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The Hankel transform of the sum of consecutive generalized Catalan numbers

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We discuss the properties of the Hankel transformation of a sequence whose elements are the sums of consecutive generalized Catalan numbers and find their values in the closed form.

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1. Introduction

The *Hankel transform* of a given sequence $A = \{a_0, a_1, a_2, \ldots\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \ldots\}$ (see [7–10]) where $h_n = |a_{i+j-2}|_{i,j=1}^{n}$, *i.e.*

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \longrightarrow \quad h = \{h_n\}_{n\in\mathbb{N}_0}: \quad h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix}. \tag{1}$$

In this paper, we will consider the sequence of the sums of two adjacent generalized Catalan numbers with parameter L:

$$a_0 = L + 1, \quad a_n = a_n(L) = c(n; L) + c(n+1; L) \ \ (n \in \mathbb{N}), \tag{2}$$

where

$$c(n; L) = T(2n, n; L) - T(2n, n-1; L) \tag{3}$$

with

$$T(n, k; L) = \sum_{j=0}^{n-k} \binom{k}{j} \binom{n-k}{j} L^j. \tag{4}$$

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The properties of $\{T(n,k;L)\}$ were studied by P. Barry in [1].

Example 1.1 Let $L=1$. Vandermonde's convolution identity implies that

$$\binom{n}{k}=\sum_j \binom{k}{j}\binom{n-k}{j}.$$

Hence

$$T(2n,n;1)=\binom{2n}{n},\quad T(2n,n-1;1)=\binom{2n}{n-1},$$

wherefrom we get Catalan numbers

$$c(n)=\binom{2n}{n}-\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}$$

and

$$a_n=c(n)+c(n+1)=\frac{(2n)!(5n+4)}{n!(n+2)!}\quad (n=0,1,2,\ldots).$$

In paper [2], Cvetković *et al.* have proved that the Hankel transform of a_n equals sequence of Fibonacci numbers with odd indices

$$h_n=F_{2n+1}=\frac{1}{\sqrt{5}\,2^{n+1}}\left\{(\sqrt{5}+1)(3+\sqrt{5})^n+(\sqrt{5}-1)(3-\sqrt{5})^n\right\}.$$

Example 1.2 For $L=2$ we get like $a_n(2)$ the next numbers

$$3, 8, 28, 112, 484, \ldots$$

and the Hankel transform h_n:

$$3, 20, 272, 7424, 405504, \ldots$$

One of us, Barry conjectured that

$$h_n(2)=2^{((n^2-n)/2)-2}\left\{(2+\sqrt{2})^{n+1}+(2-\sqrt{2})^{n+1}\right\}.$$

In general, Barry made the conjecture, which we will prove through this paper.

THEOREM 1.3 (The main result) *For the generalized Pascal triangle associated to the sequence $n\mapsto L^n$, the Hankel transform of the sequence*

$$c(n;L)+c(n+1;L)$$

is given by

$$\begin{aligned}h_n=\frac{L^{(n^2-n)/2}}{2^{n+1}\sqrt{L^2+4}}\cdot&\Big\{(\sqrt{L^2+4}+L)(\sqrt{L^2+4}+L+2)^n\\&+(\sqrt{L^2+4}-L)(L+2-\sqrt{L^2+4})^n\Big\}.\end{aligned}\tag{5}$$

From now till the end, let us denote by

$$\xi = \sqrt{L^2+4}, \quad t_1 = L+2+\xi, \quad t_2 = L+2-\xi. \tag{6}$$

Now, we can write

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot ((\xi+L)t_1^n + (\xi-L)t_2^n).$$

Or, introducing

$$\varphi_n = t_1^n + t_2^n, \quad \psi_n = t_1^n - t_2^n \quad (n \in \mathbb{N}_0), \tag{7}$$

the final statement can be expressed by

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot (L\psi_n + \xi\varphi_n). \tag{8}$$

