The Hankel Transform of the Sum of Consecutive Generalized Catalan Numbers

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Abstract. *We discuss the properties of the Hankel transformation of a sequence whose elements are the sums of consecutive generalized Catalan numbers and find their values in the closed form.*

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1. Introduction

The *Hankel transform* of a given sequence $A = \{a_0, a_1, a_2, ...\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ (see Layman [7]) where $h_n = |a_{i+j-2}|_{i,j=1}^{n}$, i.e

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \to \quad h = \{h_n\}_{n\in\mathbb{N}_0} : \quad h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix} \tag{1}$$

In this paper, we will consider the sequence of the sums of two adjacent generalized Catalan numbers with parameter L:

$$a_0 = L+1, \qquad a_n = a_n(L) = c(n;L) + c(n+1;L) \qquad (n \in \mathbb{N}), \tag{2}$$

where

$$c(n;L) = T(2n,n;L) - T(2n,n-1;L), \tag{3}$$

with

$$T(n,k;L) = \sum_{j=0}^{n-k} \binom{k}{j}\binom{n-k}{j} L^j. \tag{4}$$

Example 1.1. Let $L = 1$. Vandermonde's convolution identity implies that

$$\binom{n}{k} = \sum_j \binom{k}{j}\binom{n-k}{j}.$$

Hence

$$T(2n, n; 1) = \binom{2n}{n}, \qquad T(2n, n-1; 1) = \binom{2n}{n-1},$$

wherefrom we get Catalan numbers

$$c(n) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

and

$$a_n = c(n) + c(n+1) = \frac{(2n)!(5n+4)}{n!(n+2)!} \qquad (n = 0, 1, 2, \ldots).$$

In the paper [3], A. Cvetković, P. Rajković and M. Ivković have proved that the Hankel transform of a_n equals sequence of Fibonacci numbers with odd indices

$$h_n = F_{2n+1} = \frac{1}{\sqrt{5}\ 2^{n+1}}\left\{(\sqrt{5}+1)(3+\sqrt{5})^n + (\sqrt{5}-1)(3-\sqrt{5})^n\right\}.$$

Example 1.2. For $L = 2$ we get like $a_n(2)$ the next numbers

$$3, 8, 28, 112, 484, \ldots,$$

and the Hankel transform h_n:

$$3, 20, 272, 7424, 405504, \ldots.$$

One of us, P. Barry conjectured that

$$h_n(2) = 2^{\frac{n^2-n}{2}-2}\left\{(2+\sqrt{2})^{n+1} + (2-\sqrt{2})^{n+1}\right\}.$$

In general, P. Barry made the conjecture, which we will prove through this paper.

Theorem 1.1. (The main result) *For the generalized Pascal triangle associated to the sequence $n \mapsto L^n$, the Hankel transform of the sequence*

$$c(n; L) + c(n+1; L)$$

is given by

$$\begin{aligned} h_n = {} & \frac{L^{(n^2-n)/2}}{2^{n+1}\sqrt{L^2+4}}\cdot \\ & \left\{(\sqrt{L^2+4}+L)(\sqrt{L^2+4}+L+2)^n + (\sqrt{L^2+4}-L)(L+2-\sqrt{L^2+4})^n\right\}. \end{aligned} \tag{5}$$

From now till the end, let us denote by

$$\xi = \sqrt{L^2+4}, \quad t_1 = L+2+\xi, \quad t_2 = L+2-\xi. \tag{6}$$

Now, we can write

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot \left((\xi+L)t_1^n + (\xi-L)t_2^n\right).$$

Or, introducing

$$\varphi_n = t_1^n + t_2^n, \quad \psi_n = t_1^n - t_2^n \qquad (n \in \mathbb{N}_0), \tag{7}$$

the final statement can be expressed by

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot \left(L\psi_n + \xi\varphi_n\right). \tag{8}$$

