = 1$, $a_1 =$F$(2k+1)$.

The matrices that represent this inverse pair of transformations are, respectively, the alternating sign versions of [A039599] and [A085478].

The latter matrix has a growing literature in which it is known as the DFF triangle [3, 4, 13]. As an element of the Riordan group, it is given by $(\frac{1}{1-x}, \frac{x}{(1-x)^2})$.

It has a 'companion' matrix with general element

$$b_{n,k} = \binom{n+k+1}{2k+1} = \frac{n+k+1}{2k+1}\binom{n+k}{2k}$$

called the DFFz triangle. This is the element

$$\left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right)$$

of the Riordan group with inverse

$$\left(\frac{1-\mathrm{c}(-x)}{x}, 1-\mathrm{c}(-x)\right) = (\mathrm{c}(-x)^2, x\mathrm{c}(-x)^2)$$

Another number triangle that is related to the above [13] has general term

$$a_{n,k} = \frac{2n}{n+k}\binom{n+k}{2k}$$

Taking $a_{0,0} = 1$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 9 & 6 & 1 & 0 & 0 & \ldots \\ 2 & 16 & 20 & 8 & 1 & 0 & \ldots \\ 2 & 25 & 50 & 35 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the element $(\frac{1+x}{1-x}, \frac{x}{(1-x)^2})$ of the Riordan group [11]. Its inverse is the matrix

$$\left(\begin{array}{rrrrrrr} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 6 & -4 & 1 & 0 & 0 & 0 & \dots \\ -20 & 15 & -6 & 1 & 0 & 0 & \dots \\ 70 & -56 & 28 & -8 & 1 & 0 & \dots \\ -252 & 210 & -120 & 45 & -10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with general element $(-1)^{n-k}\binom{2n}{n-k}$ which is the element

$$\left(\frac{1}{\sqrt{1+4x}}, \frac{1+2x-\sqrt{1+4x}}{2x}\right)$$

of the Riordan group. We note that $\frac{1+2x-\sqrt{1+4x}}{2x^2}$ is the generating function of $(-1)^n\mathrm{C}(n)$.

Applying $\mathbf{Bin}^2$ to this matrix yields the matrix

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 0 & 1 & 0 & 0 & \dots \\ 6 & 0 & 4 & 0 & 1 & 0 & \dots \\ 0 & 10 & 0 & 5 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which is the element $(\frac{1}{\sqrt{1-4x^2}}, x\mathrm{c}(x^2))$ of the Riordan group.

13 An Associated Transformation

We briefly examine a transformation associated to the Ballot transformation. Unlike other transformations in this study, this is not invertible. However, it transforms some 'core sequences' to other 'core sequences', and hence deserves study. An example of this transformation is given by

$$\mathrm{M}_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \mathrm{C}(k)$$

where M_n is the nth Motzkin number [A001006]. In general, if a_n is the general term of a sequence with generating function $\mathcal{A}(x)$ then we define its transform to be

$$b_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} a_k$$

which has generating function

$$\frac{1}{1-x}\mathcal{A}(\frac{x^2}{(1-x)^2})$$

The opening assertion concerning the Motzkin numbers follows from the fact that

$$\frac{1}{1-x}\mathrm{c}\left(\frac{x^2}{(1-x)^2}\right)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$$

which is the generating function of the Motzkin numbers.

The effect of this transform on the power sequences $n \to k^n$ is easy to describe. We have

$$\frac{1}{1-kx} \to \frac{1-x}{1-2x-(k-1)x^2}$$

In other words, the sequences $n \to k^n$ are mapped to the solutions of the one parameter family of recurrences

$$a_n = 2a_{n-1} + (k-1)a_{n-2}$$

satisfying $a_0 = 1$, $a_1 = 1$.

For instance,

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^k = 1, 1, 3, 7, 17, \ldots = ((1+\sqrt{2})^n + (1-\sqrt{2})^n)/2 = [\text{A001333}]$$

A consequence of this is the following formula for the Pell numbers [A000129]

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k+1}2^k = \mathrm{Pell}(n)$$

Under this mapping, the central binomial numbers $\binom{2n}{n}$ are mapped to the central trinomial numbers [A002426] since

$$\frac{1}{\sqrt{1-4x}} \to \frac{1}{1-x}\frac{1}{\sqrt{1-4x^2/(1-x)^2}} = \frac{1}{\sqrt{1-2x-3x^2}}$$

This is an interesting result, as the central trinomial numbers are also the inverse binomial transform of the central binomial numbers :

$$\frac{1}{\sqrt{1-4x}} \to \frac{1}{1+x}\frac{1}{\sqrt{1-4x/(1+x)}} = \frac{1}{\sqrt{1-2x-3x^2}}$$