LEMMA 1.4 *The values φ_n and ψ_n satisfy the next relations*:

$$\varphi_j \cdot \varphi_k = \varphi_{j+k} + (4L)^j \varphi_{k-j}, \quad \psi_j \cdot \psi_k = \varphi_{j+k} - (4L)^j \varphi_{k-j} \quad (0 \le j \le k), \tag{9}$$

$$\varphi_j \cdot \psi_k = \psi_{j+k} + (4L)^j \psi_{k-j}, \quad \psi_j \cdot \varphi_k = \psi_{j+k} - (4L)^j \psi_{k-j} \quad (0 \le j \le k). \tag{10}$$

COROLLARY 1.5 *Assuming that the main theorem is true, the function $h_n = h_n(L)$ is the next polynomial*

$$\begin{aligned} h_n(L) &= 2^{-n} L^{n(n-1)/2} \\ &\quad \cdot \left\{ \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L+2)^{n-2i-1}(L^2+4)^i + \sum_{i=0}^{[n/2]} \binom{n}{2i} (L+2)^{n-2i}(L^2+4)^i \right\}. \end{aligned}$$

Proof By previous notation, we can write

$$\begin{aligned} &(L+\xi)(L+2+\xi)^n - (L-\xi)(L+2-\xi)^n \\ &\quad = (L+\xi)\sum_{k=0}^{n} \binom{n}{k} (L+2)^{n-k}\xi^k - (L-\xi)\sum_{k=0}^{n} (-1)^k \binom{n}{k} (L+2)^{n-k}\xi^k \\ &\quad = \sum_{k=0}^{n} (1-(-1)^k) \binom{n}{k} L(L+2)^{n-k}\xi^k + \sum_{k=0}^{n} (1+(-1)^k) \binom{n}{k} (L+2)^{n-k}\xi^{k+1} \\ &\quad = 2\sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L+2)^{n-2i-1}\xi^{2i+1} + 2\sum_{i=0}^{[n/2]} \binom{n}{2i} (L+2)^{n-2i}\xi^{2i+1} \\ &\quad = 2\xi \left\{ \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L+2)^{n-2i-1}\xi^{2i} + \sum_{i=0}^{[n/2]} \binom{n}{2i} (L+2)^{n-2i}\xi^{2i} \right\}, \end{aligned}$$

wherefrom immediately follows the polynomial expression for h_n. ∎

2. The generating function for the sequences of numbers and orthogonal polynomials

The Jacobi polynomials are given by

$$P_n^{(a,b)}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+a}{k}\binom{n+b}{n-k}(x-1)^{n-k}(x+1)^k \quad (a,b>-1).$$

Also, they can be written in the form

$$P_n^{(a,b)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n}\binom{n+a}{k}\binom{n+b}{n-k}\left(\frac{x+1}{x-1}\right)^k.$$

From the fact

$$L = \frac{x+1}{x-1} \quad \Longleftrightarrow \quad x = \frac{L+1}{L-1} \quad (x \neq 1,\ L \neq 1),$$

we conclude that:

$$T(2n,n;L) = (L-1)^n \cdot P_n^{(0,0)}\left(\frac{L+1}{L-1}\right),$$

$$T(2n+2,n;L) = (L-1)^n \cdot P_n^{(2,0)}\left(\frac{L+1}{L-1}\right).$$

The generating function $G(x,t)$ for the Jacobi polynomials is

$$G^{(a,b)}(x,t) = \sum_{n=0}^{\infty} P_n^{(a,b)}(x)t^n = \frac{2^{a+b}}{\phi\cdot(1-t+\phi)^a\cdot(1+t+\phi)^b}, \tag{11}$$

where

$$\phi = \phi(x,t) = \sqrt{1-2xt+t^2}.$$

Now,

$$\sum_{n=0}^{\infty} T(2n,n;L)t^n = \sum_{n=0}^{\infty} P_n^{(0,0)}\left(\frac{L+1}{L-1}\right)((L-1)t)^n = G^{(0,0)}\left(\frac{L+1}{L-1},(L-1)t\right),$$

$$\sum_{n=0}^{\infty} T(2n+2,n;L)t^n = \sum_{n=0}^{\infty} P_n^{(2,0)}\left(\frac{L+1}{L-1}\right)((L-1)t)^n = G^{(2,0)}\left(\frac{L+1}{L-1},(L-1)t\right).$$