Lemma 1.1. *The values φ_n and ψ_n satisfy the next relations*

$$\varphi_j \cdot \varphi_k = \varphi_{j+k} + (4L)^j \varphi_{k-j}, \quad \psi_j \cdot \psi_k = \varphi_{j+k} - (4L)^j \varphi_{k-j} \qquad (0 \le j \le k) \tag{9}$$

$$\varphi_j \cdot \psi_k = \psi_{j+k} + (4L)^j \psi_{k-j}, \qquad \psi_j \cdot \varphi_k = \psi_{j+k} - (4L)^j \psi_{k-j} \qquad (0 \le j \le k). \tag{10}$$

Corollary 1.1. *Assuming that the main theorem is true, the function $h_n = h_n(L)$ is the next polynomial*

$$h_n(L) = 2^{-n} L^{n(n-1)/2}$$
$$\cdot \left\{ \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L+2)^{n-2i-1}(L^2+4)^i + \sum_{i=0}^{[n/2]} \binom{n}{2i} (L+2)^{n-2i}(L^2+4)^i \right\}.$$

Proof. By previous notation, we can write

$$\begin{aligned}
&(L+\xi)(L+2+\xi)^n - (L-\xi)(L+2-\xi)^n \\
&= (L+\xi)\sum_{k=0}^{n} \binom{n}{k}(L+2)^{n-k}\xi^k - (L-\xi)\sum_{k=0}^{n}(-1)^k\binom{n}{k}(L+2)^{n-k}\xi^k \\
&= \sum_{k=0}^{n}\left(1-(-1)^k\right)\binom{n}{k}L(L+2)^{n-k}\xi^k + \sum_{k=0}^{n}\left(1+(-1)^k\right)\binom{n}{k}(L+2)^{n-k}\xi^{k+1} \\
&= 2\sum_{i=0}^{[(n-1)/2]}\binom{n}{2i+1}L(L+2)^{n-2i-1}\xi^{2i+1} + 2\sum_{i=0}^{[n/2]}\binom{n}{2i}(L+2)^{n-2i}\xi^{2i+1} \\
&= 2\xi\left\{\sum_{i=0}^{[(n-1)/2]}\binom{n}{2i+1}L(L+2)^{n-2i-1}\xi^{2i} + \sum_{i=0}^{[n/2]}\binom{n}{2i}(L+2)^{n-2i}\xi^{2i}\right\},
\end{aligned}$$

wherefrom immediately follows the polynomial expression for h_n.□

2. The generating function for the sequences of numbers and orthogonal polynomials

The Jacobi polynomials are given by

$$P_n^{(a,b)}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+a}{k}\binom{n+b}{n-k}(x-1)^{n-k}(x+1)^k \quad (a,b>-1).$$

Also, they can be written in the form

$$P_n^{(a,b)}(x) = \left(\frac{x-1}{2}\right)^n\sum_{k=0}^{n}\binom{n+a}{k}\binom{n+b}{n-k}\left(\tfrac{x+1}{x-1}\right)^k.$$

From the fact

$$L=\frac{x+1}{x-1} \quad\Leftrightarrow\quad x=\frac{L+1}{L-1} \qquad (x\neq 1,\ L\neq 1).$$

we conclude that

$$T(2n,n;L) = (L-1)^n\cdot P_n^{(0,0)}\left(\tfrac{L+1}{L-1}\right),$$
$$T(2n+2,n;L) = (L-1)^n\cdot P_n^{(2,0)}\left(\tfrac{L+1}{L-1}\right).$$

The generating function $G(x,t)$ for the Jacobi polynomials is

$$G^{(a,b)}(x,t)=\sum_{n=0}^{\infty}P_n^{(a,b)}(x)t^n=\frac{2^{a+b}}{\phi\cdot(1-t+\phi)^a\cdot(1+t+\phi)^b}, \tag{11}$$

where

$$\phi=\phi(x,t)=\sqrt{1-2xt+t^2}.$$

Now,

$$\sum_{n=0}^{\infty}T(2n,n;L)\ t^n = \sum_{n=0}^{\infty}P_n^{(0,0)}\left(\tfrac{L+1}{L-1}\right)\big((L-1)t\big)^n = G^{(0,0)}\left(\tfrac{L+1}{L-1},(L-1)t\right),$$
$$\sum_{n=0}^{\infty}T(2n+2,n;L)\ t^n = \sum_{n=0}^{\infty}P_n^{(2,0)}\left(\tfrac{L+1}{L-1}\right)\big((L-1)t\big)^n = G^{(2,0)}\left(\tfrac{L+1}{L-1},(L-1)t\right).$$

