Some conjectures on the ratio of Hankel transforms for sequences and series reversion

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Abstract

For each element of certain families of sequences, we study the term-wise ratios of the Hankel transforms of three sequences related to that element by series reversion. In each case, the ratios define well-known sequences, and in one case, we recover the initial sequence.

1 Introduction

The Hankel transform for sequences (defined below) has attracted an increasing amount of attention in recent years. The paper [4] situated its study within the mainstream of research into integer sequences, while papers such as [2] hinted at how the study of certain Hankel transforms can lead to results concerning classical sequences. That paper exploited a link between continued fractions and the Hankel transform, as explained by Krattenhaler [3]. The best known example of a Hankel transform for sequences is that of the Catalan numbers. One of the earlier contributors to our stock of knowledge about the Hankel transform, Christian Radoux, had published several proofs of this result, along with other interesting examples [7], [8],[9],[10],[11]. One should also note the interesting umbral interpretation of the Hankel transform given in [13]. In this paper we indicate that the term-wise ratio of Hankel transforms of shifted sequences are noteworthy objects of study, giving us more insight into the processes involved in the Hankel transform.

2 Integer Sequences and Transforms on them

In this note, we shall consider integer sequences

$$a : \mathbf{N}_0 \to \mathbf{Z}$$

with general term $a_n = a(n)$. Normally, sequences will be described by their ordinary generating function (o.g.f.), that is, the function $g(x)$ such that

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We shall study the Hankel transform of sequences in this note. This is a transformation on the set of integer sequences defined as follows. Given a sequence $a_n = a(n)$ as described above, we form the $(n+1)\times(n+1)$ matrix H_n with general term $a(i+j)$, where $0 \le i, j \le n$. Then the Hankel transform h_n of the sequence a_n is defined by

$$h_n = \det(H_n).$$

Since the elements of the matrix H_n are elements of an integer sequence, it is clear that h_n is again an integer sequence. We shall see later that this transformation is not invertible.

Example 1. The Catalan numbers $1, 1, 2, 5, 14, 42 \ldots$, defined by $C(n) = \frac{\binom{2n}{n}}{n+1}$ have o.g.f. $\frac{1-\sqrt{1-4x}}{2x}$. The Hankel transform of the Catalan numbers is the sequence of all 1's. Thus each of the determinants

$$|1|, \qquad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{vmatrix}, \qquad \ldots$$

has value 1. A unique feature of the Catalan numbers is that the shifted sequence $C(n+1)$ also has a Hankel transform of all 1's. An interesting feature of the Catalan numbers is that the sequence $C(n) - 0^n$, or $0, 1, 2, 5, 14, 42, \ldots$ has Hankel transform n. Of direct relevance to this note is the fact that the Hankel transform of the sequence $0, 1, 1, 2, 5, 14, 42, \ldots$ with o.g.f. $\frac{1-\sqrt{1-4x}}{2}$ is $-n$. This sequence is defined by the series reversion of the logistic function $x(1-x)$.

Example 2. The central binomial coefficients $1, 2, 6, 20, 70, 252, \ldots$, defined by $a_n = \binom{2n}{n}$, have o.g.f. $\frac{1}{\sqrt{1-4x}}$. The Hankel transform of the central binomial coefficients is given by $h_n = 2^n$. That is,

$$|1| = 1, \qquad \begin{vmatrix} 1 & 2 \\ 6 & 20 \end{vmatrix} = 2, \qquad \begin{vmatrix} 1 & 2 & 6 \\ 2 & 6 & 20 \\ 20 & 70 & 252 \end{vmatrix} = 4, \qquad \ldots$$

The sequence $0, 1, 2, 6, 20, \ldots$ with o.g.f. $\frac{x}{\sqrt{1-4x}}$ has Hankel transform $-n2^{n-1}$. This is the negative of the binomial transform (see below) of n.

An important transformation on integer sequences that is invertible is the so-called *Binomial transform*. Given an integer sequence a_n, this transformation returns the sequence with general term

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

If we consider the sequence with general term a_n to be the vector $\mathbf{a} = (a_0, a_1, \ldots)$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector by the lower-triangle matrix $\mathbf{B}$ whose (n,k)-th element is equal to $\binom{n}{k}$:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & 0 & \dots \\ 1 & 5 & 10 & 10 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The inverse transformation is given by

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k.$$

If we use ordinary generating functions to describe a sequence, then the sequence with o.g.f. $g(x)$ will have a binomial transform whose o.g.f. is given by $\frac{1}{1-x} g(\frac{x}{1-x})$.