Also,

$$\sum_{n=0}^{\infty} T(2n,n-1;L)t^n = t\cdot\left\{G^{(2,0)}\left(\frac{L+1}{L-1},(L-1)t\right)-1\right\},$$

$$\sum_{n=0}^{\infty} T(2n+2,n+1;L)t^n = \frac{1}{t}\cdot\left\{G^{(0,0)}\left(\frac{L+1}{L-1},(L-1)t\right)-1\right\}.$$

The generating function $\mathcal{G}(t;L)$ for the sequence $\{a_n\}_{n\geq 0}$ is given by

$$\begin{aligned}\mathcal{G}(t;L) = \sum_{n=0}^{\infty} a_n t^n &= \frac{t+1}{t}G^{(0,0)}\left(\frac{L+1}{L-1},(L-1)t\right)\\ &\quad -(t+1)G^{(2,0)}\left(\frac{L+1}{L-1},(L-1)t\right)-\frac{1}{t}.\end{aligned} \tag{12}$$

After some computation, we prove the following theorem.

THEOREM 2.1 *The generating function $\mathcal{G}(t;L)$ for the sequence $\{a_n\}_{n\geq 0}$ is*

$$\mathcal{G}(t;L)=\frac{t+1}{\rho(t;L)}\left\{\frac{1}{t}-\frac{4}{(1-(L-1)t+\rho(t;L))^2}\right\}-\frac{1}{t}, \tag{13}$$

where

$$\rho(t;L)=\phi\left(\frac{L+1}{L-1},(L-1)t\right)=\sqrt{1-2(L+1)t+(L-1)^2t^2}. \tag{14}$$

The function $\rho(t;L)$ has domain

$$D_\rho=\left(-\infty,\frac{1-2\sqrt{L}+L}{1-2L+L^2}\right)\cup\left(\frac{1+2\sqrt{L}+L}{1-2L+L^2},+\infty\right)\quad (L\neq 1)$$

and

$$D_\rho=\left(-\infty,\frac{1}{4}\right)\quad (L=1).$$

Example 2.2 For $L=1$, we get

$$\mathcal{G}(t;1)=\sum_{n=0}^{\infty}a_n(1)\,t^n=\frac{1}{t}\left(\frac{(1-\sqrt{1-4t})(1+t)}{2t}-1\right) \tag{15}$$

and for $L=2$, we find

$$\mathcal{G}(t;2)=\sum_{n=0}^{\infty}a_n(2)\,t^n=-\frac{1}{t}+\frac{t+1}{\sqrt{t^2-6t+1}}\left\{\frac{1}{t}-\frac{4}{(1-t+\sqrt{t^2-6t+1})^2}\right\}. \tag{16}$$

3. The weight function corresponding to the functional

It is known (for example, see Krattenthaler [3]) that the Hankel determinant h_n of order n of the sequence $\{a_n\}_{n\geq 0}$ equals

$$h_n=a_0^n\beta_1^{n-1}\beta_2^{n-2}\cdots\beta_{n-2}^2\beta_{n-1}, \tag{17}$$

where $\{\beta_n\}_{n\geq 1}$ is the sequence given by:

$$\mathcal{G}(x)=\sum_{n=0}^{\infty}a_nx^n=\frac{a_0}{1+\alpha_0x-(\beta_1x^2/(1+\alpha_1x-(\beta_2x^2/(1+\alpha_2x-\cdots))))}. \tag{18}$$

The sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 1}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x)=(x-\alpha_n)Q_n(x)-\beta_nQ_{n-1}(x), \tag{19}$$

where $\{Q_n(x)\}_{n\geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional $\mathcal{U}$ determined by

$$\mathcal{U}[x^n]=a_n\quad (n=0,1,2,\ldots). \tag{20}$$

In this section, the functional will be constructed for the sum of consecutive generalized Catalan numbers. We would like to express $\mathcal{U}[f]$ in the form:

$$\mathcal{U}[f(x)]=\int_R f(x)\,\mathrm{d}\psi(x),$$

where $\psi(x)$ is a distribution, or, even more, to find the weight function $w(x)$ such that $w(x)=\psi'(x)$.