Also,

$$\sum_{n=0}^{\infty}T(2n,n-1;L)\ t^n = t\cdot\left\{G^{(2,0)}\left(\tfrac{L+1}{L-1},(L-1)t\right)-1\right\},$$
$$\sum_{n=0}^{\infty}T(2n+2,n+1;L)\ t^n = \frac{1}{t}\cdot\left\{G^{(0,0)}\left(\tfrac{L+1}{L-1},(L-1)t\right)-1\right\}.$$

The generating function $\mathcal{G}(t;L)$ for the sequence $\{a_n\}_{n\geq 0}$ is given by

$$\begin{aligned}\mathcal{G}(t;L) &= \sum_{n=0}^{\infty}a_nt^n\\ &= \frac{t+1}{t}G^{(0,0)}\left(\tfrac{L+1}{L-1},(L-1)t\right)-(t+1)G^{(2,0)}\left(\tfrac{L+1}{L-1},(L-1)t\right)-\frac{1}{t}.\end{aligned} \tag{12}$$

After some computation, we prove the next theorem.

Theorem 2.1. *The generating function* $\mathcal{G}(t;L)$ *for the sequence* $\{a_n\}_{n\geq 0}$ *is*

$$\mathcal{G}(t;L) = \frac{t+1}{\rho(t;L)}\ \left\{\frac{1}{t} - \frac{4}{(1-(L-1)t+\rho(t;L))^2}\right\} - \frac{1}{t}, \tag{13}$$

where

$$\rho(t;L) = \phi\Big(\tfrac{L+1}{L-1}, (L-1)t\Big) = \sqrt{1-2(L+1)t+(L-1)^2t^2} \tag{14}$$

The function $\rho(t;L)$ has domain

$$D_\rho = \left(-\infty, \frac{1-2\sqrt{L}+L}{1-2L+L^2}\right) \cup \left(\frac{1+2\sqrt{L}+L}{1-2L+L^2}, +\infty\right) \qquad (L\neq 1),$$

and

$$D_\rho = (-\infty, 1/4) \qquad (L=1).$$

Example 2.1. For $L=1$, we get

$$\mathcal{G}(t;1) = \sum_{n=0}^{\infty} a_n(1)\ t^n = \frac{1}{t}\left(\frac{(1-\sqrt{1-4t})(1+t)}{2t} - 1\right). \tag{15}$$

and for $L=2$, we find

$$\mathcal{G}(t;2) = \sum_{n=0}^{\infty} a_n(2)\ t^n = -\frac{1}{t} + \frac{t+1}{\sqrt{t^2-6t+1}}\left\{\frac{1}{t} - \frac{4}{(1-t+\sqrt{t^2-6t+1})^2}\right\}. \tag{16}$$

3. The weight function corresponding to the functional

It is known (for example, see Krattenthaler [6]) that the Hankel determinant h_n of order n of the sequence $\{a_n\}_{n\geq 0}$ equals

$$h_n = a_0^n \beta_1^{n-1}\beta_2^{n-2}\cdots\beta_{n-2}^2\beta_{n-1}, \tag{17}$$

where $\{\beta_n\}_{n\geq 1}$ is the sequence given by:

$$\mathcal{G}(x) = \sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1+\alpha_0 x - \cfrac{\beta_1 x^2}{1+\alpha_1 x - \cfrac{\beta_2 x^2}{1+\alpha_2 x - \cdots}}} \tag{18}$$

The sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 1}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x) = (x-\alpha_n)Q_n(x) - \beta_n Q_{n-1}(x), \tag{19}$$

where $\{Q_n(x)\}_{n\geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional $\mathcal{U}$ determined by

$$\mathcal{U}[x^n] = a_n \quad (n=0,1,2,\ldots). \tag{20}$$

In this section the functional will be constructed for the sum of consecutive generalized Catalan numbers.