It is shown in [4] that if b_n is the binomial transform of the sequence a_n, then both sequences have the same Hankel transform. Thus the Hankel transform is not invertible.

3 On the series reversion of certain families of generating functions of sequences

In this note we shall be concerned mainly with the Hankel transform of sequences whose o.g.f. will be defined as the series reversion of the o.g.f.'s of certain basic sequences. Thus in this section, we will briefly recall facts about the sequences with o.g.f.'s of the forms given by $\frac{x}{1+\alpha x+\beta x^2}$, $\frac{x(1-\alpha x)}{1-\beta x}$ and $x(1-\alpha x)$ as well as their reversions. The first two families have been studied in [1].

Example 3. The family $\frac{x}{1+\alpha x+\beta x^2}$.

The sequence with o.g.f. $\frac{x}{1+\alpha x+\beta x^2}$ has general term given by

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} (-\alpha)^{n-2k} (-\beta)^k.$$

The reversion of the series $\frac{x}{1+\alpha x+\beta x^2}$, that is, the solution $u = u(x)$ to the equation

$$\frac{u}{1+\alpha u+\beta u^2} = x$$

is given by

$$u(x) = \frac{1 - \alpha x - \sqrt{1 - 2\alpha x + (\alpha^2 - 4\beta)x^2}}{2\beta x}.$$

The sequence u_n with this o.g.f. has general term

$$u_n = \sum_{k=0}^{\frac{\lfloor n-1 \rfloor}{2}} \binom{n-1}{2k} C(k) \alpha^{n-2k-1} \beta^k.$$

In this note, we shall be interested in the termwise ratios of the Hankel transforms of the three sequences u_n, $u_n^* = u_{n+1}$ and $u_n^{**} = u_{n+2}$.

We will take the case $\alpha = -3$ and $\beta = -5$ to illustrate our results. Thus let a_n be the sequence with o.g.f. $\frac{x}{1-3x-5x^2}$. Then a_n begins $0, 1, 3, 14, 57, 241, \ldots$ with

$$a_n = \sum_{k=0}^{\frac{\lfloor n-1 \rfloor}{2}} \binom{n-k-1}{k} 3^{n-2k-1} 5^k.$$

The series reversion of $\frac{x}{1-3x-5x^2}$ is $\frac{\sqrt{1+6x+29x^2}-3x-1}{10x}$ which generates the sequence u_n which begins $0, 1, -3, 4, 18, -139, 357, \ldots$ where

$$u_n = \sum_{k=0}^{\frac{\lfloor n-1 \rfloor}{2}} \binom{n-1}{2k} C(k) (-3)^{n-2k-1} (-5)^k.$$

We now form the shifted sequences

$$u_n^* = u_{n+1} = \sum_{k=0}^{\frac{\lfloor n \rfloor}{2}} \binom{n}{2k} C(k) \alpha^{n-2k} \beta^k$$

and

$$u_n^{**} = u_{n+2} = \sum_{k=0}^{\frac{\lfloor n+1 \rfloor}{2}} \binom{n+1}{2k} C(k) \alpha^{n-2k+1} \beta^k.$$

We now let h_n, h_n^* and h_n^{**}, respectively, be the Hankel transforms of these sequences. Numerically, we find that the following:

Sequence	Hankel transform
u_n	$0, -1, -15, 1750, 890625, -2353515625, \ldots$
u_n*	$1, -5, -125, 15625, 9765625, -30517578125, \ldots$
u_n^{**}	$-3, -70, 7125, 3765625, -9843750000, -129058837890625, \ldots$

These results suggest that $h_n^* = (-5)^{\binom{n+1}{2}}$, and

$$\frac{(-1)^{n+1} h_{n+1}}{h_n^*} = a_{n+1}$$

along with

$$\frac{(-1)^{n+1} h_n^{**}}{h_n^*} = a_{n+2}.$$

Thus in this case we obtain

$$h_{n+1} = (-1)^{n+1}(-5)^{\binom{n+1}{2}} a_{n+1} = (-1)^{n+1}(-5)^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} 3^{n-2k} 5^k$$

from which we infer that

$$h_n = (-1)^n (-5)^{\binom{n}{2}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} 3^{n-2k-1} 5^k.$$

Similarly, we find

$$h_n^{**} = (-1)^{n+1}(-5)^{\binom{n+1}{2}} a_{n+2} = (-1)^{n+1}(-5)^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k} 3^{n-2k+1} 5^k.$$