Denote by $F(z; L)$ the function

$$F(z; L) = \sum_{k=0}^{\infty} a_k z^{-k-1}.$$

From the generating function (13), we have:

$$F(z; L) = z^{-1}\, \mathcal{G}\big(z^{-1}; L\big) \tag{21}$$

and after some simplifications we obtain that

$$\begin{aligned} F(z; L) &= -1 + \frac{2(z+1)}{L - 1 + z + \sqrt{L^2 + (z-1)^2 - 2L(z+1)}} \\ &= -1 + \frac{2(z+1)}{L - 1 + z(1 + z\rho(\frac{1}{z}, L))} \end{aligned}$$

Example 3.1 From (15) and (16), we yield:

$$\begin{aligned} F(z; 1) &= z^{-1}\, \mathcal{G}(z^{-1}; 1) = \frac{1}{2}\left\{ z - 1 - (z+1)\sqrt{1 - \frac{4}{z}} \right\}, \\ F(z; 2) &= \frac{-1}{2z}\left\{ 1 + z\left(2 - z + (z+1)\sqrt{1 - \frac{6}{z} + \frac{1}{z^2}} \right) \right\}. \end{aligned}$$

Notice that

$$\begin{aligned} \int F(z; 2)\, \mathrm{d}z &= z + \frac{1}{4} z(z-1)\rho\left(\frac{1}{z}, 2\right) + \log(z) \\ &\quad - \frac{1}{2}\log\left(1 + z\left(\rho\left(\frac{1}{z}, 2\right) - 3\right)\right) - \frac{7}{2}\log\left(z - 3 + z\rho\left(\frac{1}{z}, 2\right)\right). \end{aligned}$$

It will be the impulse for further discussion.

Denote by

$$R(z; L) = z\rho\left(\frac{1}{z}, L\right) = \sqrt{L^2 + (z-1)^2 - 2L(z+1)}.$$

From the theory of distribution functions (see Chihara [4]), especially by the Stieltjes inversion formula

$$\psi(t) - \psi(0) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_0^t \Im F(x + iy; L)\, \mathrm{d}x, \tag{22}$$

we conclude that holds

$$\mathcal{F}(z; L) = \int F(z; L)\, \mathrm{d}z = \frac{1}{4}\big[z^2 - 2Lz - (z - L + 1)R(z; L) - l_1(z) + l_2(z)\big], \tag{23}$$

where

$$l_1(z) = 2(3L+1)\log[z-(L+1)+R(z;L)],$$
$$l_2(z) = 2(L-1)\log\left[\frac{-(L-1)R(z;L)-(L-1)^2+z(L+1)}{z^2(L-1)^3}\right].$$

Rewriting the function $R(z;L)$ in the form

$$R(z;L) = \sqrt{(z-L-1)^2-4L}$$

and replacing $z = x+iy$, we have

$$R(x;L) = \lim_{y\to 0^+} R(x+iy;L) = \begin{cases} i\sqrt{4L-(x-L-1)^2}, & x\in(a,b), \\ \sqrt{(x-L-1)^2-4L}, & \text{otherwise}, \end{cases}$$

where

$$a = (\sqrt{L}-1)^2, \quad b = (\sqrt{L}+1)^2. \tag{24}$$

In the case when $x \notin \big((\sqrt{L}-1)^2, (\sqrt{L}+1)^2\big)$, value $R(x;L)$ is real. Therefore we can calculate imaginary part of $\mathcal{F}(x;L) = \lim_{y\to 0^+}\mathcal{F}(x+iy;L)$:

$$\Im\mathcal{F}(x;L) = \Im[l_2(x)-l_1(x)] = 0.$$

Otherwise, if $x \in \big((\sqrt{L}-1)^2, (\sqrt{L}+1)^2\big)$ we have that:

$$l_1(x) = 2(3L+1)\log\Big[x-(L+1)\pm i\sqrt{4L-(x-L-1)^2}\Big],$$

$$\Im l_1(x) = \begin{cases} 2(3L+1)\arctan\dfrac{\sqrt{4L-(x-L-1)^2}}{x-(L+1)}, & x\ge L+1, \\ 2(3L+1)\left(\pi+\arctan\dfrac{\sqrt{4L-(x-L-1)^2}}{x-(L+1)}\right), & x<L+1, \end{cases}$$

$$l_2(x) = 2(L-1)\log\left[\frac{-(L-1)^2+2x(L+1)-i(L-1)\sqrt{4L-(x-L-1)^2}}{x^2(L-1)^3}\right],$$

$$\Im l_1(x) = \begin{cases} 2(L-1)\left(2\pi+\arctan\dfrac{x(L+1)-(L-1)^2}{\sqrt{4L-(x-L-1)^2}}\right), & x\ge \dfrac{(L-1)^2}{L+1}, \\ 2(L-1)\left(\pi+\arctan\dfrac{x(L+1)-(L-1)^2}{\sqrt{4L-(x-L-1)^2}}\right), & x< \dfrac{(L-1)^2}{L+1}. \end{cases}$$

After substituting all considered cases in (23), we finally obtain the value

$$\Im\mathcal{F}(x;L) = \lim_{y\to 0^+}\Im\mathcal{F}(x+iy;L) = \Im l_2(x)-\Im l_1(x)-(x-L+1)\sqrt{4L-(x-L-1)^2}.$$

From the relation (22), we conclude that

$$\omega(x;L) = \psi'(x;L) = -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\Im\mathcal{F}(x;L) \tag{25}$$

and finally, we obtain

$$\begin{aligned}\omega(x;L) &= \frac{1}{2\pi}\left(1+\frac{1}{x}\right)\sqrt{4L-(x-L-1)^2}\\ &= \frac{\sqrt{L}}{\pi}\left(1+\frac{1}{x}\right)\sqrt{1-\left(\frac{x-L-1}{2\sqrt{L}}\right)^2}.\end{aligned} \tag{26}$$

The previous formula holds for $x \in (a,b)$, and otherwise is $\omega(x;L)=0$.

4. Determining the three-term recurrence relation

The crucial moment in our proof of the conjecture is to determine the sequence of polynomials $\{Q_n(x)\}$ orthogonal with respect to the weight $w(x;L)$ given by (26) on the interval (a,b) and to find the sequences $\{\alpha_n\}$ $\{\beta_n\}$ in the three-term recurrence relation.

Example 4.1 For $L=4$, we can find the first members:

$$\begin{aligned}
&Q_0(x)=1, && \|Q_0\|^2=5,\\
&Q_1(x)=x-\frac{24}{5}, && \|Q_1\|^2=\frac{104}{5},\\
&Q_2(x)=x^2-\frac{127}{13}x+\frac{256}{13}, && \|Q_2\|^2=\frac{1088}{13},\\
&Q_3(x)=x^3-\frac{541}{17}x^2+\frac{1096}{17}x-\frac{1344}{17}, && \|Q_3\|^2=\frac{5696}{17},
\end{aligned}$$

wherefrom

$$\alpha_0=\frac{24}{5},\quad \beta_0=5,\quad \alpha_1=\frac{323}{65},\quad \beta_1=\frac{104}{25},\quad \alpha_2=\frac{1104}{221},\quad \beta_2=\frac{680}{169}.$$

Hence

$$h_1=a_0=5,\quad h_2=a_0^2\beta_1 =104,\quad h_3=a_0^3\beta_1^2\beta_2=5^3\left(\frac{104}{25}\right)^2\frac{680}{169}=8704.$$

At the beginning, we will notice that in the definition of the weight function appears the square root member.