We would like to express $\mathcal{U}[f]$ in the form:

$$\mathcal{U}[f(x)] = \int_R f(x) d\psi(x),$$

where $\psi(x)$ is a distribution, or, even more, to find the weight function $w(x)$ such that $w(x) = \psi'(x)$.

Denote by $F(z; L)$ the function

$$F(z; L) = \sum_{k=0}^{\infty} a_k z^{-k-1}.$$

From the generating function (13), we have:

$$F(z; L) = z^{-1}\ \mathcal{G}\big(z^{-1}; L\big). \tag{21}$$

and after some simplifications we obtain that

$$\begin{aligned} F(z; L) &= -1 + \frac{2(z+1)}{L - 1 + z + \sqrt{L^2 + (z-1)^2 - 2L(z+1)}} \\ &= -1 + \frac{2(z+1)}{L - 1 + z(1 + z\rho(\frac{1}{z}, L))} \end{aligned}$$

Example 3.1. From (15) and (16), we yield

$$\begin{aligned} F(z; 1) &= z^{-1}\ \mathcal{G}\big(z^{-1}; 1\big) = \frac{1}{2}\left\{ z - 1 - (z+1)\sqrt{1 - \frac{4}{z}} \right\}, \\ F(z; 2) &= \frac{-1}{2z}\left\{ 1 + z\left(2 - z + (z+1)\sqrt{1 - \frac{6}{z} + \frac{1}{z^2}} \right) \right\}. \end{aligned}$$

Notice that

$$\begin{aligned} \int F(z; 2) dz = z + \frac{1}{4} z(z-1)\rho(1/z, 2) + \log(z) \\ - \frac{1}{2}\log\Big(1 + z\big(\rho(1/z, 2) - 3\big)\Big) - \frac{7}{2}\log\big(z - 3 + z\rho(1/z, 2)\big). \end{aligned}$$

It will be the impulse for further discussion.

Denote by

$$R(z; L) = z\rho(\frac{1}{z}, L) = \sqrt{L^2 + (z-1)^2 - 2L(z+1)}.$$

From the theory of distribution functions (see Chihara [2]), especially by the Stieltjes inversion formula

$$\psi(t) - \psi(0) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_0^t \Im F(x + iy; L) dx, \tag{22}$$

we conclude that holds

$$\mathcal{F}(z;L)=\int F(z;L)dz=\frac{1}{4}\Big[z^2-2Lz-(z-L+1)R(z;L)-l_1(z)+l_2(z)\Big], \quad (23)$$

where

$$\begin{aligned} l_1(z) &= 2(3L+1)\log\Big[z-(L+1)+R(z;L)\Big] \\ l_2(z) &= 2(L-1)\log\Big[\frac{-(L-1)R(z;L)-(L-1)^2+z(L+1)}{z^2(L-1)^3}\Big] \end{aligned}$$

Rewriting the function $R(z;L)$ in the form

$$R(z;L)=\sqrt{(z-L-1)^2-4L}$$

and replacing $z=x+iy$, we have

$$R(x;L)=\lim_{y\to 0^+}R(x+iy;L)=\begin{cases} i\sqrt{4L-(x-L-1)^2}, & x\in(a,b);\\ \sqrt{(x-L-1)^2-4L}, & otherwise,\end{cases}$$

where

$$a=(\sqrt{L}-1)^2, \qquad b=(\sqrt{L}+1)^2. \quad (24)$$

In the case when $x\notin\Big((\sqrt{L}-1)^2,(\sqrt{L}+1)^2\Big)$, value $R(x;L)$ is real. Therefore we can calculate imaginary part of $\mathcal{F}(x;L)=\lim_{y\to0^+}\mathcal{F}(x+iy;L)$:

$$\Im\mathcal{F}(x;L)=\Im[l_2(x)-l_1(x)]=0.$$

Otherwise, if $x\in\Big((\sqrt{L}-1)^2,(\sqrt{L}+1)^2\Big)$ we have that:

$$\begin{aligned} l_1(x) &= 2(3L+1)\log\Big[x-(L+1)\pm i\sqrt{4L-(x-L-1)^2}\Big] \\ \Im l_1(x) &= \begin{cases} 2(3L+1)\arctan\frac{\sqrt{4L-(x-L-1)^2}}{x-(L+1)}, & x\ge L+1;\\ 2(3L+1)\Big(\pi+\arctan\frac{\sqrt{4L-(x-L-1)^2}}{x-(L+1)}\Big), & x<L+1\end{cases} \\ l_2(x) &= 2(L-1)\log\Big[\frac{-(L-1)^2+2x(L+1)-i(L-1)\sqrt{4L-(x-L-1)^2}}{x^2(L-1)^3}\Big] \\ \Im l_1(x) &= \begin{cases} 2(L-1)\Big(2\pi+\arctan\frac{x(L+1)-(L-1)^2}{\sqrt{4L-(x-L-1)^2}}\Big), & x\ge\frac{(L-1)^2}{L+1};\\ 2(L-1)\Big(\pi+\arctan\frac{x(L+1)-(L-1)^2}{\sqrt{4L-(x-L-1)^2}}\Big), & x<\frac{(L-1)^2}{L+1}\end{cases} \end{aligned}$$

After substituting all considered cases in (23), we finally obtain the value

$$\Im\mathcal{F}(x;L)=\lim_{y\to0^+}\Im\mathcal{F}(x+iy;L)=\Im l_2(x)-\Im l_1(x)-(x-L+1)\sqrt{4L-(x-L-1)^2}$$

From the relation (22), we conclude that

$$\omega(x;L)=\psi'(x;L)=-\frac{1}{\pi}\frac{d}{dx}\Im\mathcal{F}(x;L) \quad (25)$$

and finally, we obtain

$$\omega(x;L)=\frac{1}{2\pi}(1+\frac{1}{x})\sqrt{4L-(x-L-1)^2}=\frac{\sqrt{L}}{\pi}\Big(1+\frac{1}{x}\Big)\sqrt{1-\Big(\frac{x-L-1}{2\sqrt{L}}\Big)^2}\tag{26}$$

Previous formula holds for $x\in(a,b)$, and otherwise is $\omega(x;L)=0$.

4. Determining the three-term recurrence relation

The crucial moment in our proof of the conjecture is to determine the sequence of polynomials $\{Q_n(x)\}$ orthogonal with respect to the weight $w(x;L)$ given by (26) on the interval (a,b) and to find the sequences $\{\alpha_n\}$ $\{\beta_n\}$ in the three-term recurrence relation.

Example 4.1. For $L=4$, we can find the first members

$$\begin{aligned} Q_0(x)&=1, && \|Q_0\|^2=5,\\ Q_1(x)&=x-\frac{24}{5}, && \|Q_1\|^2=\frac{104}{5},\\ Q_2(x)&=x^2-\frac{127}{13}x+\frac{256}{13}, && \|Q_2\|^2=\frac{1088}{13},\\ Q_3(x)&=x^3-\frac{541}{17}x^2+\frac{1096}{17}x-\frac{1344}{17}, && \|Q_3\|^2=\frac{5696}{17},\end{aligned}$$

wherefrom

$$\alpha_0=\frac{24}{5},\quad \beta_0=5,\qquad \alpha_1=\frac{323}{65},\quad \beta_1=\frac{104}{25},\qquad \alpha_2=\frac{1104}{221},\quad \beta_2=\frac{680}{169}.$$

Hence

$$h_1=a_0=5,\quad h_2=a_0^2\beta_1\ =104,\qquad h_3=a_0^3\beta_1^2\beta_2=5^3\left(\frac{104}{25}\right)^2\frac{680}{169}=8704.$$

At the beginning, we will notice that in the definition of the weight function appears the square root member.