Summarizing, we thus have

Sequence	Hankel transform
u_n	$h_n = (-1)^n(-5)^{\binom{n}{2}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} 3^{n-2k-1} 5^k$
u_n*	$h_n^* = (-5)^{\binom{n+1}{2}}$
u_n^{**}	$h_n^{**} = (-1)^{n+1}(-5)^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k} 3^{n-2k+1} 5^k$

We note that we have been able to recover the sequence a_n in this example. Since the reversion of the reversion of a series is the original series, we can now posit the

Conjecture 4. Let u_n be the sequence

$$u_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C(k) \alpha^{n-2k-1} \beta^k$$

with integer parameters α and β, and o.g.f.

$$u(x) = \frac{1 - \alpha x - \sqrt{1 - 2\alpha x + (\alpha^2 - 4\beta)x^2}}{2\beta x}.$$

Let h_n be the Hankel transform of u_n, h_n^* the Hankel transform of u_{n+1}, and h_n^{**} be the Hankel transform of u_{n+2}. Further, let a_n be the sequence with o.g.f. the series reversion of $u(x)$, with

$$a_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} \alpha^{n-2k-1} \beta^k.$$

Then

1. $h_n^* = \beta^{\binom{n+1}{2}}$

2. $\frac{(-1)^{n+1}h_{n+1}}{h_n^*} = a_{n+1} \Rightarrow h_n = (-1)^n \beta^{\binom{n}{2}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} \alpha^{n-2k-1}\beta^k$

3. $\frac{(-1)^{n+1}h_n^{**}}{h_n^*} = a_{n+2} \Rightarrow h_n^{**} = (-1)^{n+1} \beta^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k} \alpha^{n-2k+1}\beta^k$ $\diamond$

Note that the Hankel transform of u_{n+1}, h_n^* is independent of α. This is due to

1. The binomial transform does not change the Hankel transform, and
2. The binomial transform of

$$\frac{1-\alpha x-\sqrt{1-2\alpha x+(\alpha^2-4\beta)x^2}}{2\beta x^2}$$

is given by

$$\frac{1-(\alpha+1)x-\sqrt{1-2(\alpha+1)x+((\alpha+1)^2-4\beta)x^2}}{2\beta x^2}.$$

Elements in this family are related to coloured Motzkin paths. For other links between lattice paths and Hankel transforms, see [12].

The recovery of the sequence a_n is an interesting feature of this family of sequences. That this is not always the case is illustrated by the next example.

Example 5. The family $\frac{x(1-\alpha x)}{1-\beta x}$ for $\beta \neq 0$.

The sequence with o.g.f. $\frac{x(1-\alpha x)}{1-\beta x}$ has general term given by

$$a_n = (\beta-\alpha)\beta^{n-1} + \frac{\alpha}{\beta}0^n.$$

Here, 0^n is used to denote the sequence beginning $1, 0, 0, 0, \ldots$ with o.g.f. 1. The reversion of the series $\frac{x(1-\alpha x)}{1-\beta x}$, that is, the solution $u = u(x)$ of the equation

$$\frac{u(1-\alpha u)}{1-\beta u} = x$$

is given by

$$u(x) = \frac{1+\beta x-\sqrt{1-(2\alpha-\beta)2x+\beta^2x^2}}{2\alpha}.$$

The sequence u_n with this o.g.f. has general term

$$u_n = \sum_{k=0}^{n-1} \binom{n+k-1}{2k} C(k)\alpha^k(-\beta)^{n-k-1}.$$

Again, we shall be interested in the term-wise ratios of the Hankel transforms h_n, h_n^* and h_n^{**} respectively of the sequences u_n, $u_n^* = u_{n+1}$ and $u_n^{**} = u_{n+2}$. We obtain

Conjecture 6. Using the notation above, we have

1. $h_n^* = (\alpha(\alpha-\beta))^{\binom{n+1}{2}}$

2. $h_{n+1}/h_n^* = \frac{(\alpha-\beta)^{n+1}-\alpha^{n+1}}{\beta} \Rightarrow h_n = \frac{(\alpha-\beta)^n-\alpha^n}{\beta}(\alpha(\alpha-\beta))^{\binom{n}{2}}$

3. $h_n^{**}/h_n^* = (\alpha-\beta)^{n+1} \Rightarrow h_n^{**} = (\alpha-\beta)^{n+1}(\alpha(\alpha-\beta))^{\binom{n+1}{2}}$ $\diamond$

We note that h_{n+1}/h_n^* is the general term of the sequence with o.g.f. $\frac{-1}{(1-\alpha x)(1-(\alpha-\beta)x)}$ while h_n^{**}/h_n^* is the general term of the power sequence with o.g.f. $\frac{\alpha-\beta}{1-(\alpha-\beta)x}$. Thus in this case we do not recover terms of the sequence with o.g.f. $\frac{x(1-\alpha x)}{1-\beta x}$.