That is why, let us consider the monic orthogonal polynomials $\{S_n(x)\}$ with respect to the $p^{(1/2,1/2)}(x)=\sqrt{1-x^2}$ on the interval $(-1,1)$. These polynomials are monic Chebyshev polynomials of the second kind:

$$S_n(x)=\frac{\sin((n+1)\arccos x)}{2^n\cdot\sqrt{1-x^2}}.$$

They satisfy the three-term recurrence relation (Chihara [4]):

$$S_{n+1}(x)=(x-\alpha_n^*)\,S_n(x)-\beta_n^* S_{n-1}(x)\quad (n=0,1,\ldots), \tag{27}$$

with initial values

$$S_{-1}(x)=0,\quad S_0(x)=1,$$

where

$$\alpha_n^* = 0 \ \ (n \geq 0) \quad \text{and} \quad \beta_0^* = \frac{\pi}{2}, \ \ \beta_n^* = \frac{1}{4} \ \ (n \geq 1).$$

If we use the weight function $\hat{w}(x) = (x - c) \ \ p^{(1/2,\ 1/2)}(x)$, then the corresponding coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ can be evaluated as follows (see, for example, Gautschi [5]):

$$\begin{aligned} \lambda_n &= S_n(c), \\ \hat{\alpha}_n &= c - \frac{\lambda_{n+1}}{\lambda_n} - \beta_{n+1}^* \frac{\lambda_n}{\lambda_{n+1}}, \\ \hat{\beta}_n &= \beta_n^* \frac{\lambda_{n-1}\lambda_{n+1}}{\lambda_n^2} \quad (n \in \mathbb{N}_0). \end{aligned} \tag{28}$$

From the relation (27), we conclude that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfies the following recurrence relation:

$$4\lambda_{n+1} - 4c\lambda_n + \lambda_{n-1} = 0 \quad (\lambda_{-1} = 0;\ \lambda_0 = 1). \tag{29}$$

The characteristic equation

$$4z^2 - 4\ c\ z + 1 = 0$$

has the solutions

$$z_{1,2} = \frac{1}{2}\bigl(c \pm \sqrt{c^2 - 1}\bigr)$$

and the integral solution of (29) is

$$\lambda_n = E_1 z_1^n + E_2 z_2^n \quad (n \in \mathbb{N}).$$

We evaluate values E_1 and E_2 from the initial conditions $(\lambda_{-1} = 0; \lambda_0 = 1)$.

In order to solve our problem, we will choose $c = -(L + 2/2\sqrt{L})$. Hence

$$z_k = \frac{-t_k}{4\sqrt{L}} \ \ (k = 1, 2), \quad \text{where} \ \ t_{1,2} = L + 2 \pm \sqrt{L^2 + 4}.$$

Finally, we obtain:

$$\lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{n/2}\sqrt{L^2 + 4}}\bigl(t_1^{n+1} - t_2^{n+1}\bigr) \quad (\lambda = -1, 0, 1, \ldots),$$

i.e.,

$$\lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{n/2}\xi}\psi_{n+1} \quad (\lambda = -1, 0, 1, \ldots).$$

After replacing in (28), we obtain:

$$\hat{\alpha}_n = -\frac{L + 2}{2\sqrt{L}} + \frac{1}{4\sqrt{L}} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + \sqrt{L} \cdot \frac{\psi_{n+1}}{\psi_{n+2}}, \tag{30}$$

$$\hat{\beta}_n = \frac{\psi_n \psi_{n+2}}{4\psi_{n+1}^2}. \tag{31}$$

If a new weight function $\tilde{w}(x)$ is introduced by

$$\tilde{w}(x) = \hat{w}(ax + b),$$

then we have

$$\tilde{\alpha}_n = \frac{\hat{\alpha}_n - b}{a}, \quad \tilde{\beta}_n = \frac{\hat{\beta}_n}{a^2} \quad (n \geq 0).$$