That's why, let us consider the monic orthogonal polynomials $\{S_n(x)\}$ with respect to the $p^{(1/2,1/2)}(x)=\sqrt{1-x^2}$ on the interval $(-1,1)$. These polynomials are monic Chebyshev polynomials of the second kind:

$$S_n(x)=\frac{\sin\big((n+1)\arccos x\big)}{2^n\cdot\sqrt{1-x^2}}$$

They satisfy the three-term recurrence relation (Chihara [2]):

$$S_{n+1}(x)=(x-\alpha_n^*)\ S_n(x)-\beta_n^*S_{n-1}(x)\quad(n=0,1,\ldots),\tag{27}$$

with initial values

$$S_{-1}(x)=0,\qquad S_0(x)=1,$$

where

$$\alpha_n^*=0\quad(n\ge 0)\qquad\text{and}\qquad \beta_0^*=\frac{\pi}{2},\quad \beta_n^*=\frac{1}{4}\qquad(n\ge 1).$$

If we use the weight function $\hat{w}(x) = (x-c)\ p^{(1/2,1/2)}(x)$, then the corresponding coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ can be evaluated as follows (see, for example, Gautschi [4])

$$\begin{aligned} \lambda_n &= S_n(c), \\ \hat{\alpha}_n &= c - \frac{\lambda_{n+1}}{\lambda_n} - \beta^*_{n+1}\frac{\lambda_n}{\lambda_{n+1}}, \\ \hat{\beta}_n &= \beta^*_n \frac{\lambda_{n-1}\lambda_{n+1}}{\lambda_n^2} \qquad\qquad (n \in \mathbb{N}_0). \end{aligned} \tag{28}$$

From the relation (27), we conclude that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfies the following recurrence relation:

$$4\lambda_{n+1} - 4c\lambda_n + \lambda_{n-1} = 0 \qquad (\lambda_{-1} = 0;\ \lambda_0 = 1). \tag{29}$$

The characteristic equation

$$4z^2 - 4\ c\ z + 1 = 0$$

has the solutions

$$z_{1,2} = \frac{1}{2}\Big(c \pm \sqrt{c^2-1}\Big).$$

and the integral solution of (29) is

$$\lambda_n = E_1 z_1^n + E_2 z_2^n \qquad (n \in \mathbb{N}).$$

We evaluate values E_1 and E_2 from the initial conditions $(\lambda_{-1} = 0;\ \lambda_0 = 1)$.

In other to solve our problem, we will choose $c = -\frac{L+2}{2\sqrt{L}}$. Hence

$$z_k = \frac{-t_k}{4\sqrt{L}} \quad (k = 1,2), \quad \text{where} \quad t_{1,2} = L + 2 \pm \sqrt{L^2+4}.$$

Finally, we obtain:

$$\lambda_n = \frac{(-1)^n}{2\cdot 4^n L^{\frac{n}{2}} \sqrt{L^2+4}}\Big(t_1^{n+1} - t_2^{n+1}\Big) \quad (\lambda = -1, 0, 1, \ldots),$$

i.e,

$$\lambda_n = \frac{(-1)^n}{2\cdot 4^n L^{\frac{n}{2}} \xi}\ \psi_{n+1} \quad (\lambda = -1, 0, 1, \ldots).$$

After replacing in (28), we obtain:

$$\hat{\alpha}_n = -\frac{L+2}{2\sqrt{L}} + \frac{1}{4\sqrt{L}}\cdot\frac{\psi_{n+2}}{\psi_{n+1}} + \sqrt{L}\cdot\frac{\psi_{n+1}}{\psi_{n+2}}, \tag{30}$$

$$\hat{\beta}_n = \frac{\psi_n \psi_{n+2}}{4\psi_{n+1}^2}. \tag{31}$$

If a new weight function $\tilde{w}(x)$ is introduced by

$$\tilde{w}(x) = \hat{w}(ax+b)$$

then we have

$$\tilde{\alpha}_n = \frac{\hat{\alpha}_n - b}{a}, \qquad \tilde{\beta}_n = \frac{\hat{\beta}_n}{a^2} \qquad (n \geq 0).$$