Example 7. The family $x(1-\alpha x)$.

We note that this is in fact the case of $\frac{x(1-\alpha x)}{1-\beta x}$ where $\beta = 0$. The sequence with o.g.f. $x(1-\alpha x)$ is the sequence $0, 1, -\alpha, 0, 0, 0, \ldots$. The reversion of the series $x(1-\alpha x)$, that is, the solution $u = u(x)$ of the equation

$$u(1-\alpha u) = x$$

is given by

$$u(x) = \frac{1-\sqrt{1-4\alpha x}}{2\alpha}.$$

For instance, the case $\alpha = 1$ is that of the Catalan numbers preceded by 0. In general, $\frac{1-\sqrt{1-4\alpha x}}{2\alpha}$ is the o.g.f. of the sequence $0, 1, \alpha, 2\alpha^2, 5\alpha^3, 14\alpha^4, \ldots$ with general term

$$a_0 = 0, \qquad a_n = C(n-1)\alpha^{n-1}, n > 0.$$

We obtain

Conjecture 8. Using the notation above, we have

1. $h_n = -n\alpha^{n^2-1}, \qquad \frac{h_{n+1}}{h_n^*} = -(n+1)\alpha^n$

2. $h_n^* = \alpha^{n(n+1)}$

3. $h_n^{**} = \alpha^{(n+1)^2}, \qquad \frac{h_n^{**}}{h_n^*} = \alpha^{n+1}$ $\diamond$

We note in particular that this generalizes the well-known result on the Hankel transforms of $C(n)$ and $C(n+1)$. We can in fact easily modify the proof of the fact that the Hankel transform of $C(n)$ is the all 1's sequence given in [7] to yield

Proposition 9. *The Hankel transform h_n^* of the sequence $C(n)\alpha^n$ is given by $h_n^* = \alpha^{n(n+1)}$.*

Proof. The coefficient of x^{i+j+1} in $(1-\alpha x)^2(1+\alpha x)^{2i+2j}$ is given by

$$\left\{\binom{2i+2j}{i+j+1} - 2\binom{2i+2j}{i+j} + \binom{2i+2j}{i+j-1}\right\}\alpha^{i+j+1} = -2C(i+j)\alpha^{i+j+1}.$$

On the other hand, the coefficient of x^k in $(1-\alpha x)(1+\alpha x)^{2i}$ is equal to

$$\left\{\binom{2i}{k} - \binom{2i}{k-1}\right\}\alpha^k = \binom{2i}{k}\frac{2i-2k+1}{2i-k+1}\alpha^k.$$

Proceeding as in [7], we obtain that

$$C(i+j)\alpha^{i+j} = \sum_{k=0}^{\min(i,j)} T_{i,k}T_{j,k}$$

where

$$T_{n,k} = \frac{\binom{2n}{n+k}(2k+1)}{n+k+1}\alpha^n.$$

Now $H_n = T_n U_n$ where U_n is the transpose of T_n, where T_n is the $(n+1)\times(n+1)$ matrix with general term $T_{i,k}$

$$\mathbf{T_n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \alpha & \alpha & 0 & 0 & 0 & 0 & \ldots \\ 2\alpha^2 & 3\alpha^2 & \alpha^2 & 0 & 0 & 0 & \ldots \\ 5\alpha^3 & 9\alpha^3 & 5\alpha^3 & \alpha^3 & 0 & 0 & \ldots \\ 14\alpha^4 & 28\alpha^4 & 20\alpha^4 & 7\alpha^4 & \alpha^4 & 0 & \ldots \\ 42\alpha^5 & 90\alpha^5 & 75\alpha^5 & 35\alpha^5 & 9\alpha^5 & \alpha^5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Hence h_n^* is the square of the product of the diagonal elements, that is

$$h_n^* = (\alpha^{\binom{n+1}{2}})^2 = \alpha^{n(n+1)}.$$

$\square$

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