Now, by using $x \mapsto (x - L - 1)/2\sqrt{L}$, *i.e.*, $a = 1/2\sqrt{L}$ and $b = -(L + 1/2\sqrt{L})$, we have the weight function

$$\tilde{w}(x) = \hat{w}\left(\frac{x - L - 1}{2\sqrt{L}}\right) = \frac{1}{2}\left(\frac{x - L - 1}{2\sqrt{L}} + \frac{L+2}{2\sqrt{L}}\right)\sqrt{1 - \left(\frac{x - L - 1}{2\sqrt{L}}\right)^2}.$$

Thus

$$\tilde{\alpha}_n = -1 + \frac{1}{2} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + 2L \cdot \frac{\psi_{n+1}}{\psi_{n+2}} \quad (n \in \mathbb{N}_0) \tag{32}$$

and

$$\tilde{\beta}_0 = (L+2)\frac{\pi}{2}, \quad \tilde{\beta}_n = L\frac{\psi_n \psi_{n+2}}{\psi_{n+1}^2} \ \ (n \in \mathbb{N}). \tag{33}$$

Example 4.2 For $L = 4$, we get

$$\begin{aligned}
&P_0(x) = 1, && \|P_0\|^2 = 3\pi,\\
&P_1(x) = x - \frac{17}{3}, && \|P_1\|^2 = \frac{32\pi}{3},\\
&P_2(x) = x^2 - \frac{43}{4}x + \frac{101}{4}, && \|P_2\|^2 = 42\pi,\\
&P_3(x) = x^3 - \frac{331}{21}x^2 + \frac{1579}{21}x - \frac{2189}{21}, && \|P_3\|^2 = \frac{3520\pi}{21},
\end{aligned}$$

wherefrom

$$\tilde{\alpha}_0 = \frac{17}{3}, \ \ \tilde{\beta}_0 = 3\pi, \quad \tilde{\alpha}_1 = \frac{61}{12}, \ \ \tilde{\beta}_1 = \frac{32}{9}, \quad \tilde{\alpha}_2 = \frac{421}{84}, \ \ \tilde{\beta}_2 = \frac{63}{16}.$$

Introducing the weight

$$\breve{w}(x) = \frac{2L}{\pi}\tilde{w}(x)$$

will not change the monic polynomials and their recurrence relations, only it will multiply the norms by the factor $2L/\pi$, *i.e.*

$$\begin{gathered}
\breve{P}_k(x) \equiv P_k(x), \quad \|\breve{P}_k\|_{\breve{w}}^2 = \int_a^b \breve{P}_k(x)\breve{w}(x)\,\mathrm{d}x = \frac{2L}{\pi}\|P_k\|^2 \quad (k \in \mathbb{N}_0),\\
\breve{\beta}_0 = L(L+2), \ \ \breve{\beta}_k = \tilde{\beta}_k \ \ (k \in \mathbb{N}), \quad \breve{\alpha}_k = \tilde{\alpha}_k \ \ (k \in \mathbb{N}_0).
\end{gathered}$$

Here is

$$\breve{\beta}_0\breve{\beta}_1 \cdots \breve{\beta}_{n-1} = \frac{L^n}{2} \cdot \frac{\psi_{n+1}}{\psi_n}. \tag{34}$$

In [6], Gautschi has treated the next problem: If we know all about the MOPS orthogonal with respect to $\breve{w}(x)$ what can we say about the sequence $\{Q_n(x)\}$ orthogonal with respect to a weight

$$w_d(x) = \frac{\breve{w}(x)}{x-d} \quad (d \notin \text{support}(\tilde{w})) \ ?$$