This transformation can be represented by the 'generalized' Riordan array $(\frac{1}{1-x}, \frac{x^2}{(1-x)^2})$. As such, it possesses two interesting factorizations. Firstly, we have

$$\left(\frac{1}{1-x}, \frac{x^2}{(1-x)^2}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)(1, x^2) = \mathbf{Bin} \circ (1, x^2)$$

Thus the effect of this transform is to 'aerate' a sequence with interpolated zeros and then follow this with a binomial transform. This is obvious from the following identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} a_k = \sum_{k=0}^{n} \binom{n}{k} \frac{1+(-1)^k}{2} a_{k/2}$$

Secondly, we have

$$\left(\frac{1}{1-x}, \frac{x^2}{(1-x)^2}\right) = \left(\frac{1-x}{1-2x+2x^2}, \frac{x^2}{1-2x+2x^2}\right)\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$$

As pointed out in [1], this transformation possesses a left inverse. Using the methods of [1] or otherwise, it is easy to see that the stretched Riordan array $(1, x^2)$ has left inverse $(1, \sqrt{x})$. Hence the first factorization yields

$$\left(\frac{1}{1-x}, \frac{x^2}{(1-x)^2}\right)^{\sim 1} = (\mathbf{Bin} \circ (1, x^2))^{\sim 1} = (1, \sqrt{x}) \circ \mathbf{Bin}^{-1} =$$

$$(1, \sqrt{x}) \left(\frac{1}{1+x}, \frac{x}{1+x}\right) = \left(\frac{1}{1+\sqrt{x}}, \frac{\sqrt{x}}{1+\sqrt{x}}\right)$$

where we have used $(.)^{\sim 1}$ to denote left-inverse.

It is instructive to represent these transformations by their general terms. We look at $(1, x^2)$ first. We have

$$[x^n](x^2)^k = [x^n]x^{2k} = [x^n]\sum_{j=0}^{\infty} \binom{0}{j} x^{2k+j} = \binom{0}{n-2k}$$

Hence

$$b_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} a_k = \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{j} \binom{0}{j-2k} a_j$$

We now wish to express b_n in terms of a_n. We have

$$[x^n](\sqrt{x})^k = [t^{2n}]t^k = [t^{2n}]\sum_{j=0}^{\infty} \binom{0}{j} t^{k+j} = \binom{0}{2n-k}$$

Hence

$$a_n = \sum_{k=0}^{2n} \sum_{j=0}^{k} \binom{0}{2n-k} \binom{k}{j} (-1)^{k-j} b_j$$

We can use the methods of [1] to further elucidate the relationship between $(1, x^2)$ and $(1, \sqrt{x})$. Letting $b(x)$ be the generating function of the image of the sequence a_n under $(1, x^2)$, we see that $b(x) = a(x^2)$ where

$a(x) = \sum_{k=0}^{\infty} a_k x^k$. We wish to find the general term a_n in terms of the b_n. We have $a(x) = [b(t)|x = t^2]$ and so

$$a_n = [x^n]a(x) = [x^n](b(t)|t = \sqrt{x}) = [w^{2n}](b(t)|t = w) =$$

$$\frac{1}{2n}[t^{2n-1}](b'(t))^{2n} = \frac{1}{2n}2n.b_{2n} = b_{2n}$$

The following table displays a list of sequences and their transforms under this transformation. Note that by the above, we can recover the original sequence a_n by taking every second element of the inverse binomial transform of the transformed sequence b_n.

Table 3. Transform pairs

a_n	b_n	a_n	b_n
0^n	1^n	A000007	1^n
1^n	$(2^n + 0^n)/2$	1^n	A011782
2^n	$\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n)}{2}$	A000079	A001333
n	$n2^{n-3} - \frac{\binom{1}{n}-\binom{0}{n}}{4}$	A001477	-
$n+1$	G.f. $\frac{(1-x)^3}{(1-2x)^2}$	A000027	A045891
$2n+1$	$2^n(n+2)/2$, $n > 1$	A005408	A087447
$(1+(-1)^n)/2$	$\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k}$	A059841	A038503
$\binom{2n}{n}$	G.f.$\frac{1}{\sqrt{1-2x-3x^2}}$	A000984	A002426
C(n)	Motzkin	A000108	A001006

14 Combining transformations

We finish this note by briefly looking at the effect of combining transformations. For this, we will take the Fibonacci numbers as an example. We look at two combinations : the Catalan transform followed by the binomial transform, and the Catalan transform followed by the inverse binomial transform. For the former, we have

$$\mathbf{Bin} \circ \mathbf{Cat} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)(1, x\mathrm{c}(x)) = \left(\frac{1}{1-x}, \frac{x}{1-x}\mathrm{c}(\frac{x}{1-x})\right)$$

while for the latter we have

$$\mathbf{Bin}^{-1} \circ \mathbf{Cat} = \left(\frac{1}{1+x}, \frac{x}{1+x}\right)(1, x\mathrm{c}(x)) = \left(\frac{1}{1+x}, \frac{x}{1+x}\mathrm{c}(\frac{x}{1+x})\right)$$