Now, by using $x \mapsto \frac{x-L-1}{2\sqrt{L}}$, i.e., $a = \frac{1}{2\sqrt{L}}$ and $b = -\frac{L+1}{2\sqrt{L}}$, we have the weight function

$$\tilde{w}(x) = \hat{w}(\frac{x-L-1}{2\sqrt{L}}) = \frac{1}{2}\Big(\frac{x-L-1}{2\sqrt{L}} + \frac{L+2}{2\sqrt{L}}\Big)\sqrt{1-\Big(\frac{x-L-1}{2\sqrt{L}}\Big)^2}.$$

Thus

$$\tilde{\alpha}_n = -1 + \frac{1}{2}\cdot\frac{\psi_{n+2}}{\psi_{n+1}} + 2L\cdot\frac{\psi_{n+1}}{\psi_{n+2}} \qquad (n \in \mathbb{N}_0), \tag{32}$$

and

$$\tilde{\beta}_0 = (L+2)\frac{\pi}{2}, \qquad \tilde{\beta}_n = L\frac{\psi_n\psi_{n+2}}{\psi_{n+1}^2} \quad (n \in \mathbb{N}). \tag{33}$$

Example 4.2. For $L = 4$, we get

$$P_0(x) = 1, \qquad \|P_0\|^2 = 3\pi,$$
$$P_1(x) = x - \frac{17}{3}, \qquad \|P_1\|^2 = \frac{32\pi}{3},$$
$$P_2(x) = x^2 - \frac{43}{4}x + \frac{101}{4}, \qquad \|P_2\|^2 = 42\pi,$$
$$P_3(x) = x^3 - \frac{331}{21}x^2 + \frac{1579}{21}x - \frac{2189}{21}, \quad \|P_3\|^2 = \frac{3520\pi}{21},$$

wherefrom

$$\tilde{\alpha}_0 = \frac{17}{3}, \quad \tilde{\beta}_0 = 3\pi, \qquad \tilde{\alpha}_1 = \frac{61}{12}, \quad \tilde{\beta}_1 = \frac{32}{9}, \qquad \tilde{\alpha}_2 = \frac{421}{84}, \quad \tilde{\beta}_2 = \frac{63}{16}.$$

Introducing the weight

$$\breve{w}(x) = \frac{2L}{\pi}\tilde{w}(x)$$

will not change the monic polynomials and their recurrence relations, only it will multiply the norms by the factor $2L/\pi$, i.e.

$$\breve{P}_k(x) \equiv P_k(x), \qquad \|\breve{P}_k\|^2_{\breve{w}} = \int_a^b \breve{P}_k(x)\breve{w}(x)\,dx = \frac{2L}{\pi}\|P_k\|^2 \qquad (k \in \mathbb{N}_0),$$

$$\breve{\beta}_0 = L(L+2), \quad \breve{\beta}_k = \tilde{\beta}_k \quad (k \in \mathbb{N}), \qquad \breve{\alpha}_k = \tilde{\alpha}_k \qquad (k \in \mathbb{N}_0).$$

Here is

$$\breve{\beta}_0\breve{\beta}_1\cdots\breve{\beta}_{n-1} = \frac{L^n}{2}\cdot\frac{\psi_{n+1}}{\psi_n}. \tag{34}$$

In the book [5], W. Gautschi has treated the next problem: If we know all about the MOPS orthogonal with respect to $\breve{w}(x)$ what can we say about the sequence $\{Q_n(x)\}$ orthogonal with respect to a weight

$$w_d(x) = \frac{\breve{w}(x)}{x-d} \qquad (d \notin \text{support}(\tilde{w}))\ ?$$