Gautshi has proved that, by the auxiliary sequence

$$r_{-1} = -\int_{\mathbb{R}} w_d(x)\,\mathrm{d}x, \quad r_n = d - \breve{\alpha}_n - \frac{\breve{\beta}_n}{r_{n-1}} \ (n = 0, 1, \ldots),$$

it can be determined

$$\alpha_{d,0} = \breve{\alpha}_0 + r_0, \quad \alpha_{d,k} = \breve{\alpha}_k + r_k - r_{k-1}, \quad \beta_{d,0} = -r_{-1}, \quad \beta_{d,k} = \breve{\beta}_{k-1}\frac{r_{k-1}}{r_{k-2}} \quad (k \in \mathbb{N}).$$

In our case it is enough to take $d = 0$ to get the final weight

$$w(x) = \frac{\breve{w}(x)}{x}.$$

Hence

$$r_{-1} = -(L+1), \quad r_n = -\left(\breve{\alpha}_n + \frac{\breve{\beta}_n}{r_{n-1}}\right) \ (n = 0, 1, \ldots). \tag{35}$$

LEMMA 4.3 *The parameters r_n have the explicit form*

$$r_n = -\frac{\psi_{n+1}}{\psi_{n+2}} \cdot \frac{L\psi_{n+2} + \xi\varphi_{n+2}}{L\psi_{n+1} + \xi\varphi_{n+1}} \quad (n \in \mathbb{N}_0). \tag{36}$$

Proof We will use the mathematical induction. For $n = 0$, we really get the expected value

$$r_0 = -\frac{L^2 + 2L + 2}{(L+1)(L+2)}.$$

Suppose that it is true for $k = n$. Now, by the properties for φ_n and ψ_n, we have

$$\tilde{\alpha}_{n+1} \cdot r_n + \tilde{\beta}_{n+1} = -\frac{\psi_{n+1}}{\psi_{n+3}} \cdot \frac{L\psi_{n+3} + \xi\varphi_{n+3}}{L\psi_{n+1} + \xi\varphi_{n+1}}.$$

Dividing with r_n, we conclude that the formula is valid for r_{n+1}. ■

Example 4.4 For $L = 4$, we get

$$r_{-1} = -5, \quad r_0 = -\frac{13}{15}, \quad r_1 = -\frac{51}{52}, \quad r_2 = -\frac{356}{357},$$

wherefrom

$$\alpha_0 = \frac{24}{5}, \ \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \ \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \ \beta_2 = \frac{680}{169},$$

just the same as in Example 4.1.

Proof of the main result Krattenthaler's formula (17) can also be written in the form

$$h_1 = a_0, \quad h_n = \beta_0\beta_1\beta_2\cdots\beta_{n-2}\beta_{n-1}\cdot h_{n-1}. \tag{37}$$

From the theory of orthogonal polynomials, it is known that

$$\|Q_{n-1}\|^2 = \beta_0\beta_1\beta_2\cdots\beta_{n-2}\beta_{n-1} \quad (n = 2, 3, \ldots), \tag{38}$$

wherefrom

$$h_1 = a_0, \quad h_n = \|Q_{n-1}\|^2 \cdot h_{n-1} \ \ (n = 2, 3, \ldots). \tag{39}$$

■

Here,

$$\|Q_{n-1}\|^2 = \beta_0 \frac{r_{n-2}}{r_{-1}} \prod_{k=0}^{n-2} \breve{\beta}_k = \frac{L^{n-1}}{2} \cdot \frac{L\psi_n + \xi\varphi_n}{L\psi_{n-1} + \xi\varphi_{n-1}}. \tag{40}$$

We will apply the mathematical induction again. The formula for h_n is true for $n = 1$. Suppose that it is valid for $k = n - 1$. Then

$$h_n = \frac{L^{n-1}}{2} \cdot \frac{L\psi_n + \xi\varphi_n}{L\psi_{n-1} + \xi\varphi_{n-1}} \cdot \frac{L^{(n-1)(n-2)/2}}{2^n\xi} \cdot (L\psi_{n-1} + \xi\varphi_{n-1}),$$

wherefrom it follows that the final statement

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot (L\psi_n + \xi\varphi_n) \quad (n \in \mathbb{N})$$

is true.

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