Applying the first combined transformation to the Fibonacci numbers yields the sequence $0, 1, 4, 15, 59, 243, \ldots$ with generating function

$$\frac{(\sqrt{5x-1} - \sqrt{x-1})}{2((x-1)\sqrt{5x-1} - x\sqrt{x-1}))}$$

or

$$\frac{\sqrt{1-6x+5x^2}-(1-5x+4x^2)}{2(1-x)(1-6x+4x^2)}$$

Applying the second combined transformation (Catalan transform followed by the inverse binomial transform) to the Fibonacci numbers we obtain the sequence $0, 1, 0, 3, 3, 13, 26, \ldots$ with generating function

$$\frac{(1+2x)\sqrt{1-2x-3x^2}-(1-x-2x^2)}{2(1+x)(1-2x-4x^2)}$$

It is instructive to reverse these transformations. Denoting the first by $\mathbf{Bin} \circ \mathbf{Cat}$ we wish to look at $(\mathbf{Bin} \circ \mathbf{Cat})^{-1} = \mathbf{Cat}^{-1} \circ \mathbf{Bin}^{-1}$. As elements of the Riordan group, we obtain

$$(1, x(1-x))\left(\frac{1}{1+x}, \frac{x}{1+x}\right) = \left(\frac{1}{1+x-x^2}, \frac{x(1-x)}{1+x-x^2}\right)$$

Applying the inverse transformation to the family of functions k^n with generating functions $\frac{1}{1-kx}$, for instance, we obtain

$$\frac{1}{1+x-x^2}\frac{1}{1-\frac{kx(1-x)}{1+x-x^2}} = \frac{1}{1-(k-1)x+(k-1)x^2}$$

In other words, the transformation $(\mathbf{Bin} \circ \mathbf{Cat})^{-1}$ takes a power k^n and maps it to the solution of the recurrence

$$a_n = (k-1)a_{n-1} - (k-1)a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = k-1$. In particular, it takes the constant sequence 1^n to 0^n.

The Jacobsthal numbers $\mathrm{J}(n)$, for instance, are transformed into the sequence with ordinary generating function $\frac{x(1-x)}{1+x-3x^2+4x^3-2x^4}$ with general term

$b_n = \sum_{k=0}^{n} \binom{k}{n-k} \sum_{j=0}^{k} \binom{k}{j}(-1)^{n-j}\mathrm{J}(j)$.

We have $b_n = 2\sqrt{3}\sin(\pi n/3+\pi/3)/9 - \frac{\sqrt{3}}{18}\left\{(\sqrt{3}-1)^{n+1} - (-1)^n(\sqrt{3}+1)^{n+1}\right\}$.

We now look at $(\mathbf{Bin}^{-1} \circ \mathbf{Cat})^{-1} = \mathbf{Cat}^{-1} \circ \mathbf{Bin}$. As elements of the Riordan group, we obtain

$$(1, x(1-x))(\frac{1}{1-x}, \frac{x}{1-x}) = \left(\frac{1}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2}\right)$$

Applying the inverse transformation to the family of functions k^n with generating functions $\frac{1}{1-kx}$, for instance, we obtain

$$\frac{1}{1-x+x^2}\frac{1}{1-\frac{kx(1-x)}{1-x+x^2}} = \frac{1}{1-(k+1)x+(k+1)x^2}$$

Thus the transformation $(\mathbf{Bin}^{-1} \circ \mathbf{Cat})^{-1}$ takes a power k^n and maps it to the solution of the recurrence

$$a_n = (k+1)a_{n-1} - (k+1)a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = k+1$. In particular, it takes the constant sequence 1^n to $2^{\frac{n}{2}}(\cos(\frac{\pi n}{4}) + \sin(\frac{\pi n}{4}))$ (the inverse Catalan transform of 2^n).

As a final example, we apply the combined transformation $\mathbf{Cat}^{-1} \circ \mathbf{Bin}$ to the Fibonacci numbers. We obtain the sequence

$$0, 1, 2, 2, 0, -5, -13, -21, -21, 0, 55, 144, 233, 233, 0, -610, -1597, \ldots$$

whose elements would appear to be Fibonacci numbers. This sequence has generating function

$$\frac{x(1-x)}{1-3x+4x^2-2x^3+x^4}$$

In closed form, the general term of the sequence is

$$\phi^n \sqrt{\frac{2}{5} + \frac{2\sqrt{5}}{25}} \sin(\frac{\pi n}{5} + \frac{\pi}{5}) - (\frac{1}{\phi})^n \sqrt{\frac{2}{5} - \frac{2\sqrt{5}}{25}} \sin(\frac{2\pi n}{5} + \frac{2\pi}{5})$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

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