W. Gautshi has proved that, by the auxiliary sequence

$$r_{-1} = -\int_{\mathbb{R}} w_d(x)\,dx, \qquad r_n = d - \breve{\alpha}_n - \frac{\breve{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots),$$

it can be determined

$$\begin{aligned} \alpha_{d,0} &= \breve{\alpha}_0 + r_0, & \alpha_{d,k} &= \breve{\alpha}_k + r_k - r_{k-1}, \\ \beta_{d,0} &= -r_{-1}, & \beta_{d,k} &= \breve{\beta}_{k-1}\frac{r_{k-1}}{r_{k-2}} & (k \in \mathbb{N}). \end{aligned}$$

In our case it is enough to take $d = 0$ to get the final weight

$$w(x) = \frac{\breve{w}(x)}{x}.$$

Hence

$$r_{-1} = -(L+1), \qquad r_n = -\left(\breve{\alpha}_n + \frac{\breve{\beta}_n}{r_{n-1}}\right) \quad (n = 0, 1, \ldots). \tag{35}$$

Lemma 4.1. *The parameters r_n have the explicit form*

$$r_n = -\frac{\psi_{n+1}}{\psi_{n+2}} \cdot \frac{L\psi_{n+2} + \xi\varphi_{n+2}}{L\psi_{n+1} + \xi\varphi_{n+1}} \qquad (n \in \mathbb{N}_0). \tag{36}$$

Proof. We will use the mathematical induction. For $n = 0$, we really get the expected value

$$r_0 = -\frac{L^2 + 2L + 2}{(L+1)(L+2)}.$$

Suppose that it is true for $k = n$. Now, by the properties for φ_n and ψ_n, we have

$$\tilde{\alpha}_{n+1} \cdot r_n + \tilde{\beta}_{n+1} = -\frac{\psi_{n+1}}{\psi_{n+3}} \cdot \frac{L\psi_{n+3} + \xi\varphi_{n+3}}{L\psi_{n+1} + \xi\varphi_{n+1}}.$$

Dividing with r_n, we conclude that the formula is valid for r_{n+1}. □

Example 4.3. For $L = 4$, we get

$$r_{-1} = -5, \quad r_0 = -\frac{13}{15}, \quad r_1 = -\frac{51}{52}, \quad r_2 = -\frac{356}{357},$$

wherefrom

$$\alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \qquad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \qquad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169},$$

just the same as in the Example 4.1.

Proof of the main result. The Krattenthaler's formula (17) can be also written in the form

$$h_1 = a_0, \qquad h_n = \beta_0\beta_1\beta_2\cdots\beta_{n-2}\beta_{n-1} \cdot h_{n-1}. \tag{37}$$

From the theory of orthogonal polynomials, it is known that

$$\|Q_{n-1}\|^2 = \beta_0\beta_1\beta_2\cdots\beta_{n-2}\beta_{n-1} \quad (n = 2, 3, \ldots), \tag{38}$$

wherefrom

$$h_1 = a_0, \qquad h_n = \|Q_{n-1}\|^2 \cdot h_{n-1} \quad (n = 2, 3, \ldots). \tag{39}$$

Here,

$$\|Q_{n-1}\|^2 = \beta_0 \frac{r_{n-2}}{r_{-1}} \prod_{k=0}^{n-2} \breve{\beta}_k = \frac{L^{n-1}}{2} \cdot \frac{L\psi_n + \xi\varphi_n}{L\psi_{n-1} + \xi\varphi_{n-1}}. \tag{40}$$

We will apply the mathematical induction again. The formula for h_n is true for $n = 1$. Suppose that it is valid for $k = n - 1$. Then

$$h_n = \frac{L^{n-1}}{2} \cdot \frac{L\psi_n + \xi\varphi_n}{L\psi_{n-1} + \xi\varphi_{n-1}} \cdot \frac{L^{(n-1)(n-2)/2}}{2^n \xi} \cdot (L\psi_{n-1} + \xi\varphi_{n-1}),$$

wherefrom it follows that the final statement

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot (L\psi_n + \xi\varphi_n) \qquad (n \in \mathbb{N})$$

is true. □

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